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SIMULTANEOUS EXTENSIONS OF SELBERG INEQUALITY AND HEINZ-KATO-FURUTA INEQUALITY

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ABSTRACT. Based on Heinz-Kato-Furuta inequality, we gave an extension of recent Lin's improvement of a generalized Schwarz inequality in our previous note. We present a simultaneous extension of Selberg and Heinz-Kato-Furuta inequalities. As a consequence, we can sharpen the Heinz-Kato-Furuta inequality and further extensions are obtained by Furuta inequality.

1. Introduction.

Throughout this note, an operator means a bounded linear one acting on a Hilbert space. An operator A is positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. We first cite the Heinz-Kato-Furuta inequality, [8] and also [7]:

The Heinz-Kato-Furuta inequality. *Let A and B be positive operators on H. If an operator T on H satisfies $T^*T \leq A^2$ and $TT^* \leq B^2$, then*

$$|(T|T|^{\alpha+\beta-1}x, y)| \leq \|A^\alpha x\| \|B^\beta y\| \tag{1}$$

for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geq 1$ and $x, y \in H$.

We here remark that the Heinz-Kato inequality is just the case $\alpha+\beta = 1$ in above. Based on (1), we have the following extension of a recent Lin's refinement [11] of the generalized Schwarz inequality. Let $T = U|T|$ be the polar decomposition of an operator T on H in the below.

Theorem A. [2] *Let T be an operator on H and $0 \neq y \in H$. For $z \in H$ satisfying $T|T|^{\alpha+\beta-1}z \neq 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$,*

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2 (|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \leq (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y) \tag{2}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and $x \in H$.

It is easily seen that Lin's theorem [11; Theorem 1] is the case $\alpha + \beta = 1$ in Theorem A. As a consequence, we have the following improvement of the Heinz-Kato-Furuta inequality via the Löwner-Heinz inequality, i.e., $A \geq B \geq 0$ implies $A^\alpha \geq B^\alpha$ for $\alpha \in [0,1]$, see [12]:

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Theorem B. *Let A and B be positive operators on H. If an operator T on H satisfies $T^*T \le A^2$ and $TT^* \le B^2$, then*

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2(|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \le \|A^\alpha x\|^2\|B^\beta y\|^2 \tag{3}$$

for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \ge 1$ and $x, y, z \in H$ such that $T|T|^{\alpha+\beta-1}z \ne 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$.

On the other hand, Kubo informed us of the Selberg inequality which is a generalization of the Bessel inequality, [10] and cf. [6], [3]: For given nonzero vectors $z_1, \cdots, z_n \in H$,

$$\sum_i \frac{|(x, z_i)|^2}{\sum_j |(z_i, z_j)|} \le \|x\|^2 \tag{4}$$

holds for all $x \in H$.

In this note, we first point out that the Selberg inequality (4) is refined as follows: If $(y, z_i) = 0$ for given $\{z_i\}$, then

$$|(y, x)|^2 + \sum_i \frac{|(x, z_i)|^2}{\sum_j |(z_i, z_j)|}\|y\|^2 \le \|x\|^2\|y\|^2 \tag{5}$$

holds for all x. Though the refinement (5) is motivated by Theorem A, it gives us further extensions of Theorem A, precisely a simultaneous extension of Selberg and Heinz-Kato-Furuta inequalities. Along with our preceding paper [2], we moreover consider its generalizations via the Furuta inequality.

2. Refinements of Selberg inequality.

We begin with the proof of (5), which is done along with Furuta's way [6]:

Lemma 1. *If $(y, z_i) = 0$ for given nonzero vectors $\{z_i; i = 1, 2, \cdots, n\}$, then*

$$|(x, y)|^2 + \sum_i \frac{|(x, z_i)|^2}{\sum_j |(z_i, z_j)|}\|y\|^2 \le \|x\|^2\|y\|^2 \tag{5}$$

holds for all x.

Proof. We put

$$u = x - \sum_i \frac{(x, z_i)}{\sum_j |(z_j, z_i)|} z_i = x - \sum_i a_i z_i.$$

Then we have

$$\begin{aligned}\|u\|^2 &= \|x - \sum_i a_i z_i\|^2 \\ &\le \|x\|^2 - 2\mathrm{Re} \sum \bar{a}_i(x, z_i) + \sum\{|a_i|^2 \sum_j |(z_i, z_j)|\} \\ &= \|x\|^2 - \sum_i \frac{|(x, z_i)|^2}{\sum_j |(z_i, z_j)|}.\end{aligned}$$

Hence it follows that

$$\begin{aligned}\|y\|^2\{\|x\|^2 - \sum_i \frac{|(x, z_i)|^2}{\sum_j |(z_i, z_j)|}\} &\ge \|y\|^2\|u\|^2 \ge |(y, u)|^2 \\ &= |(y, x - \sum_i \frac{(x, z_i)}{\sum_j |(z_j, z_i)|} z_i)|^2 = |(y, x)|^2.\end{aligned}$$

Now Furuta [6; Theorem 2] showed the following extension of the Selberg inequality: Let T be an operator on H with the kernel ker (T). For given $\{z_i\} \not\subset \ker(T^*)$,

$$\sum_i \frac{|(Tx, z_i)|^2}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|} \leq \||T|^\alpha x\|^2 \tag{6}$$

holds for all $x \in H$ and $\alpha \in [0,1]$.

Thus we have a refinement of (6) by Lemma 1.

Theorem 2. *Let $T = U|T|$ be the polar decomposition of an operator T on H, $\{z_i; i = 1, 2, \cdots, n\} \not\subset \ker(T^*)$ and $\alpha \in [0,1]$. If $(U|T|^{1-\alpha} y, z_i) = 0$ for all i, then*

$$|(|T|^\alpha x, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|} \|y\|^2 \leq \||T|^\alpha x\|^2 \|y\|^2 \tag{7}$$

holds for all $x \in H$.

Proof. We replace x and z_i by $|T|^\alpha x$ and $|T|^{1-\alpha} U^* z_i$ in Lemma 1 respectively. Then we have (7).

Next we propose another refinement of (6). As a matter of fact, it is contained in (9) below.

Theorem 3. *Suppose that $\{z_i; i = 1, 2, \cdots, n\} \not\subset \ker(T^*)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1 \geq \alpha$. If $(|T^*|^{\beta+1-\alpha} y, z_i) = 0$ for all i, then*

$$|(T|T|^{\alpha+\beta-1} x, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2 \||T^*|^\beta y\|^2}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|} \leq \||T|^\alpha x\|^2 \||T^*|^\beta y\|^2 \tag{8}$$

holds for all $x \in H$. In particular, if $(|T^*|^{2(1-\alpha)} y, z_i) = 0$ for $\alpha \in [0,1]$, then

$$|(Tx, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2 \||T^*|^{1-\alpha} y\|^2}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|} \leq \||T|^\alpha x\|^2 \||T^*|^{1-\alpha} y\|^2 \tag{9}$$

holds for all $x \in H$.

Proof. As in the proof of Lemma 1, we put $u = |T|^\alpha x - \sum_i \frac{|T|^{1-\alpha} U^* z_i}{\sum |(|T^*|^{2(1-\alpha)} z_i, z_j)|}$. Since

$$\|u\|^2 \leq \||T|^\alpha x\|^2 - \sum_i \frac{|(Tx, z_i)|^2}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|}$$

we have

$$\begin{aligned}
&\|T^*|^\beta y\|^2 \{\||T|^\alpha x\|^2 - \sum_i \frac{|(Tx, z_i)|^2}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|}\} \\
&\geq \||T^*|^\beta y\|^2 \|u\|^2 \\
&\geq |(|T^*|^\beta y, Uu)|^2 \\
&= |(|T^*|^\beta y, U|T|^\alpha x - \sum_i \frac{U|T|^{1-\alpha} U^* z_i}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|})|^2 \\
&= |(|T^*|^\beta y, U|T|^\alpha x - \sum_i \frac{|T^*|^{1-\alpha} z_i}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|})|^2
\end{aligned}$$

Moreover it follows from the assumption $(|T^*|^{\beta+1-\alpha} y, z_i) = 0$ that

$$|(T|T|^{\alpha+\beta-1} x, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2 \||T^*|^\beta y\|^2}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|} \leq \||T|^\alpha x\|^2 \||T^*|^\beta y\|^2.$$

By a similar way to Lemma 1, we have an alternative simultaneous extension of Selberg and generalized Schwarz inequalities:

Theorem 4. *Suppose that* $\{z_i; i = 1, 2, \cdots, n\} \not\subset \ker(T)$ *and* $\alpha, \beta \geq 0$ *with* $\alpha + \beta \geq 1$. *If* $(T|T|^{\alpha+\beta-1} z_i, y) = 0$ *for all* i, *then*

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \sum_i \frac{|(|T|^{2\alpha}x, z_i)|^2 \||T^*|^\beta y\|^2}{\sum_j |(|T|^{2\alpha}z_i, z_j)|} \leq \||T|^\alpha x\|^2 \||T^*|^\beta y\|^2 \tag{10}$$

holds for all $x \in H$.

Proof. The proof is quite similar to that of Theorem 3. We put

$$u = x - \sum_i \frac{(|T|^{2\alpha}x, z_i)}{\sum_j |(|T|^{2\alpha}z_j, z_i)|} z_i = x - \sum_i a_i z_i.$$

Since

$$\begin{aligned}
\||T|^\alpha u\|^2 &= \||T|^\alpha x - \sum_i a_i |T|^\alpha z_i\|^2 \\
&\leq \||T|^\alpha x\|^2 - 2\mathrm{Re} \sum \overline{a_i} (|T|^{2\alpha}x, z_i) + \sum_i \{|a_i|^2 \sum_j |(|T|^{2\alpha}z_i, z_j)|\} \\
&= \||T|^\alpha x\|^2 - \sum_i \frac{|(|T|^{2\alpha}x, z_i)|^2}{\sum_j |(|T|^{2\alpha}z_i, z_j)|},
\end{aligned}$$

we have

$$\begin{aligned}
&\||T^*|^\beta y\|^2 \{\||T|^\alpha x\|^2 - \sum_i \frac{|(|T|^{2\alpha}x, z_i)|^2}{\sum_j |(|T|^{2\alpha}z_i, z_j)|}\} \\
&\geq \||T^*|^\beta y\|^2 \||T|^\alpha u\|^2 \\
&\geq |(|T^*|^\beta y, U|T|^\alpha u)|^2 \\
&= |(|T^*|^\beta y, U|T|^\alpha x - \sum_i \frac{(|T|^{2\alpha}x, z_i)}{\sum_j |(|T|^{2\alpha}z_j, z_i)|} U|T|^\alpha z_i)|^2 \\
&= |(|T^*|^\beta y, U|T|^\alpha x)|^2 \\
&= |(y, T|T|^{\alpha+\beta-1}x)|^2.
\end{aligned}$$

3. Refinements of Heinz-Kato-Furuta inequality. In our previous note, we pointed out that Theorem A implies Theorem B as a refinement of the Heinz-Kato-Furuta inequality. We now obtain extensions of it which includes Theorem B.

Corollary 5. *Suppose that* $\alpha, \beta \geq 0$ *with* $\alpha + \beta \geq 1 \geq \alpha - \beta$ *and* $z_i \notin \ker(T^*)$ *satisfies* $(|T^*|^{\beta+1-\alpha}y, z_i) = 0$ *for* $i = 1, \cdots, n$. *If* $|T|^2 \leq A^2$ *and* $|T^*|^2 \leq B^2$ *for* $A, B \geq 0$, *then*

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2 \||T^*|^\beta y\|^2}{\sum_j |(|T^*|^{2(1-\alpha)}z_i, z_j)|} \leq \|A^\alpha x\|^2 \|B^\beta y\|^2 \tag{11}$$

holds for all $x \in H$. *In particular, if* $(|T^*|^{2(1-\alpha)}y, z_i) = 0$ *for* $\alpha \in [0, 1]$, *then*

$$|(Tx, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2 \||T^*|^{1-\alpha} y\|^2}{\sum_j |(|T^*|^{2(1-\alpha)}z_i, z_j)|} \leq \|A^\alpha x\|^2 \|B^{1-\alpha} y\|^2 \tag{12}$$

holds for all $x \in H$.

Proof. The Löwner-Heinz inequality says that the assumptions $|T|^2 \leq A^2$ and $|T^*|^2 \leq B^2$ for $A, B \geq 0$ imply

$$\||T|^\alpha x\| \leq \|A^\alpha x\| \quad \text{and} \quad \||T^*|^\beta y\| \leq \|B^\beta y\|$$

for all $x, y \in H$ respectively. Combining with Theorem 3, we have the conclusion.

Similarly we have the following corollary by Theorem 4:

Corollary 6. *Suppose that $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1 \geq \alpha - \beta$ and $z_i \notin \ker(T)$ satisfies $(T|T|^{\alpha+\beta-1} z_i, y) = 0$ for $i = 1, 2, \cdots, n$. If $|T|^2 \leq A^2$ and $|T^*|^2 \leq B^2$ for $A, B \geq 0$, then*

$$(13) \qquad |(T|T|^{\alpha+\beta-1}x, y)|^2 + \sum_i \frac{|(|T|^{2\alpha}x, z_i)|^2 \| |T^*|^\beta y\|^2}{\sum_j |(|T|^{2\alpha} z_i, z_j)|} \leq \|A^\alpha x\|^2 \|B^\beta y\|^2$$

holds for all $x \in H$.

To give further extensions of the Heinz-Kato-Furuta inequality, we apply the Furuta inequality [4], see also [1],[5],[9], which is cited for convenience:

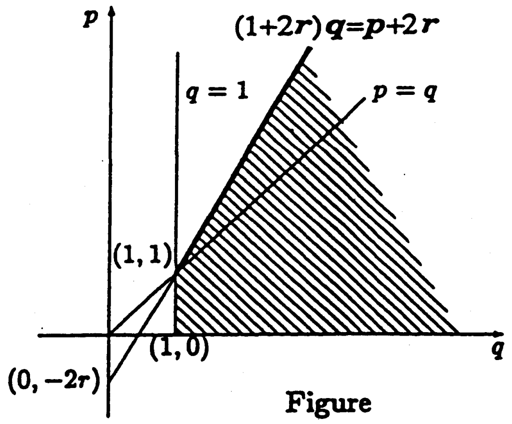
The Furuta inequality. *If $A \geq B \geq 0$, then for each $r \geq 0$,*

$$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$$

holds for $p \geq 0$ and $q \geq 1$ with

$$(*) \qquad (1 + 2r)q \geq p + 2r.$$

The domain determined by (*) is expressed in the right.



Figure

Theorem 7. *Let A and B be positive operators on H and T an operator such that $T^*T \leq A^2$ and $TT^* \leq B^2$. Then for each $r,\ s \geq 0$*

$$(14) \qquad |(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2 (|T^*|^{2(1+2s)\beta} y, y)}{\sum_j (|T|^{2(1-\alpha-2r\alpha)} z_i, z_j)}$$

$$\leq ((|T|^{2r} A^{2p} |T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x)((|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y)$$

for all $p,\ q \geq 1$, $\alpha, \beta \in [0, 1]$ with $(1+2r)\alpha + (1+2s)\beta \geq 1 \geq (1+2r)\alpha$ and $x, y, z_1, \cdots, z_n \in H$ such that $z_i \notin \ker(T^*)$ and $(|T^*|^{(1+2s)\beta+1-(1+2r)\alpha} y, z_i) = 0$ for $i = 1, \cdots, n$.

Proof. By replacing α (resp. β) to $\alpha_1 = (1+2r)\alpha$ (resp. $\beta_1 = (1+2s)\beta$) in Theorem 3, we have

$$|(T|T|^{\alpha_1+\beta_1-1}x, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2 (|T^*|^{2\beta_1} y, y)}{\sum_j (|T|^{2(1-\alpha_1)} z_i, z_j)} \leq (|T|^{2\alpha_1} x, x)(|T^*|^{2\beta_1} y, y).$$

Next we use the Furuta inequality for $|T|^2 \leq A^2$ and $|T^*|^2 \leq B^2$; namely (for the former) we replace $A,\ B;\ q$ in the Furuta inequality to $A^2,\ |T|^2;\ \frac{p+2r}{(1+2r)\alpha}$ respectively. Then we have

$$|T|^{2\alpha_1} = |T|^{2(1+2r)\alpha} \leq (|T|^{2r} A^{2p} |T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}$$

and similarly

$$|T^*|^{2\beta_1} = |T^*|^{2(1+2s)\beta} \leq ((|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}.$$

Combining them, we obtain the inequality (14).

Similarly we have the following further extensions by Theorem 4:

Theorem 8. *Let A and B be positive operators on H and T an operator such that $T^*T \leq A^2$ and $TT^* \leq B^2$. Then for each $r,\ s \geq 0$*

$$(15) \qquad |(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x, y)|^2 + \sum_i \frac{|(|T|^{2(1+2r)\alpha}x, z_i)|^2(|T^*|^{2(1+2s)\beta}y, y)}{\sum_j(|T|^{2(1+2r)\alpha}z_i, z_j)}$$

$$\leq ((|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x, x)(((|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}y, y)$$

for all $p,\ q \geq 1$, $\alpha, \beta \in [0, 1]$ with $(1+2r)\alpha + (1+2s)\beta \geq 1$ and $x, y, z_1, \cdots, z_n \in H$ such that $z_i \notin ker(T)$ and $(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z_i, y) = 0$ for $i = 1, \cdots, n$.

Proof. By replacing α (resp. β) to $\alpha_1 = (1+2r)\alpha$ (resp. $\beta_1 = (1+2s)\beta$) in Theorem 4, we have

$$|(T|T|^{\alpha_1+\beta_1-1}x, y)|^2 + \sum_i \frac{|(|T|^{2\alpha_1}x, z_i)|^2(|T^*|^{2\beta_1}y, y)}{\sum_j(|T|^{2\alpha_1}z_i, z_j)} \leq (|T|^{2\alpha_1}x, x)(|T^*|^{2\beta_1}y, y).$$

By the use of the Furuta inequality for $|T|^2 \leq A^2$ and $|T^*|^2 \leq B^2$, we have

$$|T|^{2\alpha_1} = |T|^{2(1+2r)\alpha} \leq (|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}$$

and similarly

$$|T^*|^{2\beta_1} = |T^*|^{2(1+2s)\beta} \leq ((|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}.$$

Combining them, we obtain the inequality (15).

Remark. (1) We remark that the condition $(1+2r)\alpha + (1+2s)\beta \geq 1$ in above is unneccessary if T is either positive or invertible.

(2) Though Theorem 7 is followed from the Furuta inequality, they are equivalent actually, that is, Theorem 7 is an alternative representation of the Furuta inequality. Because Theorem 7 is an extension of [2; Theorem 3] which is equivalent to the Furuta inequality, see Remark after Theorem 3 in [2].

(3) Theorem 7 will be discussed under the chaotic order, which will be appeared in a separate paper.

4. A concluding remark. By a similar way to proofs of results in § 2, we give a simple proof to an extension of Diaz-Metcalf inequality due to Fujii-Yamada [3].

Theorem 9. *Let $z_1, \cdots, z_n$ be non zero vectors in H satisfying*

$$0 \leq r_k \leq \frac{Re\ (x_i, z_k)}{\|x_i\|} \text{ for all } i, k$$

for $x_1, \cdots, x_m \in H$. If $(y, z_i) = 0$ for all i, then

$$|(x_1 + \cdots + x_m, y)|^2 + (\sum \frac{r_k^2}{c_k})(\|x_1\| + \cdots + \|x_m\|)^2\|y\|^2 \leq \|x_1 + \cdots + x_m\|^2\|y\|^2$$

where $c_k = \sum_j |(z_j, z_k)|$.

Proof. Put $x = x_1 + \cdots + x_n$. Then we have

$$\begin{aligned}
&\|y\|^2\{\|x\|^2 - \sum \frac{r_k^2}{c_k}(\|x_1\| + \cdots + \|x_m\|)^2\} \\
&\geq \|y\|^2\{\|x\|^2 - \sum \frac{|\mathrm{Re}\ (x, z_k)|^2}{c_k}\} \\
&\geq \|y\|^2\{\|x\|^2 - \sum \frac{|(x, z_k)|^2}{c_k}\} \\
&\geq \|y\|^2\|x - \sum \frac{(x, z_k)}{c_k} z_k\|^2 \\
&\geq |(y, x - \sum \frac{(x, z_k)}{c_k} z_k)|^2 \\
&= |(y, x)|^2.
\end{aligned},$$

as desired.

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