NOTES ON VERTEX ATLAS OF DANZER TILING

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ABSTRACT. In this note, we study in detail the remark in the appendix of Danzer [6]. We find that planer Danzer tilings have many different aspects than Penroze tilings. For e.g., we observe that Danzer tiling with 7-fold symmetry does not belong to the topological closure of tilings generated by up-down generation.

1. Introduction

In 1982 a quasi-crystal with 5-fold rotational symmetry was discovered by Shechtman et al.(published in 1984 [11]). Before that, it had been believed that the structure of crystals was periodic, like a wallpaper pattern. Periodicity is another name for translational symmetry. Translational symmetry is another name of periodicity. 5-fold rotational symmetry is incompatible with translational symmetry and therefore quasi-crystals are not periodic. The most famous 2-dimensional mathematical model for a quasi-crystal may be Penrose tiling with 5-fold rotational symmetry ([7],[8]). In addition, there are Ammann-Beenker tiling with 8-fold rotational symmetry([1],[2]) and Danzer tiling with 7-fold rotational symmetry([6]) in typical tilings.

We prepare several basic definitions (cf.[10]). A planar tiling $\mathcal{T}$ is a countable family of polygons T_i called tiles: $\mathcal{T} = \{T_i \,|\, i = 1, 2, \cdots \}$ such that $\bigcup_{i=1}^{\infty} T_i = \mathbf{R}^2$ and $\mathrm{Int}\, T_i \cap \mathrm{Int}\, T_j = \emptyset$ if $i \neq j$, where $\mathbf{R}^2$ denotes the 2-dimensional Euclidean space. A nonperiodic tiling is one that admits no translation isomorphisms to itself. A patch of a tiling is a finite subset of the tiles in a tiling. A vertex atlas of a vertex in a tiling is the patch consisting of all tiles which contains the vertex.

Let $\mathcal{S} = \{S_1, S_2, \cdots, S_l\}$ be a finite set of polygons S_j. When each tile T in a tiling $\mathcal{T}$ is congruent to some $S_i \in \mathcal{S}$, $\mathcal{S}$ is called a prototile set of $\mathcal{T}$.

A set of matching rules for a prototile set $\mathcal{S}$ is a finite set of patches that may appear in the tilings admitted by $\mathcal{S}$.

Fix $\lambda(> 1)$. For $\mathcal{S} = \{S_1, S_2, \cdots, S_l\}$, any S_k $(\in \mathcal{S})$ is decomposed into λ^{-1} scale-down copies $\lambda^{-1}\mathcal{S} = \{\lambda^{-1}S_1, \lambda^{-1}S_2, \cdots, \lambda^{-1}S_l\}$ of $\mathcal{S}$. This decomposition is called a substitution rule of $\mathcal{S}$ if such a decomposition is possible. Let Φ denote a substitution rule of $\mathcal{S}$, and $\Phi[S_k]$ denote the decomposition of $S_k \in \mathcal{S}$. Similarly, let $\hat{\Phi}[S_k]$ denote the decomposition of λS_k into $\mathcal{S}$. A patch in a tiling corresponding to $\hat{\Phi}^n[S_k]$ is called a supertile with level n of S_k.

We can construct nonperiodic tilings with a given prototile set by up-down generation using substitution rule (cf.[3]).

The prototiles of Danzer tiling are six types of triangles with arrows on the edges (three triangles a, b, c in Figure 1 and their reflections). The substitution rule of prototiles is given as in Figure 2 and their mirror images (cf.[6]).
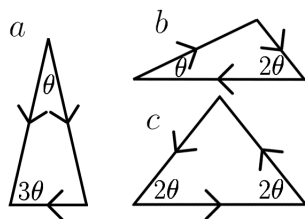

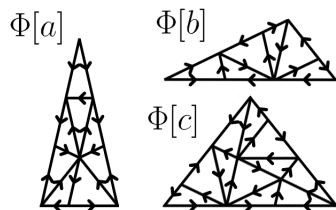FIGURE 1. Three prototiles with arrows of Danzer tiling $(\theta = \pi/7)$

FIGURE 2. The substitution rule of Danzer tiling

One of the purpose of this note is to analyze construction of tilings with rotational symmetry obtained by substitution rules (cf.[5]). This note is motivated by the following remark in the appendix of [6]: "29 kinds of vertex atlases appear in Danzer tiling, and these vertex atlases may serve as a matching rule." We study details of his remark, and obtain the following theorem.

Theorem 1.1. *Danzer tiling satisfies the following properties* (1) – (4)*:*

(1) *In Danzer tilings generated by up-down generation, 39 kinds of vertex atlases with arrows appear, and 29 kinds of vertex atlases appear by erasing arrows.*

(2) *The Danzer tiling with 7-fold symmetry cannot be generated by up-down generation. It is necessary to expand to whole plane by using reflection and rotation.*

(3) *Danzer tiling with 7-fold symmetry does not belong to the topological closure of tilings generated by up-down generation.*

(4) *39 kinds of vertex atlases with arrows serve as a matching rule in the set of tilings generated by up-down generation, but cannot do in the set of tilings generated by up-down generation and reflection and rotation.*

We compare the results of Theorem with the following properties of Penrose tiling:

(1) 8 kinds of vertex atlases with arrows appear in Penrose tilings generated by up-down generation (cf. [4],[10]).

(2) Penrose tiling with 5-fold symmetry cannot be generated by up-down generation. It is necessary to expand to whole plane by using reflection and rotation (folklore. We prove in the section 3).

(3) Penrose tiling with 5-fold symmetry belongs to the topological closure of tilings generated by up-down generation. (folklore. We prove in the section 3).

(4) 8 kinds of vertex atlases with arrows serve as a matching rule in the set of tilings generated by up-down generation and reflection and rotation (cf. [4],[10]).

Our theorem says that substitution rule of Danzer tiling is exotic in contrast to Penrose tiling's one.

This note is arranged as follows. In the section 2 we prove Theorem. In the section 3 we tell about our remarks.

2. Proof of Theorem

(1) A tiling is generated by up-down generation if and only if every patch of the tiling is found in some supertile. $\hat{\Phi}^2[a]$ includes any of prototiles a, b, c and these reflections for the prototile a. So, it suffices to check up vertex atlases which appear in $\hat{\Phi}^n[a]$. New vertex atlases of the number 1 in Figure 3 appear in $\hat{\Phi}[a]$. In addition, new vertex atlases of the number from 2 to 9 and 1'and 2' in Figure 3 appear in $\hat{\Phi}^2[a]$, where k' denotes the reflection of k. When we subdivide further, we have new vertex atlases of the number from 10 to 19 and from 3' to 6' and 13' in Figure 3 in $\hat{\Phi}^3[a]$, new vertex atlases of the number from 20 to 26 and from 14' to 17' in Figure 3 in $\hat{\Phi}^4[a]$ and new vertex atlases of the number 20' and 21' in $\hat{\Phi}^5[a]$. Because new vertex atlas do not come out in $\hat{\Phi}^6[a]$, this subdivision process finishes at this point. Note that the same shape is on the block in Figure 3, and that there are some vertex

atlases with reflection of different shape. Then we obtain that 39 kinds of vertex atlases with arrows appear, and that 29 kinds of vertex atlases appear by erasing arrows.
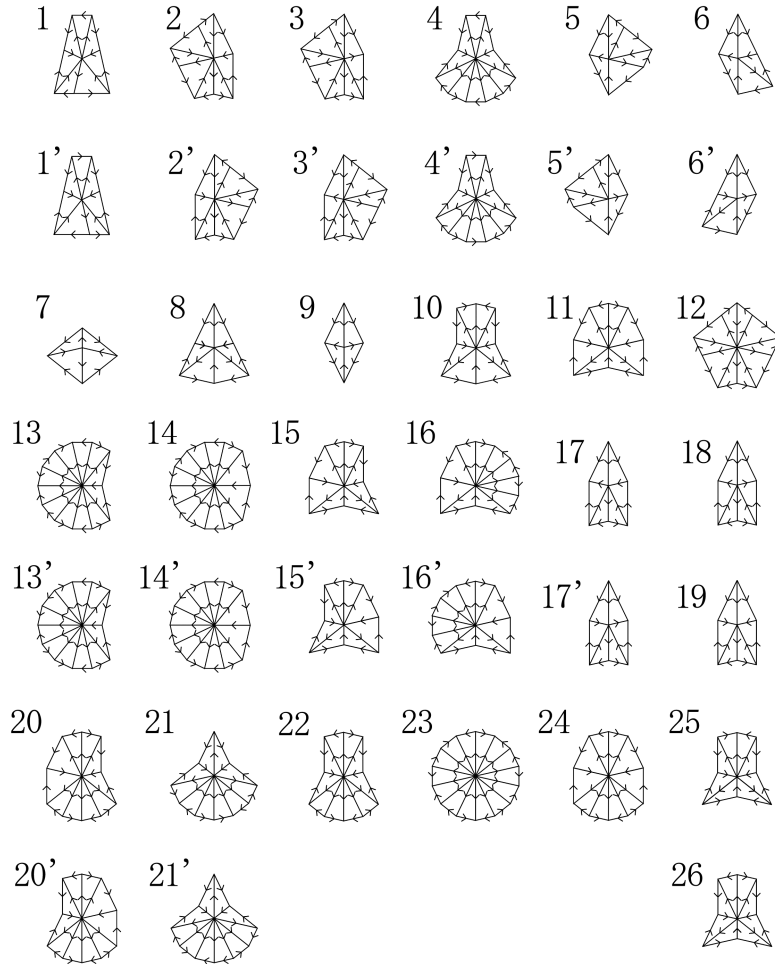

FIGURE 3. The list of vertex atlases

(2) We can construct the Danzer tiling with 7-fold symmetry in Figure 5 using up-down generation and reflection and rotation(cf.[5],[6]). When we take a sequence $\{\hat{\Phi}^n[a]\} = \{a, \hat{\Phi}[a], \hat{\Phi}^2[a], \cdots\}$ of up-down generation procedure, we get the unbounded configuration in Figure 4. We expand this unbounded configuration to whole plane by using reflection and rotation. Then Danzer tiling with 7-fold symmetry appears as in Figure 5. Note that the Danzer tiling obtained above is only tiling with 7-fold symmetry (cf. [5]).

The arrowed vertex atlas of Figure 6 is not in the list of vertex atlases of Figure 3. So, the arrowed vertex atlas of Figure 6 cannot appear in tilings generated by
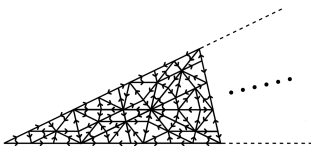
FIGURE 4. The unbounded configuration obtained from a sequence $\{\hat{\Phi}^n[a]\}$
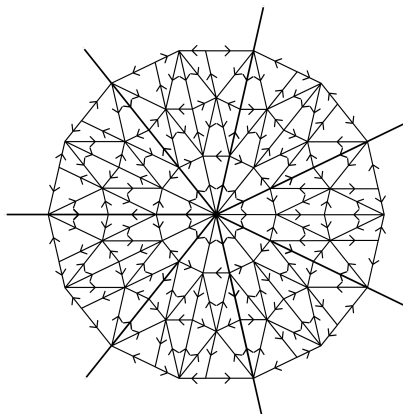

FIGURE 5. Danzer tiling with 7-fold symmetry

up-down generation. Since the arrowed vertex atlas of Figure 6 is the arrowed vertex atlas at the center of 7-fold symmetry, the arrowed Danzer tiling with 7-fold symmetry cannot be generated by up-down generation.

We find an arrowed local configuration around a vertex that doesn't exist in Figure 3 whenever we draw arrows on edges in the patch of Figure 7. So, the patch of Figure 7 cannot appear in tilings generated by up-down generation. Even when the arrows are erased, Danzer tiling with 7-fold symmetry cannot be generated by up-down generation.

The proof of (2) is completed.

(3) Assume that Danzer tiling with 7-fold symmetry is obtained as a limit of sequence of tilings generated by up-down generation. Since the substitution matrix of Danzer tiling is primitive, we obtain that every patch in Danzer tiling with 7-fold symmetry appears in any of sequence of tilings by the definition of the tiling metric. In the proof of Theorem (2) we obtain that the arrowed vertex atlas of Figure 6 and the patch in Figure 7 cannot appear in tilings generated by up-down generation.

By contradiction the proof of (3) is completed.

(4) This follows immediately by the result of (1) and the proof of (2).
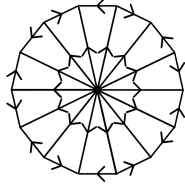
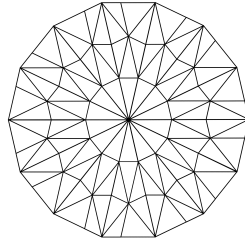FIGURE 6. Arrowed vertex atlas at the center of 7-fold symmetry

FIGURE 7. Patch around the center of 7-fold symmetry

3.   Remarks and Folklores

3.1.   Singular vertex atlases

We find vertex atlases (i), (ii) in Figure 8 like impurities in the material. These do not exist in the list of Figure 3.
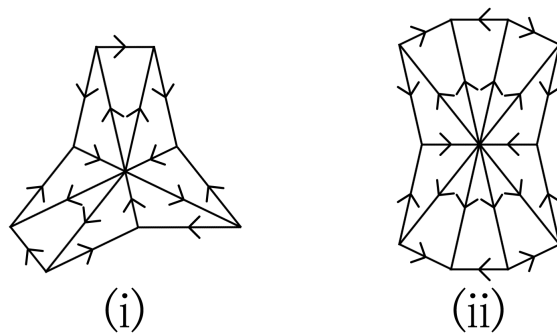

FIGURE 8. Vertex atlases (i), (ii)

When we subdivides vertex atlas (i) (resp. (ii)) and rescaling, we have patch (I) (resp. (II)) in Figure 9 (resp. 10) which includes (i) (resp. (ii)). By repeating

subdivision and rescaling, we get a tiling with vertex atlas (i) (resp. (ii)). Vertex atlas (i) (resp. (ii)) appears only in one place, and other vertex atlases are in 39 vertex atlases. In addition, this tiling remains unchanged by the subdivision for each tile and rescaling.
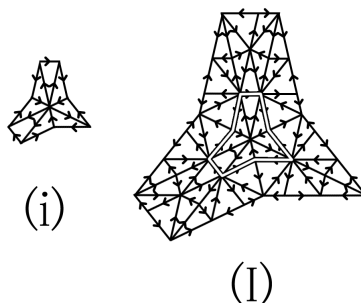

(i)

(I)

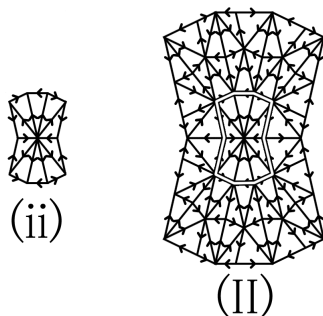FIGURE 9. The Patch (I)

(ii)

(II)

FIGURE 10. The Patch (II)

Starting from vertex atlas (i) (resp. (ii)) we join 39 vertex atlases and generate larger patch. By trial and error, we can show that a patch (I) (resp. (II)) is necessarily included in suffcient large patches.

Note that the above results remain true even if some of the 'a' tiles are substituted by their reflections in (i) and (ii).

3.2. Folklores

It seems that the properties (2),(3) of Penrose tiling in the section 1 are folklores and unpublished. We give the proof of (2),(3).

(2) Penrose tiling with 5-fold symmetry cannot be generated by up-down generation. It is necessary to expand to whole plane by using reflection and rotation

Proof. Assume that Penrose tiling with 5-fold symmetry can be generated by up-down generation. Then we obtain that there exists a supertile such that the center

of 5-fold symmetry is in the interior of the union of all tiles in the supertile. When we can get unique scale-up tiling by regarding level 1 supertiles as prototiles, this property is called unique composition property. Penrose tilings have unique composition property (cf. [4],[10]). We see that all of Penrose tilings with 5-fold symmetry are two tilings in Figure 11 (cf. [5]). In the scale-up procedure, these two tilings
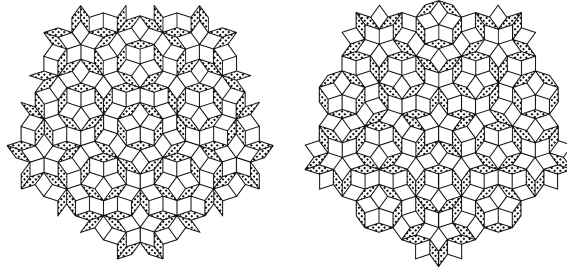

FIGURE 11. All of Penrose tilings with 5-fold symmetry

appear alternately(see Figure 12). By repeating the scale-up procedure, vertex at-
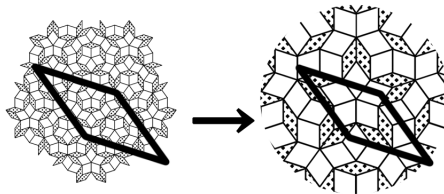

FIGURE 12. Example of the scale-up procedure

lases at the center of 5-fold symmetry must be found in supertiles of level 1. This contradicts the substitution rule of Penrose tiles.

Hence, the proof of (2) is completed. □

In general, we obtain that the following proposition by the similar argument.

Proposition 3.1. *Assume that tilings has unique composition property. If any of minimal patches around the center of rotational symmetry is not in supertiles of level 1, then tilings with rotational symmetry cannot be generated by up-down generation.*

(3) Penrose tiling with 5-fold symmetry is obtained as a limit of sequence of tilings generated by up-down generation.

Proof. The arrowed vertex atlas at the center of 5-fold symmetry is in 8 kinds of vertex atlases with arrows of the property (1) of Penrose tiling in the section 1. So, we can choose a tiling $\mathcal{T}$ with the same arrowed vertex atlas as one at the center of 5-fold symmetry. We may assume that $\mathcal{T}$ can be generated by up-down generation. By the subdivision for each tile in the tiling $\mathcal{T}$ and rescaling, we get a new tiling $\hat{\Phi}(\mathcal{T})$. Inductively, we have the sequence of tilings $\hat{\Phi}^n(\mathcal{T})$ ($n = 2, 3, \ldots$). We obtain that $\hat{\Phi}^n(\mathcal{T})$ ($n = 1, 2, 3, \ldots$) can be generated by up-down generation since $\mathcal{T}$ can be generated by up-down generation. For some sufficient large n, $\hat{\Phi}^n[\mathcal{T}]$ has the same patch as any patch which includes the center in each of Penrose tilings with 5-fold symmetry. In fact, we can choose convergent sequences $\{\mathcal{T}, \hat{\Phi}^2[\mathcal{T}], \hat{\Phi}^4[\mathcal{T}], \cdots\}$ and $\{\hat{\Phi}[\mathcal{T}], \hat{\Phi}^3[\mathcal{T}], \hat{\Phi}^5[\mathcal{T}], \cdots\}$ to Penrose tilings with 5-fold symmetry. $\qquad\square$

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