Nihonkai Math. J.
Vol.22(2011), 39–47

GARDEN REPRESENTATION AND INTERIOR VARIATION OF REAL RATIONAL FUNCTIONS

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ABSTRACT. In this paper, we show that the fundamental surgeries of the graph representation for real rational functions can be achieved by classical interior variations essentially due to Schiffer. As an application we give a constructive proof of the main theorem of Natanzon, Shapiro and Vainshtein in [2].

1. Introduction

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree n. We say that f is *in general position* if the preimage $f^{-1}(p)$ of any point $p \in \widehat{\mathbb{C}}$ consists of either n or $n-1$ points. The points of the latter type are called *simple ramification points*. The set of all simple ramification points of f is denoted by $\Sigma(f)$ and consists of $2n-2$ points.

Next, let τ be an anti-holomorphic involution of $\widehat{\mathbb{C}}$. A rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\overline{f(\tau p)} = f(p)$ for any $p \in \widehat{\mathbb{C}}$ is called to be *real*. Clearly, $\overline{\Sigma(f)} = \Sigma(f)$ for any real rational function f.

We use an object, called a *garden*, which consists of a weighted labeled directed planar chord diagram and a set of weighted rooted trees each of which corresponds to faces of the diagram. We start with the definition of it. See [2].

Definition 1.1. By a *planar chord diagram* (of *order* $2l$) we mean a circle drawn on the plane together with $2l$ points on this circle partitioned into l pairs in such a way that, for any two pairs, the chords joining the points from the same pair do not intersect. The above $2l$ points are called the *vertices* of the chord diagram. The chords joining the vertices from the same pair and the arcs of the circle joining adjacent vertices are called the *edges*. The notion of its *faces* is defined in a usual way (except for the outer face of the graph, which is not a face of the diagram).

We say that a planar chord diagram is *directed* if its edges are directed in such a way that the boundary of each face becomes a directed circle. A planar chord

2000 *Mathematics Subject Classification.* 30C15, 26C15, 32G05.
Key words and phrases. Real rational functions, interior variation.

diagram is said to be *weighted* if each edge is equipped with a nonnegative integer, and *labeled* if there exists a bijection β which maps the set of all vertices to $\{1, 2, \dots, 2l\}$. The value $\beta(v)$ is called the *label* of v. Two labelings β_1 and β_2 are said to be *cyclically equivalent* if $\beta_1(v) - \beta_2(v)$ is a constant $\bmod 2l$ not depending on the choice of a vertex v.

Definition 1.2. A *rooted tree* is, by definition, a tree with one distinguished vertex called the *root*. All other vertices of the tree are said to be *inner*. We say that a rooted tree is *weighted* if each vertex is equipped with a positive integer.

Definition 1.3. A *garden* is a weighted labeled directed planar chord diagram with a set of weighted rooted trees (possibly consisting of its roots only) associated with faces of the diagram. The weights of the inner vertices of the trees are arbitrary positive integers, and the weight of the root associated with a face j equals t_j defined below. The (*total*) *weight* of the garden equals the sum of the weights of all roots and two times of the weights of all inner vertices of trees.

Here, for any face j of a weighted labeled directed planar chord diagram, d_j denotes the number of descents in the sequence of the labels of vertices ordered cyclically along the boundary of the face j, and t_j denotes the sum of d_j and the weights of all edges on the boundary of the face j.

Two gardens are said to be *equivalent* if there exists a bijection of the sets of the vertices which preserves chords, orientation, the weights of edges, labels (up to the cyclic equivalence), the rooted trees, and the weights of inner vertices. In the sequel, we identify equivalent gardens.

Here, we recall how to get a garden from a real rational function. Take a real rational function f of degree n in general position, and represent $\Sigma = \Sigma(f)$ as $\Sigma_R \cup \Sigma_I$, where Σ_R and Σ_I are the sets of all real critical values and of all non-real ones of f, respectively. The number of elements in Σ_R is denoted by $2l(\Sigma)$.

Let $S(f)$ be the preimage of the extended real line $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ under f. For every point in Σ_R, $S(f)$ contains exactly 4 arcs corresponding to a neighborhood of it. These arcs define a 2-dimensional cell complex on $\widehat{\mathbb{C}}$. The 2-cells of this complex are called the *faces* of $S(f)$. Here, $S(f)$ may contain simple closed curves called *ovals* as connected components.

To construct the garden $G(f)$ representing f, we start with defining a planar chord diagram of order $2l(\Sigma)$. The vertices of the diagram correspond to the critical points with real critical values, and the chords correspond to arcs in $S(f)$ lying above $\widehat{\mathbb{R}}$. Thus, the faces of the diagram correspond to the faces of $S(f)$ lying above $\widehat{\mathbb{R}}$. The orientation of edges are induced by that of $\widehat{\mathbb{R}}$. To define the labeling of the chord diagram, consider the natural order $<$ on Σ_R. Here, if ∞ belongs to Σ_R, we assume that it is the biggest one. The label of a vertex equals the ordinal number of the corresponding critical value under this order. To define the weights, for any given arc or oval, let $w(x)$ be the number of preimages of $x \in \widehat{\mathbb{R}} - \Sigma_R$ lying on this arc or oval. The weight of the arc or oval is then defined as the minimum of $w(x)$ over all $x \in \widehat{\mathbb{R}} - \Sigma_R$.

The root of a tree corresponds to the boundary of a face, and inner vertices correspond to ovals contained in the face. The weight of an inner vertex is equal to the weight of the corresponding oval.

Example 1.1. A garden G with weight 2 is either

(i) a garden of order 0 with one root only, or
(ii) a garden of order 2 with two roots.

Here, note that the weights of the edges are 0 in the second garden, and two kinds of orientation for each garden are equivalent. And we can find an explicit real rational function of degree 2 with $G(f) = G$.

Lemma 1.1. *The first garden represents any real rational function in*

$$\left\{ z + \frac{b}{z} \;\middle|\; b < 0 \right\},$$

and the second garden does any one in

$$\left\{ z + \frac{b}{z} \;\middle|\; b > 0 \right\}.$$

The proof is elementary. Here, note that the opposite orientation can be obtained from the family

$$\left\{ -z - \frac{b}{z} \right\},$$

i.e., the functions in Lemma 1.1 with pre-composition of $\phi(z) = -z$.

2. Interior variation as surgery of gardens

One of the simplest way to increase the degree of a given rational function f is to add another rational function h to f. Such operation with h of degree 1 was used to deform the complex structure, and is well-known as a typical kind of Schiffer's *interior variation*. See, for instance, [3] and [1, Appendix 1]. In the present context, by using such interior variation, we can increase the weight of a given garden.

In the sequel, fix a rational function f with real coefficients of degree $n \geq 2$ in general position such that ∞ is neither a fixed point nor a critical point of f, and let $G(f)$ be the garden representing f.

Theorem 2.1 (Variation of attaching an edge or of increasing the weight of a circular edge)**.** *Let $a \in \mathbb{R}$ be neither a critical point nor a pole of f. Assume that $f'(a)$ is positive.*

Then for every sufficiently small $\epsilon > 0$,

(i)
$$g^-(z) = f(z) - \frac{\epsilon}{z-a}$$
is a real rational function of degree $n+1$ in general position, and corresponds to the same garden as $G(f)$ except that the weight of the edge corresponding to a increases by 1*.*

(ii)
$$g^+(z) = f(z) + \frac{\epsilon}{z-a}$$
is a real rational function of degree $n+1$ in general position, and corresponds to the garden $G(g^+)$ such that one edge is added to the edge of $G(f)$ corresponding to a.

Remark 2.1. In this theorem, if $f'(a)$ is negative, then the conclusions are interchanged. Also the orientation and the labels are determined uniquely under the circumstances (Cf., Remark 3.1).

Proof. Set
$$f'(z) = \frac{P(z)}{Q(z)^2}$$
with real polynomials $P(z), Q(z)$ without nontrivial common factors. Then from the assumptions, $Q(z)$ is of degree n and $P(z)$ is of degree $2n-2$.

Since $f'(a) > 0$, $P(x) > 0$ with real x sufficiently near a. Hence, if $\epsilon > 0$ is sufficiently small,
$$(x-a)^2 P(x) + \epsilon Q(x)^2 = 0$$
has the same number, say m, of real solutions as that for $P(x) = 0$. While,
$$(x-a)^2 P(x) - \epsilon Q(x)^2 = 0$$
has $m+2$ real solutions, and the additional two solutions locate near $x = a$.

In both cases, the garden representing $g^\pm$ are the same outside the edge corresponding to a, if ϵ is sufficiently small. And from the definitions, we can conclude the assertions. □

Remark 2.2. Lemma 1.1 can be shown by a similar argument as in the above proof with the identical function $f(z) = z$ (i.e., a typical real rational function of degree 1).

Next, by using interior variation with h of degree 2, we can prove the following theorem.

Theorem 2.2 (Variation of attaching an inner vertex)**.** *Let $f(z)$ be as above. Fix a point $z_0 = a + ib$ with $a, b \in \mathbb{R}$ such that $f(z_0) \notin \widehat{\mathbb{R}}$. Then for every sufficiently small $\epsilon > 0$,*

$$g(z) = f(z) + \epsilon \frac{2(z-a)}{(z-a)^2 + b^2}$$

is a real rational function of degree $n+2$ in general position, and corresponds to the garden $G(g)$ obtained from $G(f)$ by adding one inner vertex corresponding to a and the associated edge to the rooted tree of $G(f)$.

Proof. Set

$$f'(z) = \frac{P(z)}{Q(z)^2}$$

with polynomials $P(z), Q(z)$ without nontrivial common factors. Then as before, $Q(z)$ is of degree n and $P(z)$ is of degree $2n-2$.

For every sufficiently small $\epsilon > 0$,

$$\{(z-a)^2 + b^2\}^2 P(z) + 2\epsilon\{b^2 - (z-a)^2\}Q(z)^2 = 0$$

has the same number of real solutions as that for $P(z) = 0$. But, $g(z)$ has a pole at $z = z_0$. Since $f(z_0) \notin \widehat{\mathbb{R}}$, z_0 should be on an oval of $g(z)$ corresponding to no inner vertices of $G(f)$, which show the assertion. $\square$

Moreover, we have the following results.

Theorem 2.3 (Variation of increasing the weight of a chordal edge or of an inner vertex)**.** *Let $f(z)$ be as above. Suppose that the garden $G(f)$ has a chordal edge L or an inner vertex V_0.*

Let ℓ be the arc or the oval in $S(f)$ corresponding to L or V_0, respectively, and $z_0 = a + ib$ with $a, b \in \mathbb{R}$ be a point on ℓ which is not a pole of f. Then there are ϵ (with sufficiently small $|\epsilon| > 0$) such that

$$g(z) = f(z) + 2\frac{(\mathrm{Re}\,\epsilon)z - \mathrm{Re}\,(\epsilon\overline{z_0})}{(z - z_0)(z - \overline{z_0})}$$

is a real rational function of degree $n+2$ in general position, and corresponds to the garden $G(g)$ obtained from $G(f)$ by increasing the weight of L or V_0 by 1.

Proof. First, recall that

$$\ell_0 = \left\{\mathrm{Im}\,\frac{\epsilon}{z - z_0} = 0\right\}$$

is a line depending $\arg \epsilon$, but not $|\epsilon|$. Hence, fixing $\arg \epsilon$ suitably, we assume that ℓ and ℓ_0 intersect transversally at z_0.

Now, outside a sufficiently small neighborhood U of z_0, $S(g)$ tend to $S(f)$ as ϵ tend to 0. On the other hand, for a fixed ϵ, we can see that there is a unique arc, say ℓ', in $S(g)$ tangent to ℓ_0 at z_0, and hence that $S(g) \cap U = \ell'$.

Thus, we can conclude that ℓ' is contained, not in a new oval of $S(g)$, but in the arc in $S(g)$ corresponding to ℓ, which implies the assertion. $\square$

3. A constructive proof of a theorem of Natanzon, Shapiro and Vainshtein

Two rational functions f_1 and f_2 are called to be *equivalent* if there exists a Möbius transformation ϕ such that $f_1 = f_2 \circ \phi$. Let $\mathbb{C}H_{0,n}$ be the set of all equivalence classes of complex rational functions of degree n in general position. The correspondence $f \mapsto \Sigma(f)$ generates a covering

$$\mathbb{C}\Phi_n : \mathbb{C}H_{0,n} \to C_{0,2n-2},$$

where $C_{0,2n-2}$ is the configuration space consisting of all $(2n-2)$-tuples of unordered distinct points on $\widehat{\mathbb{C}}$. We assume that $\mathbb{C}H_{0,n}$ is provided with the weakest topology among those making the map $\mathbb{C}\Phi_n$ continuous.

Two real rational functions (τ_1, f_1) and (τ_2, f_2) are called to be *equivalent* if there exists a Möbius transformation ϕ such that

$$f_1 = f_2 \circ \phi, \quad \phi \circ \tau_1 = \tau_2 \circ \phi.$$

Every real rational function is equivalent to another (J, g), where J is the complex conjugation and $g(z)$ a rational function with real coefficients. Let $RH_{0,n}$ denote the space of all equivalence classes of real rational functions of degree n in general position. The topology of $CH_{0,n}$ induces a topology on $RH_{0,n}$. In this topology $RH_{0,n}$ is not connected.

Since there will be no confusions, we assume that $\tau = J$ and f has real coefficients. We write (J, f) simply as f. Now, the main result of Natanzon, Shapiro, and Vainshtein in [2] is the following theorem.

Theorem 3.1 ([2]). *The set of all connected components of the space $RH_{0,n}$ can be identified with the set of all gardens of weight n.*

In this section, we will give a constructive proof of this theorem as an application of the main theorems in Section 2.

First, it is easy to see that every garden G with weight $n\,(> 2)$ can be obtained by starting from a suitable garden G_0 stated in Lemma 1.1 (with weight 2) and applying inductively a suitable sequence of surgeries in the following list.

List of surgeries

(i) To add an edge L' with end points on a circular edge L of the diagram in such a way that L' and one other new edge bound a new face, and that the total weight increases by 1.

(ii) To increase the weight of an edge by 1.
(iii) To add an inner vertex and the associated edge to a rooted tree.
(iv) To increase the weight of an inner vertex by 1.
(v) To rearrange labels to the given ones, keeping the weights of all edges unchanged, when (more than two faces exist and) $t_j = 1$ for every face j.

Remark 3.1. In the surgery (i), the assumption implies that both of L' and the other new edge L'' have 0 as their weights. Here, the orientation of L' and L'' are determined uniquely. We attach $s+1, s+2$ as the labels of new vertices $L' \cap L''$ with suitable order, if the starting vertex of L has s as the label, and change the labels s' to $s'+2$ if $s' > s$. In particular, the number d_j is 1 for the new face j.

If the weight of L is $m > 0$, then we can splits L into L'' and other two edges with weights m_1, m_2, where $m_1 + m_2 = m$.

Actually, by using surgeries in the list, we can proceed as follows.

- Add suitable edges inductively to the weighted labeled directed planar chord diagram of a suitably chosen starting garden G_0 with weight 2, and we have a diagram D_1, which is the same as the weighted labeled directed planar chord diagram D of G except for the labeling and the weights. (In particular, $t_j = 1$ for every face j if the order is positive.) Let G_1 be the garden consisting of D_1 and the roots only.
- Rearrange the labels for D_1 to the same ones for D, we have a new weighted labeled directed planar chord diagram, say D_2. Let G_2 be the garden consisting of D_2 and the roots only.
- Increase the weight of all edges of D_2, and we obtain the diagram D.
- Add inner vertices and edges suitably so that the resulting rooted trees are the same as those of G except for the weights. Finally, increase the weights of inner vertices, and we obtain the weighted rooted trees of G.

Thus, we need to show that every surgery in the list is possible. Here, the surgeries (i), (iii), and (iv) can be achieved by applying inner variation in Theorems 2.1, 2.2, and 2.3, respectively. Also the surgery (ii) can be achieved by applying inner variation in Theorem 2.1 or 2.3. Hence, it remains the case of surgery (v) only, which can be achieved by using the following classical result.

Proposition 3.1 (Branch point variation)**.** *Let L be an egde in $G(f)$, ℓ the corresponding arc in $S(f)$, and p the starting point of ℓ. Also let s_p and δ be the critical value and the connected subset of $\widehat{\mathbb{R}}$ corresponding to p and ℓ, respectively. For any proper connected subset δ' of δ having s_p as the starting point and containing no other critical values, we can deform f so that s_p moves to the end point of δ' continuously without changing all other critical values.*

This proposition is proved by using so-called branch point variation by cutting a curve of $S(f)$ and repasting differently. Such a variation is useful not only for deformation of holomorphic functions but also for that of the complex structure. Cf., for instance, [4, Section 1].

Here, we can conclude the same result also in case that p is the ending point of ℓ.

Lemma 3.1. *The surgery* (v) *can be achieved by applying interior variation in Theorems 2.1 and 2.3, and branch point variation.*

Proof. From the assumption, the weights of edges are 0. We start with a face, say j_1, having at least two chordal edges on the boundary. Let $v_1, \cdots, v_{2k}$ be the vertices ordered along the boundary of j_1, and L_i the edge for the vertices $\{v_i, v_{i+1}\}$ for every i, where $v_{2k+1} = v_1$. Let $e_i \in \{1, \cdots, 2k\}$ be the desired label of v_i for every i and fix the critical value S_i correponding to e_i. By applying branch point variation if necessary, we may assume that the critical values s_i corresponding to v_i are monotonously increasing.

Firstly, let $i_1 \geq 1$ be the smallest number satisfying $S_i > S_{i+1}$. If $i = 2k$, then by applying branch point variation, we can move every critical value s_i to S_i without changing the weights of all edges on the boundary of j_0, where we also move the other critical values suitably. If not, we apply inner variation in Theorem 2.1 or 2.3, and change the weight of L_{i_1} to 1. And then, by applying branch point variation, we can move the critical value s_i to S_i for every $i = 1, \cdots, i_1 + 1$ and also the others suitably.

Here, if L_i is circular, then there exist two circlular edges, say L_+ and L_-, having one common vertex v_i and v_{i+1}, respectively, with L_i. Let $j_\pm$ be the other face having $L_\pm$ as a boundary edge. By applying branch point variation as before if necessary, we can assume that the critical values on the boundary of $j_\pm$ do not lie between s_{i_1} and S_{i_1} and s_{i_1+1} and S_{i_1+1}, respectively. Hence, we can move the value s_{i_1} and s_{i_1+1} to S_{i_1} and S_{i_1+1} as desired. Here and in the sequel, we always use the same notations for any resulting function.

Secondly, let i_2 be the smallest number satisfying $i > i_1$ and $S_i > S_{i+1}$. Then we deform f so that, keeping $S_1, \cdots, S_{i_1+1}$ fixed, the critical values s_i are moved to S_i for $i = i_1 + 2, \cdots, i_2 + 1$, and the others suitably as before. Repeating this process till we reach to $2k$, we finish to rearrange the critcal values to the desired ones $\{S_i\}$ on the boundary of j_1.

Now, as the next step, we take a face, say j_2 adjacent to j_1 along some chordal edge, say L_j. Then, we may assume that critical values corresponding to vertices on the boudnary of j_2 different from $\{S_j, S_{j+1}\}$ are monotonously increasing, and greater than S_j and S_{j+1}. Starting from the critical value S_{j+1}, we proceed similarly as above, and we can rearrange the labels as desired on the boundaries of faces j_1 and j_2.

Repeating this rearrangement for all other faces step by step, we obtain the desired correspondance between the critical values and vertices, and hence the desired labeling. $\square$

Finally, since we can show uniqueness of the garden on a component of $RH_{0,n}$ by using a standard continuity argument, we finish to prove Theorem 3.1.

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Received April 15, 2011