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ORLICZ NORM ESTIMATES FOR POISSON MAXIMAL OPERATORS

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ABSTRACT. A condition for Poisson maximal operator to be of weak type L_ϕ are chracterized in terms of the Orlicz norm. This operator unifies various maximal operators cited in the literatures.

I. Introduction

For a given function f on $\mathbb{R}^n$, set

$$\mathcal{M}f(x,t) = \sup_{Q} \frac{1}{|Q|} \int_Q |f|\, dx, \quad (x \in \mathbb{R}^n,\ t \geq 0),$$

where the supremum is taken over the cubes Q in $\mathbb{R}^n$ centered at x with sides parallel to the x-axis and has side length at least t. It is well known tha this operator plays important role in studying the Poisson integral on the upper half-space.

For a given positive measure ν on $\mathbb{R}^n \times [0,\infty)$, under what condition on ν can we assert that $\mathcal{M}$ is bounded from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n \times [0,\infty),\nu)$? Carleson[C] showed this is equivalent to the Carleson condition and later Feffermann-Stein[FS] found a sufficient condition, and later Ruiz[R], and Ruiz-Torrea[RT] unified all these results. Further, Gallardo[G] and Chen[Ch] obtained a characterization in terms of the Orlicz norm.

On the other hand, Sueiro[Su] studied a maximal operator $\mathcal{M}_\Omega$ to study Poisson-Szegö integral. This operator generalizes the standard Hardy-Littlewood maximal operator.

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In this paper we define a maximal operator $\mathcal{M}_\Omega f(x,t)$ that generalizes the Poisson integral operator $\mathcal{M}$ and chracterize a condition to be $\mathcal{M}_\Omega$ of weak type (ϕ,ϕ) in terms of the Orlicz norm.

II. Terminologies

Definition 2.1. Let X be a topological space and $d : X \times X \to [0,\infty)$ be a map satisfying

i) $d(x,y) \geq 0$; $d(x,y) = 0$ if and only if $x = y$;

ii) $d(x,y) = d(y,x)$;

iii) $d(x,y) \leq K[d(x,z) + d(z,y)]$, where K is a fixed constant.

iv) the balls $B(x,r) = \{y \in X : d(x,y) < r\}$ form a basis of open neighborhoods at $x \in X$ and that μ is a Borel measure on X such that

v) $0 < \mu(B(x,2r)) \leq A\mu(B(x,r)) < \infty$, where A is some fixed constant.

Then the triple (X,d,μ) is called a space of homogeneous type.

Definition 2.2. Assume for each $x \in X$, we are given $\Omega_x \subset X \times [0,\infty)$. Let Ω be the set $\{\Omega_x\}$. For each $t \geq 0$ set

$$\Omega_{(x,t)} = \Omega_x \cap (X \times [t,\infty))$$

and for each $\alpha > 0$

$$\mathcal{R}_\alpha(x,t) = \{(y,r) \in X \times [0,\infty) : \Omega_{(y,r)}(2t) \cap B(x,\alpha t) \neq \phi\},$$

where

$$\Omega_{(x,r)}(t) = \{z \in X : (z,t) \in \Omega_{(x,r)}\}$$

is the cross section of $\Omega_{(x,r)}$ at height t. We assume that $\mathcal{R}_\alpha(x,t)$ is measurable for each x and t.

For $f \in L^1(d\mu)$ and $x \in X,\ t \geqslant 0$ set

$$\mathcal{M}_\Omega f(x,t) = \sup_{(y,s)\in\Omega_{(x,t)}} \frac{1}{\mu(B(y,s))} \int_{B(y,s)} |f|\, d\mu.$$

We also assume that $\mathcal{M}_\Omega f(x,t)$ is measurable for each x and t.

Example 2.1. Let $X = \mathbb{R}^n$ and $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. If

$$\Omega_x = \{(y,r) \in \mathbb{R}^n \times [0,\infty) : |x-y| < r\},$$

then for any $\alpha > 0$, the set $\mathcal{R}_\alpha(x,t)$ is given by

$$\mathcal{R}_\alpha(x,t) = B(x,(\alpha+2)t) \times [0,2t].$$

Definition 2.3. Let $\phi : [0,\infty) \to \mathbb{R}$ be a continuous and convex function satisfying

i) $\phi(s) > 0$ for all $s \geqslant 0$;

ii) $\lim_{s\to 0} \phi(s)/s = 0$.

iii) $\lim_{s\to\infty} \phi(s)/s = \infty$.

Then ϕ is called an N-function. Each N-function has the integral represention: $\phi(s) = \int_0^s \varphi(t)\,dt$, where $\varphi(s) > 0$ for $s > 0$, $\varphi(0) = 0$, and $\varphi(s) \to \infty$ as $s \to \infty$. Further, φ is right-continuous and nondecreasing. φ is called the density function of ϕ.

Define $\rho : [0,\infty) \to \mathbb{R}$ by $\rho(t) = \sup\{s : \varphi(s) \leq t\}$. Then ρ is called the generalized inverse of φ. Finally, define

$$\psi(t) = \int_0^t \rho(s)\,ds$$

and ψ is called the complementary N-function of ϕ. For further details, see Musielak[Mu].

Definition 2.4. An N-function ϕ is said to satisfy the Δ_2-condition in $[0,\infty)$ if $\sup_{s>0} \phi(2s)/\phi(s) < \infty$.

Remark 1. If ψ is the complementary N–function ϕ, then $st \leq \phi(s) + \psi(t)$ for all $s, t \geq 0$.

Futher the equality holds if and only if $\varphi(s-) \leq t \leq \varphi(s)$ or else $\rho(t-) \leq s \leq \rho(t)$.

Definition 2.5. Let $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and if ϕ is an N-function, then the Orlicz spaces $L_\phi(d\mu)$ and $L_\phi^*(d\mu)$ are defined by

$$L_\phi(d\mu) = \{f : \int_X \phi(|f|)d\mu < \infty\}$$

and

$$L_\phi^*(d\mu) = \{f : fg \in L_1(d\mu) \quad \text{for all} \quad g \in L_\psi\},$$

where ψ is the complementary N-function of ϕ.

Keeping these definitions and notations, the following properties about the Orlicz space will be used in the proof of theorem 3.1.

Proposition 2.1. i) *The Orlicz space $L^*_\phi(d\mu)$ is a Banach space with the Orlicz norm*

$$\|f\|_\phi = \sup\{\int |fg| d\mu : g \in S_\psi\},$$

where $S_\psi = \{g \in L_\psi : \int \psi(|g|)d\mu \leq 1\}$, or with the Luxemburg norm

$$\|f\|_{(\phi)} = \inf\{\lambda > 0 : \int \phi(|f|/\lambda) d\mu \leq 1\}.$$

ii) *(Hölder's inequality) If $f \in L^*_\phi(d\mu)$ and $g \in L^*_\psi(d\mu)$, then $\|fg\|_\phi \leq 2\|f\|_{(\phi)}\|g\|_{(\psi)}$.*

Definition 2.6. Fix $\alpha > 0$. Let ν be a Borel measure on $X \times [0,\infty)$ and w a nonnegative measurable function on X. The pair (ν, w) is said to satify the $C_\phi(\Omega, \alpha)$ condition if there is constant $C(K, A, \phi, \alpha)$ such that

$$\text{(1)} \qquad \frac{\epsilon\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))} \leq \frac{C(K,A,\phi,\alpha)}{\varphi([[(\rho(1/\epsilon w)) : B(x,r)]])}$$

for any $(x,r) \in X \times [0,\infty)$ and any $\epsilon > 0$, where $[[f : B]] = \frac{1}{\mu(B)} \int_B f d\mu$.

Definition 2.7. $\mathcal{M}_\Omega$ is of weak type (ϕ, ϕ) with respect to (ν, w) if there is a constant $C = C(K, A, \phi)$ such that

$$\nu(\{(x,r) \in X \times [0,\infty) : \mathcal{M}_\Omega f(x,r) > \lambda\}) \leq \frac{C}{\phi(\lambda)} \int_X \phi(|f|) w d\mu$$

for every $\lambda > 0$.

III. Results

The following lemma is given in [CW]. Also see [Su].

Lemma 3.1. *(Vitali-Wiener type covering lemma) Let E be a bounded subset of X and for each $x \in X$, assign $r(x) > 0$. Then there is a sequence of pairwise disjoint balls $B(x_i, r(x_i))$, $x_i \in E$, such that the balls $B(x_i, 4Kr(x_i))$ cover E, where K is the constant in the definition 2.1. Further, every $x \in E$ is contained in some ball $B(x_i, 4Kr(x_i))$ satisfying $r(x) \leq 2r(x_i)$.*

The following lemma is given in [Ch].

Lemma 3.2. *For any N-function ϕ, $t \le \varphi(\rho(t))$ and $\phi(t) \le t\varphi(t)$. If ϕ satisfies the Δ_2-condition, then there is a constant $C(\phi)$ such that $\varphi(\rho(t)) \le C(\phi)t$ and $t\varphi(t) \le C(\phi)\phi(t)$.*

Theorem 3.1. *Assume that an N-function ϕ satisfies the Δ_2-condition and assume further that Ω satisfies that if $x \in X$, $(y,r) \in \Omega_x$ and $s \ge r$, then $(y,s) \in \Omega_x$.*

i) If $\mathcal{M}_\Omega$ is of weak type (ϕ,ϕ) with respect to (ν,w), then (ν,w) satifies the condition $C_\phi(\Omega,\alpha)$ for all $\alpha > 0$.

ii) If (ν,w) satifies the condition $C_\phi(\Omega,\alpha)$ for some $\alpha \ge 4K$, $\mathcal{M}_\Omega$ is of weak type (ϕ,ϕ) with respect to (ν,w).

Proof. Suppose that $\mathcal{M}_\Omega$ is of weak type (ϕ,ϕ) with respect to (ν,w). Let f be a nonnegative measurable function on X. If $(x_0,t) \in \mathcal{R}_\alpha(x,r)$, then $\Omega_{(x_0,t)}(2r) \cap B(x,\alpha r) \ne \phi$ and so we can choose $y \in \Omega_{(x_0,t)}(2r) \cap B(x,\alpha r)$. From this observation and the triangle inequality, it follows that

$$(2)\quad B(x,r) \subset B(y,K(\alpha+1)r) \subset B(y,2K(\alpha+1)r) \subset B(x,(2K^2\alpha+K\alpha+2K^2)r).$$

Since $(y,2K(\alpha+1)r) \in \Omega_{(x_0,t)}$ by the hypothesis on Ω, we have

$$(3)\qquad [[f : B(y,2K(\alpha+1)r)]] \le \mathcal{M}_\Omega(f \cdot \chi_{B(y,2K(\alpha+1)r)})(x_0,t)$$

For all λ with $0 < \lambda < [[f : B(y,2K(\alpha+1)r)]]$, if we write

$$(4)\qquad E_\lambda = \{\mathcal{M}_\Omega(f \cdot \chi_{B(y,2K(\alpha+1)r)}) > \lambda\},$$

then $\mathcal{R}_\alpha(x,r) \subset E_\lambda$ and so

$$(5)\qquad \phi(\lambda)\nu(\mathcal{R}_\alpha(x,r)) \le C \int_X \phi(f\chi_{B(y,2K(\alpha+1)r})wd\mu.$$

Since $0 < \lambda < [[f : B(y,2K(\alpha+1)r)]]$, we have

$$(6)\quad \phi\Big([[f : B(y,2K(\alpha+1)r)]]\Big)\nu(\mathcal{R}_\alpha(x,r)) \le C \int_{B(y,2K(\alpha+1)r)} \phi(f)wd\mu.$$

Invoking (2) there is a constant C_1 so that

$$C_1 \le \frac{\mu(B(x,r))}{\mu(B(y,2K(\alpha+1)r))}$$

for $y \in B(x, \alpha r)$. Then by (2), note that C_1 depends only on K, A, and α. If we replace f by $\frac{\rho(1/w)\chi_{B(x,r)}}{C_1}$ in (6), by lemma 3.2 we then obtain

(7)
$$\phi\Big([[f : B(y, 2K(\alpha+1)r)]]\Big) \geq \phi\Big([[\rho(1/w) : B(x,r)]]\Big) \geq \frac{1}{C(\phi)}[[\rho(1/w) : B(x,r)]]\varphi\Big([[\rho(1/w) : B(x,r)]]\Big).$$

and by the Δ_2-condition of ϕ and lemma 3.2, we also have

(8)
$$\begin{aligned}\int_{B(y,2K(\alpha+1)r)} \phi(f) w d\mu &= \int_{B(x,r)} \phi\Big(\rho(1/w)/C_1\Big) w d\mu \\ &\leq C_2 \int_{B(x,r)} \phi(\rho(1/w)) w d\mu \\ &\leq C_2 \int_{B(x,r)} \rho(1/w)\varphi(\rho(1/w)) w d\mu \\ &\leq C_2 C(\phi) \int_{B(x,r)} \rho(1/w) d\mu \\ &\leq C_2 C(\phi)\mu(B(x,r))[[\rho(1/w)) : B(x,r)]].\end{aligned}$$

Combining (6), (7), and (8), we get

(9)
$$\frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))} \leq \frac{CC_2C(\phi)^2}{\varphi([[(\rho(1/w)) : B(x,r)]])},$$

which gives (1). This completes the proof of i).

To prove ii), suppose there is a constant C so that (1) holds. We follow the idea of Sueiro[Su]. For each $\lambda > 0$, define

$$E_\lambda = \{(x,t) \in X \times [0,\infty) : \mathcal{M}_\Omega f(x,t) > \lambda\}$$

and

$$E'_\lambda = \{x \in X : \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| d\mu > \lambda\}.$$

Also for each $x \in E'_\lambda$, if we put

$$r(x) = \sup\{r > 0 : \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| d\mu > \lambda\},$$

then $r(x) > 0$ and

$$\frac{1}{\mu(B(x,r(x)))} \int_{B(x,r(x))} |f| \, d\mu \geq \lambda.$$

Assume for a moment that E'_λ is bounded and that $r(x)$ is everywhere finite. Then by lemma 3.1, there exists a sequence of pairwise disjoint balls $\{B(x_i,r_i)\}$ so that $E'_\lambda \subset \cup_i B(x_i, 4Kr_i)$. Now we want to verify

$$E_\lambda \subset \cup_i \mathcal{R}_{4K}(x_i, r_i) \tag{10}$$

To do this, let $(x,t) \in E_\lambda$. Then

$$\frac{1}{\mu(B(y,r))} \int_{B(y,r)} |f| \, d\mu > \lambda$$

for some $(y,r) \in \Omega_{(x,t)}$. So $y \in E'_\lambda$ and $t \leq r \leq r(y)$. By the last part of the lemma 3.1, $y \in B(x_i, 4Kr_i)$ for some i such that $r(y) \leq 2r_i$. Therefore $t \leq r \leq r(y) \leq 2r_i$ and so by the hypothesis $(y, 2r_i) \in \Omega_{(x,t)}$. Thus $y \in \Omega_{(x,t)}(2r_i) \cap B(x_i, 4Kr_i)$, and so $(x,t) \in \mathcal{R}_{4K}(x_i,r_i)$, and so (10) holds.

Let $\varepsilon > 0$. By Hölder's inequality, we obtain

$$\int_{B(x_i,r_i)} |f| \, d\mu \leq 2\|f \cdot \chi_{B(x_i,r_i)}\|_{(\phi),\varepsilon w} \, \|\frac{\chi_{B(x_i,r_i)}}{\varepsilon w}\|_{(\psi),\varepsilon w}. \tag{11}$$

To estimate $\|\frac{\chi_{B(x_i,r_i)}}{\varepsilon w}\|_{(\psi),\varepsilon w}$, let $\delta > 0$. Since $(\nu, w) \in C_\phi(\Omega)$,

$$\varphi([[\rho(1/\delta\varepsilon w) : B(x_i,r_i)]]) \leq \frac{C\mu(B(x_i,r_i))}{\delta\varepsilon\nu(\mathcal{R}_{4K}(x_i,r_i))},$$

and so we have

$$\frac{1}{\mu(B(x_i,r_i))} \int_{B(x_i,r_i)} \rho(1/\delta\varepsilon w) \, d\mu \leq \rho\left(\frac{C\mu(B(x_i,r_i))}{\delta\varepsilon\nu(\mathcal{R}_{4K}(x_i,r_i))}\right)$$

and so we have

(12)

$$\int_{B(x_i,r_i)} \psi(1/\delta\varepsilon w)\varepsilon w d\mu \le \frac{1}{\delta}\int_{B(x_i,r_i)} \rho(1/\delta\varepsilon w)d\mu$$
$$\le \frac{C_1\mu(B(x_i,r_i))}{\delta}\rho\left(\frac{C\mu(B(x_i,r_i))}{\delta\varepsilon\nu(\mathcal{R}_{4K}(x_i,r_i))}\right).$$

The constant C_1 in (12) is due to the doubling property of μ. If we take

$$\delta = C\mu(B(x_i,r_i))\phi^{-1}\left(\frac{1}{\varepsilon\nu(\mathcal{R}_{4K}(x_i,r_i))}\right),$$

then from the fact $s \le \phi^{-1}(s)\psi^{-1}(s)$ for $s \ge 0$ and (12), we obtain

$$\int_X \psi\left(\frac{\chi_{B(x_i,r_i)}}{\delta\varepsilon w}\right)\varepsilon w d\mu$$

(13)

$$\le \frac{C_2}{\phi^{-1}(\frac{1}{\varepsilon\nu(\mathcal{R}_{4K}(x_i,r_i))})}\rho\left(\frac{\frac{1}{\varepsilon\nu(\mathcal{R}_{4K}(x_i,r_i))}}{\phi^{-1}\left(\frac{1}{\varepsilon\nu(\mathcal{R}_{4K}(x_i,r_i))}\right)}\right)$$
$$\le \kappa C_2\varepsilon\nu(\mathcal{R}_{4K}(x_i,r_i)\psi\left(\frac{\frac{1}{\varepsilon\nu(\mathcal{R}_{4K}(x_i,r_i))}}{\phi^{-1}\left(\frac{1}{\varepsilon\nu(\mathcal{R}_{4K}(x_i,r_i))}\right)}\right)$$
$$\le \kappa C_2,$$

where $\kappa > 1$ satisfies $s\rho(s) \le \kappa\psi(s), s \ge 0$. Here we may assume that $\kappa C_2 \le 1$ so that

(14)
$$\|\frac{\chi_{B(x_i,r_i)}}{\varepsilon w}\|_{(\psi),\varepsilon w} \le C\mu(B(x_i,r_i))\phi^{-1}\left(\frac{1}{\varepsilon\nu(\mathcal{R}_{4K}(x_i,r_i))}\right).$$

Now take $1/\varepsilon = \int_{B(x_i,r_i)} \phi(|f|)w\ d\mu$. Then by the direct computation we have $\|f\chi_{B(x_i,r_i)}\|_{(\phi),\varepsilon w} = 1$ and so by (11), we get

$$\int_{B(x_i,r_i)} |f|\ d\mu \le 2C\mu(B(x_i,r_i))\phi^{-1}\left(\frac{\int_{B(x_i,r_i)} \phi(|f|)w\ d\mu}{\nu(\mathcal{R}_{4K}(x_i,r_i))}\right)$$

or

(15)
$$\frac{\nu(\mathcal{R}_{4K}(x_i,r_i))}{\mu(B(x_i,r_i))} \le C\frac{[[\phi(|f|)w : B(x_i,r_i)]]}{\phi([[|f|, B(x_i,r_i)]])}.$$

Since $\phi([[|f| : B(x_i, r_i)]]) \geq \phi(\lambda)$, it follows from (10), (15), and the disjointness of $\{B(x_i, r_i)\}$ that

$$\begin{aligned}
\nu(E_\lambda) &\leq \sum_i \nu(\mathcal{R}_{4K}(x_i, r_i)) \\
&\leq C \sum_i \frac{\mu(B(x_i, r_i))}{\phi(\lambda)} [[\phi(|f|)w : B(x_i, r_i)]] \\
&\leq \frac{C'}{\phi(\lambda)} \sum_i \int_{B(x_i, r_i)} \phi(|f|) w d\mu \\
&\leq \frac{C''}{\phi(\lambda)} \int_X \phi(|f|) w d\mu.
\end{aligned} \tag{16}$$

This completes the proof for the case E'_λ is bounded and each r_i is finite.

If $r(x) = \infty$ for some $x \in X$, then there is a sequence $\{r_n\}$ such that

$$\frac{1}{\mu(B(x, r_n))} \int_{B(x, r_n)} |f| \, d\mu \geq \lambda.$$

and $r_n \uparrow \infty$ as $n \to \infty$. For these r_n, if we apply the inequality (15) and n tends to infinity, then

$$\nu(X \times [0, \infty)) \leq \frac{C}{\phi(\lambda)} \int_X \phi(|f|) w d\mu,$$

as desired.

Next, assume that E'_λ is unbounded. Let $a \in X$ and $R > 0$. Then the set $E'_\lambda \cap B(a, R)$ is bounded. The above argument shows the same estimate since we can apply the covering lemma to the balls $B(x, r) : x \in E'_\lambda \cap B(a, R)$. This completes the proof. □

Example 3.1. Let $\phi(t) = t^p$, $p > 1$. Then the the complementary N-function ψ is given by $\psi(t) = t^q$, where q is the conjugate exponent of p. Also the corresponding density functions are given by $\varphi(t) = pt^{p-1}$ and $\rho(t) = qt^{q-1}$. Let $\Omega_x = \{(y, t) : |x - y| < t\}$. Then the condition $C_\phi(\Omega)$ says

$$\frac{\nu(\mathcal{R}_1(x, r))}{|B(x, r)|^p} \left(\int_{B(x, r)} w^{-1/(p-1)} d\mu \right)^{p-1} \leq C,$$

which is equivalent to the condition C_p given by Ruiz[R] since

$$\mathcal{R}_1(x, r) = B(x, 3r) \times [0, 2r].$$

Example 3.2. Let $X = \mathbb{R}^n$. and $d\nu = udx \otimes d\delta_o$, where $d\delta_o$ is the Dirac measure concentrated on $t = 0$. Since $\mathcal{R}_1(x,r) = B(x,4r) \times [0,2r)$, our condition implies

$$\frac{1}{|B(x,r)|}\left(\int_{B(x,r)} \epsilon u dx\right)\phi\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} \rho(1/\epsilon w)dx\right) \le C,$$

which is the A_ϕ condition obtained by Gallardo[G].

Definition 3.1. Set

$$\widehat{\Omega}_{(x_o,t)} = \{(x,r) \in X \times [t,\infty) : (x,s) \in \Omega_{x_o} \text{for some } s \le r\}$$

and

$$\widehat{\mathcal{R}}_\alpha(x,r) = \{(x_o,t) \in X \times [0,\infty) : \widehat{\Omega}_{(x_o,t)}(2r) \cap B(x,\alpha r) \neq \phi\}.$$

With this definition, we can define $\widehat{\Omega}$ and the $C_\phi(\widehat{\Omega},\alpha)$-condition in the same fashion as $C_\phi(\Omega,\alpha)$-condition and we have the following

Theorem 3.2. *Assume that an N-function satisfies the Δ_2-condition.*

i) If $\mathcal{M}_{\widehat{\Omega}}$ is of weak type (ϕ,ϕ) with respect to (ν,w), then (ν,w) satifies the condition $C_\phi(\widehat{\Omega},\alpha)$ for all $\alpha > 0$.

ii) If (ν,w) satifies the condition $C_\phi(\widehat{\Omega},\alpha)$ for some $\alpha \ge 4K$, then $\mathcal{M}_{\widehat{\Omega}}$ is of weak type (ϕ,ϕ) with respect to (ν,w).

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