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Local uniqueness of the unknown diffusivity in some heat conduction problem

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1. Introduction

Let us consider some heat conduction problem in a infinite composite material in $\mathbf{R}^3$. Suppose that the region $z > 0$ is of one material and $z < 0$ of another, and that there is no contact resistance at the boundary $z = 0$. Assume that a line source (which is identical with the y-axis) is emitting at the rate Q per unit length per unit time from time 0 to T. Measuring the temperature at the boundary, we want to determine the physical properties of one material when the physical properties of the other material are known. The uniqueness of the diffusivity of a finite homogeneous conductor in $\mathbf{R}^1$ was proved in [3]. In this note we shall give mathematical rigorous justification of the local uniqueness studied experimentally and theoretically in [1], [2], [6]. Our mathematical model is as follows:

$$(\partial_t - \kappa_1 \Delta) u_1 = \frac{Q}{c_1 \rho_1} \delta(x) \delta(z) \{H(t) - H(t-T)\}, -\infty < x, y < +\infty, z > 0, t > 0, T > 0, \quad (1.1)$$

$$(\partial_t - \kappa_2 \Delta) u_2 = 0, \qquad -\infty < x, y < +\infty, z < 0, t > 0, \quad (1.2)$$

$$u_1(x, y, z, 0) = 0, u_2(x, y, z, 0) = 0, \quad (1.3)$$

$$u_1(x, y, 0, t) = u_2(x, y, 0, t), \quad (1.4)$$

$$\rho_1 c_1 \kappa_1 \partial_z u_1(x, y, 0, t) = \rho_2 c_2 \kappa_2 \partial_z u_2(x, y, 0, t), \quad (1.5)$$

where κ_1 (a positive constant) is the known diffusivity, ρ_1 and ρ_2 (positive constants) are the known density, c_1 and c_2 (positive constants) are the known specific heat, κ_2 (a positive constant) is the unknown diffusivity, and $H(t)$ is the Heaviside function. We measure the temperature at the boundary $z = 0$ excepting the y-axis (a line source).

$$\theta(x_0, y_0, T) \equiv u_2(x_0, y_0, 0, T), \ x_0 \neq 0. \quad (1.6)$$

Knowing $d\theta / d \log T$, we study the determination of κ_2. Our main theorem is as follows:

Theorem. Suppose that $x_0^2/4\kappa_1 T < 1$, then the unknown diffusivity κ_2 is uniquely determined locally (in the neighbourhood of κ_1) from $d\theta/d\log T$. The following inequality holds

$$\left|\frac{d\theta}{d\log T} - \frac{Q}{2\pi}\frac{1}{(\rho_1 c_1\kappa_1 + \rho_2 c_2\kappa_2)}\right| \le C\left|\frac{x_0^2}{T}\right|, \tag{1.7}$$

where C is a positive constant.

Remark. From (1.7), assuming that x_0^2/T is sufficiently small, then κ_2 is approximately $\frac{1}{\rho_2 c_2}(\frac{Q}{2\pi}\frac{d\log T}{d\theta} - \rho_1 c_1 \kappa_1)$.

2. Proof of theorem

To construct the solutions u_1 and u_2 satisfying (1.1)-(1.5), we define w_1 and w_2 as the solutions of the following system:

$$(\partial_t - \kappa_1\Delta)w_1 = \delta(x)\delta(y)\delta(z-z')\delta(t), \qquad -\infty < x, y < +\infty, z > 0, z' > 0, t > 0, \tag{2.1}$$

$$(\partial_t - \kappa_2\Delta)w_2 = 0, \qquad -\infty < x, y < +\infty, z < 0, t > 0, \tag{2.2}$$

$$w_1(x,y,z,0) = 0, w_2(x,y,z,0) = 0, \tag{2.3}$$

$$w_1(x,y,0,t) = w_2(x,y,0,t), \tag{2.4}$$

$$\rho_1 c_1\kappa_1\partial_z w_1(x,y,0,t) = \rho_2 c_2\kappa_2\partial_z w_2(x,y,0,t). \tag{2.5}$$

Then it is easily seen that

$$u_2(x,y,0,T) = \frac{Q}{c_1\rho_1}\int_0^T\int_{-\infty}^{+\infty}\lim_{z'\downarrow 0} w_2(x, y-y', 0, t-t')dy'dt'. \tag{2.6}$$

Carslaw and Jaeger constructed the solutions satisfying (2.1)-(2.5).

Lemma 2.1([4]). The solution w_2 is expressed as follows:

$$w_2(x,y,z,t) = H(t)\frac{k^2}{8\pi^2(\kappa_1 t)^{3/2}}\int_0^1 \frac{e^{-k^2R^2/4(k^2u+1-u)}}{(k^2u+1-u)u^{1/2}(1-u)^{1/2}} f_2(Z, Z', k, \sigma, u)du, \tag{2.7}$$

where

$$f_2 = \frac{kZ(1-u)+\sigma^3 uZ'}{(1-u+\sigma^2 u)^2} e^{-\frac{Z'^2(1-u)+k^2Z^2u}{4u(1-u)}} + \frac{\sigma\pi^{1/2}u^{1/2}(1-u)^{1/2}}{(1-u+\sigma^2u)^{3/2}} \left\{1 - \frac{(kZ-\sigma Z')^2}{2(1-u+\sigma^2u)}\right\} \times$$

$$\times \mathrm{erfc}\left[\frac{(1-u)Z'+\sigma kZu}{2u^{1/2}(1-u)^{1/2}(1-u+\sigma^2u)^{1/2}}\right] e^{-\frac{(\sigma Z'-kZ)^2}{4(1-u+\sigma^2u)}},$$

$$Z = -\frac{z}{(\kappa_1 t)^{1/2}}, Z' = \frac{z'}{(\kappa_1 t)^{1/2}}, R = \frac{(x^2+y^2)^{1/2}}{(\kappa_1 t)^{1/2}}, \sigma = \rho_2 c_2 \kappa_2^{1/2} \rho_1^{-1} c_1^{-1} \kappa_1^{-1/2}, \text{and } k = \kappa_1^{1/2}\kappa_2^{-1/2}.$$

Proof of theorem. Noting that $H(T-t') = 1, 0 \le t' \le T$ and $f_2(0,0,k,\sigma,u) = \sigma\pi^{1/2}u^{1/2}(1-u)^{1/2}(1-u+\sigma^2u)^{-3/2}$, from (2.6) and (2.7), we get

$$u_2(x,y,0,T) = \frac{Qk^2\sigma}{c_1\rho_1}\int_0^T\int_{-\infty}^{+\infty}\left(\frac{1}{4\pi\kappa_1(T-t')}\right)^{3/2}\int_0^1 \frac{e^{-\frac{k^2(x^2+(y-y')^2)}{4\kappa_1(T-t')(k^2u+1-u)}}}{(k^2u+1-u)(1-u+\sigma^2u)^{3/2}} du dy' dt'$$

$$= \frac{Qk\sigma}{4\pi\kappa_1 c_1\rho_1}\int_0^1\left(\int_0^T \frac{1}{T-t'} e^{-\frac{k^2x^2}{4\kappa_1(T-t')}} dt'\right)\frac{1}{(k^2u+1-u)^{1/2}(1-u+\sigma^2u)^{3/2}} du$$

$$= \frac{Qk\sigma}{4\pi\kappa_1 c_1\rho_1}\int_0^1 \frac{1}{(k^2u+1-u)^{1/2}(1-u+\sigma^2u)^{3/2}}\int_{\frac{k^2x^2}{4\kappa_1 T}}^{+\infty} e^{-s}s^{-1} ds du$$

$$= \frac{Qk\sigma}{4\pi\kappa_1 c_1\rho_1}\int_0^1 \frac{1}{(k^2u+1-u)^{1/2}(1-u+\sigma^2u)^{3/2}}\Gamma(0, \frac{k^2x^2}{4\kappa_1 T}) du,$$

where $\Gamma(\nu,x) = \int_x^{+\infty} e^{-s}s^{\nu-1}ds$. Using the formula $\Gamma(0,x) = -\gamma - \log x - \sum_{n=1}^{+\infty}\frac{(-x)^n}{n!n}$ (γ is the Euler constant),we have

$$\theta = \frac{Qk\sigma}{4\pi\kappa_1 c_1\rho_1}\int_0^1 \frac{1}{(k^2u+1-u)^{1/2}(1-u+\sigma^2u)^{3/2}}(-\gamma - \log k^2x_0^2$$

$$+\log(4\kappa_1 T(k^2u+1-u)) - \sum_{n=1}^{+\infty}\frac{\left(-\frac{k^2x_0^2}{4\kappa_1 T(k^2u+1-u)}\right)^n}{n!n}) du.$$

Hence

$$\frac{d\theta}{d\log T} = \frac{Qk\sigma}{4\pi\kappa_1 c_1\rho_1}\int_0^1 \frac{1}{(k^2u+1-u)^{1/2}(1-u+\sigma^2u)^{3/2}} T\frac{d}{dT}(-\gamma - \log k^2x_0^2$$

$$+\log(4\kappa_1 T(k^2u+1-u)) - \sum_{n=1}^{+\infty}\frac{\left(-\frac{k^2x_0^2}{4\kappa_1 T(k^2u+1-u)}\right)^n}{n!n}) du$$

$$= \frac{Qk\sigma}{4\pi\kappa_1 c_1\rho_1}\int_0^1 \frac{1}{(k^2u+1-u)^{1/2}(1-u+\sigma^2u)^{3/2}}\left(1+\sum_{n=1}^{+\infty}\frac{1}{n!}\left(-\frac{k^2x_0^2}{4\kappa_1 T(k^2u+1-u)}\right)^n\right)du$$

$$= \frac{Qk\sigma}{4\pi\kappa_1 c_1\rho_1}\int_0^1 \frac{1}{(k^2u+1-u)^{1/2}(1-u+\sigma^2u)^{3/2}}e^{-\frac{k^2x_0^2}{4\kappa_1 T(k^2u+1-u)}}du$$

$$= \frac{Q\rho_2 c_2}{4\pi\rho_1^2c_1^2\kappa_1}\kappa_2^{1/2}\int_0^1 \frac{1}{(\kappa_1 u+(1-u)\kappa_2)^{1/2}(1-u+\rho_2^2c_2^2\rho_1^{-2}c_1^{-2}\kappa_1^{-1}\kappa_2 u)^{3/2}}e^{-\frac{x_0^2}{4T(\kappa_1 u+(1-u)\kappa_2)}}du.$$

To prove the local uniqueness, it is sufficient to show that $\partial_{\kappa_2}\left(\frac{d\theta}{d\log T}\right)|_{\kappa_2=\kappa_1}\neq 0$. We set

$$F(s) = C_1 s^{1/2}\int_0^1 \frac{1}{(\kappa u+(1-u)s)^{1/2}(1-u+A^2\kappa^{-1}su)^{3/2}}e^{\frac{B}{\kappa u+(1-u)s}}du,$$

here $C_1 = \frac{Q\rho_2c_2}{4\pi\rho_1^2c_1^2\kappa_1}$, $\kappa=\kappa_1$, $s=\kappa_2$, $A=\rho_2c_2\rho_1^{-1}c_1^{-1}$, and $B=(-x_0^2/4T)$. If we show that $F'(\kappa)<0$, then the local uniqueness will be proved. Now we are going to compute $F'(\kappa)$.

$$F'(\kappa) = \frac{C_1}{2\kappa}e^{B/\kappa}\int_0^1 \frac{u}{(1-u+A^2u)^{3/2}}du + \frac{3C_1A^2}{2\kappa}e^{B/\kappa}\int_0^1\frac{-u}{(1-u+A^2u)^{5/2}}du$$
$$+\frac{C_1B}{\kappa^2}e^{B/\kappa}\int_0^1\frac{u-1}{(1-u+A^2u)^{3/2}}du.$$

Inserting $\int_0^1 \frac{u}{(1-u+A^2u)^{3/2}}du = 2A^{-1}(A+1)^{-2}$, $\int_0^1 \frac{-u}{(1-u+A^2u)^{5/2}}du = (-2/3)(2A+1)A^{-3}(A+1)^{-2}$, and $\int_0^1 \frac{u-1}{(1-u+A^2u)^{3/2}}du = -2(A+1)^{-2}$ in this equality, we have

$$F'(\kappa) = 2C_1\kappa^{-1}(A+1)^{-2}e^{B/\kappa}(-1-\kappa^{-1}B).$$

From the assumption $x_0^2/4\kappa_1 T<1$, we obtain $F'(\kappa)<0$. Therefore the local uniqueness has been proved. Let us derive the inequality (1.7). Using the estimates $|e^{-x}-1|\leq C|x|$ (for$x\geq 0$), we get

$$\left|\frac{d\theta}{d\log T} - \frac{Q\rho_2c_2}{4\pi\rho_1^2c_1^2\kappa_1}\int_0^1\frac{1}{(k^2+1-u)^{1/2}(1-u+\sigma^2u)^{3/2}}du\right|$$
$$\leq C\left|\int_0^1 \frac{1}{(k^2+1-u)^{1/2}(1-u+\sigma^2u)^{3/2}}\frac{k^2x_0^2}{4\kappa_1T(k^2u+1-u)}du\right|.$$

Inserting $\int_0^1(k^2u+1-u)^{-1/2}(1-u+\sigma^2u)^{-3/2}du = 2(k+\sigma)^{-1}\sigma^{-1}$ and $\int_0^1(k^2u+1-u)^{-3/2}(1-u+\sigma^2u)^{-3/2}du = 2(k\sigma+1)(k+\sigma)^{-2}k^{-1}\sigma^{-1}$ in this inequality, we have

$$\left|\frac{d\theta}{d\log T} - \frac{2Q\rho_2c_2}{4\pi\rho_1^2c_1^2\kappa_1(k+\sigma)\sigma}\right| \leq C\left|\frac{x_0^2}{T}\right|.$$

Combining this with $\sigma = \rho_2 c_2 \kappa_2^{1/2} \rho_1^{-1} c_1^{-1} \kappa_1^{-1/2}$ and $k = \kappa_1^{1/2} \kappa_2^{-1/2}$, we can derive (1.7). The proof is completed.

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