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DUNKL–WILLIAMS INEQUALITY FOR OPERATORS ASSOCIATED WITH p-ANGULAR DISTANCE

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Abstract. We present several operator versions of the Dunkl–Williams inequality with respect to the p-angular distance for operators. More precisely, we show that if $A, B \in \mathbb{B}(\mathscr{H})$ such that $|A|$ and $|B|$ are invertible, $\frac{1}{r} + \frac{1}{s} = 1$ ($r > 1$) and $p \in \mathbb{R}$, then

$$\left| A|A|^{p-1} - B|B|^{p-1} \right|^2 \leq |A|^{p-1} \left(r|A-B|^2 + s \left| |A|^{1-p}|B|^p - |B| \right|^2 \right) |A|^{p-1}.$$

In the case that $0 < p \leq 1$, we remove the invertibility assumption and show that if $A = U|A|$ and $B = V|B|$ are the polar decompositions of A and B, respectively, $t > 0$, then

$$\left| (U|A|^p - V|B|^p) |A|^{1-p} \right|^2 \leq \left(1+t\right) |A-B|^2 + \left(1 + \frac{1}{t}\right) \left||B|^p|A|^{1-p} - |B|\right|^2.$$

We obtain several equivalent conditions, when the case of equalities hold.

1. Introduction

In 1964, Dunkl and Williams [3] showed that, for any two nonzero vectors x and y in a normed space $(\mathcal{X}, \|\cdot\|)$,

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\| + \|y\|}. \tag{1.1}$$

In the same paper, the authors proved that the constant 4 can be replaced by 2 if $\mathcal{X}$ is an inner product space. This inequality has some applications in the study of geometry of Banach spaces. Kirk and Smiley [7] showed that inequality (1.1) with 2 instead of 4 characterizes inner product spaces. Thus, the smallest number which can replace 4 in inequality (1.1) measures "how much" this space is close (or far) to be a Hilbert space, cf. [6].

Now the inequality (1.1) is regarded as an estimation of the angular distance between given vectors x and y. It has many interesting refinements which have

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obtained over the years, e.g., Maligranda [8], Mercer [9], Dragomir [2], and Pečarić and Rajić [11].

Now we pay our attention to the following improvement of Dunkl–Williams inequality due to Pečarić and Rajić:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{(2\|x-y\|^2 + 2(\|x\| - \|y\|)^2)^{\frac{1}{2}}}{\max\{\|x\|, \|y\|\}}. \tag{1.2}$$

Also they introduced an operator version of (1.2) by estimating $|\, A|A|^{-1} - B|B|^{-1}\,|$, where A and B are Hilbert space operators such that $|A|$ and $|B|$ are invertible (see Corollary 2.4 below).

In [8], Maligranda considered the p-angular distance ($p \in \mathbb{R}$), as a generalization of the concept of angular distance (when $p = 0$), between nonzero elements x and y in a normed space $(\mathcal{X}, \|\cdot\|)$ as $\alpha_p[x, y] := \|\|x\|^{p-1}x - \|y\|^{p-1}y\|$; see also [1].

In this paper, we introduce an operator version of the p-angular distance for Hilbert space operators as a generalization of the Pečarić–Rajić inequality presented in [12]. Thus we will obtain the following estimation of it: If $|A|$ and $|B|$ are invertible, $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$ and $p \in \mathbb{R}$, Then

$$\left|\, A|A|^{p-1} - B|B|^{p-1}\,\right|^2 \le |A|^{p-1}\left(r|A-B|^2 + s\left|\,|A|^{1-p}|B|^p - |B|\,\right|^2\right)|A|^{p-1}.$$

On the other hand, Saito and Tominaga [13] recently generalized Pečarić and Rajić inequality by deleting the invertibility condition on $|A|$ and $|B|$. We also discuss their result.

Our basic tool is the generalized parallelogram law for operators;

$$|A-B|^2 + \frac{1}{t}|tA+B|^2 = \left(1+t\right)|A|^2 + \left(1+\frac{1}{t}\right)|B|^2$$

for any nonzero $t \in \mathbb{R}$. We, in addition, consider several equivalent conditions when the case of equality holds in the obtained inequality. The reader is referred to [4, 10] for undefined notation and terminology related to Hilbert space operators.

2. Dunkl–Williams inequality for operators

In this section, we consider Dunkl-Williams inequality for operators as an application of the generalized parallelogram law of operators (GPL):

$$|A-B|^2 + \frac{1}{t}|tA+B|^2 = \left(1+t\right)|A|^2 + \left(1+\frac{1}{t}\right)|B|^2$$

for any nonzero $t \in \mathbb{R}$. This equality can be easily verified by using $|C|^2 = C^*C$ $(C \in \mathbb{B}(\mathscr{H}))$.

The following lemma follows from it easily, cf. [5].

Lemma 2.1. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$. Then for each $t > 0$*

$$|A - B|^2 \leq \Big(1 + t\Big)\,|A|^2 + \left(1 + \frac{1}{t}\right)|B|^2.$$

The equality holds if and only if $tA + B = 0$.

We now state our main results, which are understood as an application of the above lemma.

Theorem 2.2. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$ and let $t > 0$ and $0 < p \leq 1$ be arbitrary. Then*

$$\big|\,(U|A|^p - V|B|^p)|A|^{1-p}\,\big|^2 \leq \Big(1 + t\Big)\,|A - B|^2 + \left(1 + \frac{1}{t}\right)\big|\,|B|^p|A|^{1-p} - |B|\,\big|^2.$$

The equality holds if and only if $t(A - B) + V(|B|^p|A|^{1-p} - |B|) = 0$.

Proof. Replace A and B in the preceding lemma by $A - B$ and $V(|B|^p|A|^{1-p} - |B|)$ respectively. Then we have

$$\begin{aligned}\big|\,A - V|B|^p|A|^{1-p}\,\big|^2 &\leq \Big(1 + t\Big)\,|A - B|^2 + \left(1 + \frac{1}{t}\right)\big|\,V(|B|^p|A|^{1-p} - |B|)\,\big|^2 \\ &= \Big(1 + t\Big)\,|A - B|^2 + \left(1 + \frac{1}{t}\right)\big|\,|B|^p|A|^{1-p} - |B|\,\big|^2\end{aligned}$$

because V^*V is a projection onto the closure of the range of B^*. Hence we have the required inequality. The equality holds if and only if $t(A-B)+V(|B|^p|A|^{1-p}-|B|) = 0$. □

Next we have an estimation of the operator p-angular distance.

Theorem 2.3. *Let $A, B \in \mathbb{B}(\mathscr{H})$ such that $|A|$ and $|B|$ are invertible, $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$ and $p \in \mathbb{R}$. Then*

$$\big|\,A|A|^{p-1} - B|B|^{p-1}\,\big|^2 \leq |A|^{p-1}\left(r|A - B|^2 + s\,\big|\,|B|^p|A|^{1-p} - |B|\,\big|^2\right)|A|^{p-1}.$$

Moreover the equality holds if and only if

$$(r - 1)(A - B)|A|^{p-1} = B\left(|A|^{p-1} - |B|^{p-1}\right).$$

Proof. The proof is similar to the above, that is, put

$$A_1 = A - B, \quad B_1 = B|B|^{p-1}|A|^{1-p} - B$$

and $t = r-1$ in Lemma 2.1. Since $r = t+1$ and so $s = 1+\frac{1}{t}$, we have the conclusion including the equality condition.

□

A special case of Theorem 2.3, where $p = 0$ gives rise to the main result of Pečarić and Rajić [12, Theorem 2.1].

Corollary 2.4. *Let $A, B \in \mathbb{B}(\mathscr{H})$ such that $|A|$ and $|B|$ are invertible and $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$. Then*

$$\left| A|A|^{-1} - B|B|^{-1} \right|^2 \leq |A|^{-1} \left(r|A - B|^2 + s\left(|A| - |B|\right)^2 \right) |A|^{-1}. \tag{2.1}$$

Further, the equality holds if and only if

$$(r-1)(A-B)|A|^{-1} = B\left(|A|^{-1} - |B|^{-1}\right).$$

We here give some conditions equivalent to the equality condition in Theorem 2.3.

Proposition 2.5. *Let $p \in \mathbb{R}$, $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$ and $A, B \in \mathbb{B}(\mathscr{H})$ such that $|A|$ and $|B|$ are invertible for the case where $p < 1$. Then the following conditions are mutually equivalent:*

- (i) $(r-1)(A-B)|A|^{p-1} = B\left(|A|^{p-1} - |B|^{p-1}\right)$;
- (ii) $(s-1)B\left(|A|^{p-1} - |B|^{p-1}\right) = (A-B)|A|^{p-1}$;
- (iii) $r(A-B)|A|^{p-1} + sB\left(|B|^{p-1} - |A|^{p-1}\right) = 0$;
- (iv) $A|A|^{p-1} - B|B|^{p-1} = sB\left(|A|^{p-1} - |B|^{p-1}\right)$.

Proof. The equivalence (i)⇒(ii)⇒(iii) ⇒(i) is easily checked.

To complete the proof, we prove (iii)⇔(iv). Putting $t = r - 1$, we have $s = \frac{t+1}{t}$, by which (iii) and (iv) are written respectively as follows:

$$t(A-B)|A|^{p-1} + B\left(|B|^{p-1} - |A|^{p-1}\right) = 0$$

and

$$t\left(A|A|^{p-1} - B|B|^{p-1}\right) = (t+1)B\left(|A|^{p-1} - |B|^{p-1}\right).$$

It is obvious that they are equivalent. □

Next we give some necessary conditions for the equality condition in Theorem 2.3.

Proposition 2.6. *Let $A, B \in \mathbb{B}(\mathscr{H})$ such that $|A|$ and $|B|$ are invertible, $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$, $p \in \mathbb{R}$ and*

$$(r-1)(A-B)|A|^{p-1} = B\left(|A|^{p-1} - |B|^{p-1}\right). \tag{2.2}$$

Then the following statements hold:

- (i) $(r-1)|A-B|^2 = \frac{1}{r}|A|^{1-p}|B|^{2p}|A|^{1-p} + \frac{1}{s}|A|^2 - |B|^2$;
- (ii) $|B| \leq \left(\frac{1}{r}|A|^{1-p}|B|^{2p}|A|^{1-p} + \frac{1}{s}|A|^2\right)^{\frac{1}{2}}$;
- (iii) $r|A-B| = s\left|\,|B|^p|A|^{1-p} - |B|\,\right|$.

Proof. Put $t = r - 1$ and then $s = \frac{t+1}{t}$.

(i) Since $t(A - B) = B\left(1 - |B|^{p-1}|A|^{1-p}\right)$ by the assumption, we have

$$tA - (t+1)B = -B|B|^{p-1}|A|^{1-p}.$$

Therefore it implies that

$$|tA - (t+1)B|^2 = |A|^{1-p}|B|^{2p}|A|^{1-p} = C.$$

On the other hand, (i) is expressed as

$$t(t+1)|A - B|^2 = C + t|A|^2 - (t+1)|B|^2.$$

So it suffices to check that

$$|tA - (t+1)B|^2 = t(t+1)|A - B|^2 - t|A|^2 + (t+1)|B|^2.$$

(ii) It follows from (i) and the Löwner-Heinz inequality.

(iii) Since $t(A - B) = B - B|B|^{p-1}|A|^{1-p}$ by the assumption, we have

$$t|A - B| = \left|B - B|B|^{p-1}|A|^{1-p}\right| = \left|\,|B| - |B|^p|A|^{1-p}\,\right|,$$

which is equivalent to (iii).

□

Remark 2.7. Assume that

$$(r-1)(A - B)|A|^{-1} = B\left(|A|^{-1} - |B|^{-1}\right).$$

This is the same equation (2.2) in the special case when $p = 0$. From (ii) of Proposition 2.6 we have

$$|B| \le \left(\frac{1}{r}|A|^2 + \frac{1}{s}|A|^2\right)^{\frac{1}{2}} = |A|$$

and so

$$\frac{r}{s}|A - B| = |A| - |B|, \text{ or } |A| = |B| + \frac{r}{s}|A - B|,$$

which has been shown by Pečarić and Rajić [12].

3. Saito-Tominaga's generalization

Very recently, Saito-Tominaga improved Pečarić and Rajić inequality without the assumption of the invertibility of the absolute value of operators.

Theorem 3.1. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$, and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$|\,(U - V)\,|A|\,|^2 \le p|A - B|^2 + q\left(|A| - |B|\right)^2.$$

The equality holds if and only if

$$p(A - B) = qV\left(|B| - |A|\right) \quad \textit{and} \quad V^*V = U^*U.$$

We here remark that it just corresponds to the case $p = 0$ in Theorem 2.2. In this section, we consider Theorem 3.1 based on the discussion in the preceding section. For this, we rewrite it as follows:

Theorem 3.2. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$, and $t > 0$. Then*

$$|\,(U-V)|A|\,|^2 \;\le\; \Big(t+1\Big)\,|A-B|^2 + \left(1+\frac{1}{t}\right)(|A|-|B|)^2 .$$

The equality holds if and only if

$$t(A-B) = V\,(|B|-|A|) \quad \text{and} \quad V^*V = U^*U.$$

Note that Theorem 3.1 is obtained by taking $t = p - 1$ in above inequality.

Now we prepare a lemma for the equality condition in above.

Lemma 3.3. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$ and $t > 0$. If $t\,(A-B) + V\,(|A|-|B|) = 0$ is satisfied, then*

$$t|A-B|^2 \le |A|^2 - |B|^2,$$

and so $|A| \ge |B|$ and $U^*U \ge V^*V$.

In addition, if $U^*U = V^*V$, then $t|A-B|^2 = |A|^2 - |B|^2$.

Proof. Since $tA - (t+1)B = -V|A|$ by the assumption, we have

$$|tA-(t+1)B|^2 = |A|V^*V|A|.$$

Adding $t|A|^2 - (t+1)|B|^2$ to both sides, we get

$$t(t+1)|A-B|^2 = |A|V^*V|A| + t|A|^2 - (t+1)|B|^2 \le (t+1)\left(|A|^2 - |B|^2\right),$$

so that

$$0 \le t|A-B|^2 \le |A|^2 - |B|^2.$$

Hence it follows that $|A| \ge |B|$ and $U^*U \ge V^*V$. Moreover, if $U^*U = V^*V$ is assumed, then $V^*V|A| = |A|$ and so $t|A-B|^2 = |A|^2 - |B|^2$. □

Proof of Theorem 3.2. We replace A and B in Lemma 2.1 by $A-B$ and $V(|A|-|B|)$ respectively. Then we have the required inequality, and the condition for which the equality holds is that

$$t(A-B) = V(|B|-|A|) \quad \text{and} \quad V^*V = U^*U.$$

The latter in above is equivalent to $|A|V^*V|A| = |A|^2$, or $V^*V|A| = |A|$, that is, $V^*V \ge U^*U$. By the help of the preceding Lemma 3.3, $|B| \le |A|$ and $V^*V \le U^*U$, so that $V^*V = U^*U$. □

Finally, along with the argument due to Saito and Tominaga [13], we investigate the equality condition in Theorem 3.2.

Theorem 3.4. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$, and $C = W|C|$ the polar decomposition of $C = A-B$. Assume that the equality*

$$|(U-V)\,|A||^2 = \Big(t+1\Big)\,|A-B|^2 + \left(1+\frac{1}{t}\right)(|A|-|B|)^2 .$$

holds for some $t > 0$.

(i) *If $t \geq 1$, then $A = B$.*

(ii) *If $0 < t < 1$, then*

$$A = B\left(I - \frac{2}{1-t}W^*W\right) \quad \text{and} \quad |A| = |B|\left(I + \frac{2t}{1-t}W^*W\right),$$

and the converse is true.

We here prepare the following two lemmas.

Lemma 3.5. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$, and $t > 0$. Suppose that $V^*V = U^*U$. Then*

$$t(A-B) = V\,(|B|-|A|)$$

if and only if

$$|A| = |B| + t|A-B| \text{ and } A - B = -V\,(|B|-|A|)\,.$$

Proof. Since $t(A-B) = -V\,(|A|-|B|)$, it follows from Lemma 3.3 that

$$t|A-B| = |\,|A|-|B|\,| = |A|-|B|$$

and moreover

$$A - B = \frac{1}{t}V\,(|B|-|A|) = -\frac{1}{t}\;tV|A-B| = -V|A-B|.$$

Conversely, since $|A|-|B| = t|A-B|$, we have

$$t(A-B) + V\,(|A|-|B|) = -tV|A-B| + tV|A-B| = 0.$$

□

Lemma 3.6. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be operators with the polar decompositions $A = U|A|$ and $B = V|B|$, and $t > 0$. Suppose that $V^*V = U^*U$. If $t(A-B) = V\,(|B|-|A|)$, then*

$$|B||A-B| + |A-B||B| = (1-t)|A-B|^2.$$

Proof. Put $C = A - B$. The preceding lemma ensures that

$$t|C| = |B+C| - |B| \text{ and } C = -V|C|.$$

Then it follows that

$$|B+C| = |B| + t|C|,$$

and that

$$B^*C = -B^*V|C| = -\left(|B|V^*V\right)|C| = -|B||C|.$$

Hence we have

$$|B+C|^2 = (|B|-|C|)^2 \text{ and } |B+C|^2 = (|B|+t|C|)^2,$$

so that

$$(t+1)\left(|B||C|+|C||B|\right) = \left(1-t^2\right)|C|^2,$$

which is equivalent to the conclusion. □

Concluding this paper, we give a proof:

Proof of Theorem 3.4. The preceding lemma leads us the fact that if positive operators S and T satisfy $ST+TS=rS^2$ for some $r\in\mathbb{R}$, then (i) $S=0$ if $r<0$, and (ii) S and T commute if $r\geq 0$. (Since $S^2T = STS - tS^3$ is selfadjoint, S^2 commutes with T and so does S.) Thus we apply it for $S=|A-B|$, $T=|B|$ and $r=1-t$.

(i) Since $r=1-t\leq 0$, we first suppose that $r<0$. Then $S=|A-B|=0$, that is, $A=B$, as desired. Next we suppose $r=0$. Then $S=|C|$ commutes with $T=|B|$ and so $ST=0$. Hence we have $|C|V^*V=0$. Moreover, since $C=-V|C|$ by Lemma 3.5, it follows that $|C|^2=|C|V^*V|C|=0$, i.e., $C=0$.

(ii) We apply the second case (ii) in above. Namely we have

$$|B||C| = |C||B| = \frac{1-t}{2}|C|^2,$$

so that

$$B|C| = V|B||C| = \frac{1-t}{2}V|C|^2 = \frac{t-1}{2}C|C| = \frac{t-1}{2}A|C| - \frac{t-1}{2}B|C|.$$

It implies that

$$A|C| = \frac{2}{t-1}\left(1+\frac{t-1}{2}\right)B|C| = \frac{t+1}{t-1}B|C|,$$

and so

$$AW^*W = \frac{t+1}{t-1}BW^*W.$$

Therefore we have

$$A = AW^*W + A\left(I-W^*W\right) = \frac{t+1}{t-1}BW^*W + B\left(I-W^*W\right) = B\left(I+\frac{2}{t-1}W^*W\right).$$

For the second equality, it suffices to show that W^*W commutes with $|B|$ because

$$\left|I-\frac{2}{1-t}W^*W\right| = I + \frac{2t}{1-t}W^*W$$

is easily seen. For this commutativity, we note that $C = A - B = \frac{2}{t-1} BW^*W$ by the first equality, $C = -V|C|$ by Lemma 3.5, and $V^*V \geq W^*W$ by $W^*W \leq \sup\{V^*V, U^*U\}$ and $V^*V = U^*U$. So we prove that

$$|B|W^*W = V^*BW^*W = -\frac{1-t}{2}V^*C = \frac{1-t}{2}V^*V|C| = \frac{1-t}{2}|C|.$$

Incidentally the converse implication in (ii) is as follows: We first note that the second equality assures the commutativity of $|B|$ and W^*W. Next it follows that

$$|A| - |B| = -\frac{2t}{1-t}|B|W^*W$$

and

$$V|A| - B = V(|A| - |B|) = -\frac{2t}{1-t}BW^*W = -t(A-B)$$

by the first equality. Hence we have

$$(U-V)|A| = A - V|A| = A + t(A-B) - B = (1+t)(A-B);$$

$$|(U-V)|A||^2 = (1+t)^2|A-B|^2.$$

On the other hand, since

$$(|A| - |B|)^2 = \left(\frac{2t}{1-t}\right)^2 B^*BW^*W = t^2|A-B|^2,$$

we have

$$\left(1+t\right)|A-B|^2 + \left(1+\frac{1}{t}\right)(|A|-|B|)^2$$

$$= \left(\left(1+t\right) + \left(1+\frac{1}{t}\right)t^2\right)|A-B|^2 = (1+t)^2|A-B|^2.$$

□

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