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AN OPERATOR VERSION OF THE WILF-DIAZ-METCALF INEQUALITY

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ABSTRACT. Diaz and Metcalf generalized the Wilf inequality, which is also a generalization of the arithmetic-geometric mean inequality, to the case of vectors in a Hilbert space. In this note, we shall consider Wilf-Diaz-Metcalf type inequalities for operators on a Hilbert space.

1. Introduction. In 1963, Wilf [11] generalized the arithmetic-geometric mean inequality for complex numbers and Diaz and Metcalf [5] advanced it to the case for vectors in Hilbert space by the similar proof to Wilf's one:

Theorem A. *Let a be a unit vector in a Hilbert space H. If nonzero vectors x_k in H satisfy*

$$0 \leq r \leq \frac{\mathrm{Re}\,(x_k, a)}{\|x_k\|}$$

for some r, then

$$r(\|x_1\| \cdots \|x_n\|)^{1/n} \leq \frac{\|x_1 + \cdots + x_n\|}{n}.$$

More presicely, they showed the following inequality,

$$r(\|x_1\| + \cdots + \|x_n\|) \leq \|x_1 + \cdots + x_n\|,$$

which implies Theorem A by the arithmetic-geometric mean inequality.

In this note, we try to generalize the above inequalities to the case for operators on a Hilbert space on a line with their proof.

2. The Wilf-Diaz-Metcalf inequality. An operator version of Theorem A would be the following one:

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Theorem 1. *If operators A_k on a Hilbert space satisfy*

$$0 \le R \le \frac{\operatorname{Re} A_k}{\|A_k\|} \tag{1}$$

for some positive operator R, then

$$(\|A_1\| \cdots \|A_n\|)^{1/n} R \le \frac{1}{n}\|A_1 + \cdots + A_n\|. \tag{2}$$

This theorem follows from the following Diaz-Metcalf type inequality:

Theorem 2. *If every A_k satisfies* (1) *for $k = 1, \ldots, n$, then*

$$(\|A_1\| + \cdots + \|A_n\|)R \le \|A_1 + \cdots + A_n\|. \tag{3}$$

Proof. By $\sum \|A_i\| R \le \sum \operatorname{Re} A_i$, we have

$$\sum_{i=1}^{n} \|A_i\| R \le \|\sum_{i=1}^{n} \operatorname{Re} A_i\| = \|\operatorname{Re} \sum_{i=1}^{n} A_i\| \le \|\sum_{i=1}^{n} A_i\|.$$

Remark 1. We can exchange all the norms in the above theorem to an order-preserving function φ satisfying $\varphi(X) \ge \varphi(\operatorname{Re} X)$ and $\varphi(\alpha 1) = \alpha$, for example, the numerical radius $w(X)$: *If every A_k satisfies*

$$0 \le R \le \frac{\operatorname{Re} A_k}{w(A_k)},$$

then

$$(w(A_1) + \cdots + w(A_n))R \le w(A_1 + \cdots + A_n).$$

Remark 2. The denominator $\|A_k\|$ in the assumption (1) cannot be omitted even for the scalar case. Moreover, (1) cannot be exchanged to

$$0 \le 1 \le \operatorname{Re} A_k.$$

In fact, put $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\sqrt{\|A_1\|\|A_2\|} = (\|A_1\| + \|A_2\|)/2 = 2$ and we can choose $R = 1$, while $\|A_1 + A_2\|/2 = 3/2$

In addition, note that Theorem 1 or 2 is not an exact generalization of Theorem A, but a formal generalization. In a seminar talk, M.Fujii pointed out that we would rather generalize the Diaz-Metcalf inequality to the following style:

Theorem B(M.Fujii). *If there exist an operator R and a projection P such that*

$$0 \leq R \leq \frac{\operatorname{Re} PA_kP}{\|A_k\|}, \tag{4}$$

for $k = 1, \ldots, n$, then

$$(\|A_1\| + \cdots + \|A_n\|)R \leq \|A_1 + \cdots + A_n\|.$$

Proof. By $\sum \|A_i\|R \leq P(\sum \operatorname{Re} A_i)P$, we have

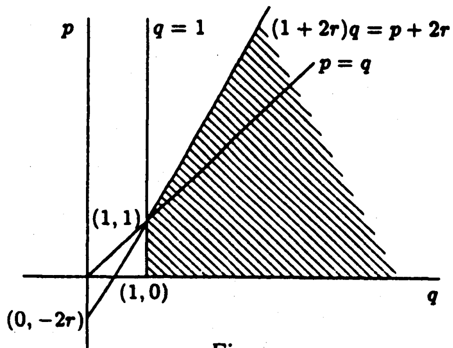
$$\sum_{i=1}^{n} \|A_i\|R \leq \|P(\sum_{i=1}^{n} \operatorname{Re} A_i)P\| \leq \|\sum_{i=1}^{n} \operatorname{Re} A_i\| = \|\operatorname{Re} \sum_{i=1}^{n} A_i\| \leq \|\sum_{i=1}^{n} A_i\|.$$

Putting $A_k = x_k \otimes a$, $P = a \otimes a$, $R = rP$ in the above theorem, we have

$$(PA_kPa, a) = (x_k, a), \quad \|A_k\| = \|x_k\|, \quad \|\sum A_k\| = \|\sum x_k\|,$$

so that we have Theorem A as a corollary.

Next, applying the Furuta inequality to the above theorems, we have variations of them. Furuta established the following result as an extension of Löwner-Heinz inequality.



Figure

Furuta Inequality [7; Theorem 1].
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^rA^pB^r)^{1/q} \geq (B^rB^pB^r)^{1/q}$$

and

(ii) $$(A^rA^pA^r)^{1/q} \geq (A^rB^pA^r)^{1/q}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$.

The domain drawn for p, q and r in the figure is the best possible one for Furuta inequality in [10]. Moreover, a function

$$f(p) = (B^rA^pB^r)^{\frac{1+2r}{p+2r}}$$

is monotone increasing for $p \geq 1$ as we see in [8]. Under the condition (4), put $A = \sum \operatorname{Re} PA_iP = \operatorname{Re} P \sum A_i P$ and $B = \sum \|A_i\|R$. By $A \geq B \geq 0$, the above monotone function shows

$$B^{1+2r} \leq B^rAB^r \leq (B^rA^pB^r)^{\frac{1+2r}{p+2r}} \leq B^{\frac{2r(1+2r)}{p+2r}} \|A\|^{\frac{p(1+2r)}{p+2r}}.$$

Thereby we have

Theorem 3. *If operators A_k on a Hilbert space satisfy* (4), *then, for each $p \geq 1$ and $r \geq 0$,*

$$\begin{aligned}(\sum_{i=1}^{n} \|A_i\|R)^{1+2r} &\leq (\sum_{i=1}^{n} \|A_i\|)^{2r} R^r (\sum_{i=1}^{n} \operatorname{Re} PA_iP) R^r \\ &\leq (\sum_{i=1}^{n} \|A_i\|)^{\frac{2r(1+2r)}{p+2r}} (R^r (\sum_{i=1}^{n} \operatorname{Re} PA_iP)^p R^r)^{\frac{1+2r}{p+2r}} \\ &\leq (\sum_{i=1}^{n} \|A_i\|R)^{\frac{2r(1+2r)}{p+2r}} \|\sum_{i=1}^{n} \operatorname{Re} PA_iP\|^{\frac{p(1+2r)}{p+2r}}. \\ &\leq (\sum_{i=1}^{n} \|A_i\|R)^{\frac{2r(1+2r)}{p+2r}} \|\sum_{i=1}^{n} \operatorname{Re} A_i\|^{\frac{p(1+2r)}{p+2r}}. \\ &\leq (\sum_{i=1}^{n} \|A_i\|R)^{\frac{2r(1+2r)}{p+2r}} \|\sum_{i=1}^{n} A_i\|^{\frac{p(1+2r)}{p+2r}}.\end{aligned}$$

3. N-ary mean inequality. Considering that the above theorems are derived from the arithmetic-geometric mean inequality, we will have such theorems from other mean inequalities. Means of positive operators have been discussed in some ways, see [2,3,4]. Based on the Kubo-Ando theory [9], Arazy [3] defined the *n-ary operator mean* $M(X_1, ..., X_n)$ of positive operators X_k on a Hilbert space as a positive operator satisfying the following axioms:

(monotonicity) $\quad 0 \leq A_k \leq B_k$ implies $M(A_1, ..., A_n) \leq M(B_1, ..., B_n)$

(continuity) $\quad A_{k,m} \downarrow A_k$ implies $M(A_{1,m}, ..., A_{n,m}) \downarrow M(A_1, ..., A_n)$

(transformer inequality) $\quad T^*M(A_1, ..., A_n)T \leq M(T^*A_1T, ..., T^*A_nT)$

(normality) $\quad M(1, ..., 1) = 1.$

Note that the transformer inequality becomes an equality if T is invertible. In particular, n-ary operator means are homegeneous:

(5) $$M(\alpha A_1, ..., \alpha A_n) = \alpha M(A_1, ..., A_n)$$

for $\alpha > 0$. By the transformer inequality, we also have

(6) $$M(\alpha_1 A, ..., \alpha_n A) = M(\alpha_1, ..., \alpha_n)A$$

For n-ary operator means M and L, we can define a natural order $M \leq L$ by

(7) $$M(A_1, ..., A_n) \leq L(A_1, ..., A_n)$$

for all $A_k \geq 0$.

Recall that the parallel sum $A : B$ for positive operators A and B, which was introduced by Anderson and Duffin [1], is defined by:

$$\langle A : Bx, x\rangle = \inf \{ \langle Ay, y\rangle + \langle Bz, z\rangle \mid y + z = x \}.$$

One of the noteworthy properties of the parallel sum is associativity: $A : (B : C) = (A : B) : C$. Since the harmonic (operator) mean h as a binary operation is defined by $AhB = 2A : B$ (cf. [9].), the harmonic mean M_h is defined by (see [2]):

$$M_h(A_1, ..., A_n) = n(A_1 : \cdots : A_n)$$

and Kosaki defined the geometric mean M_g (see also [6]):

$$M_g(A_1, ..., A_n) = \int (t_1 A_1 : \cdots : t_{n-1}A_{n-1} : A_n) d\mu(t_1, ..., t_n)$$

where $d\mu(t_1, ..., t_n) = \Gamma(1/n)^{-n} \prod_{j=1}^{n-1} t_j^{-(n+1)/n} dt_j$. Then the following harmonic-geometric-arithmetic mean inequality holds.

$$M_h(A_1, ..., A_n) \leq M_g(A_1, ..., A_n) \leq M_a(A_1, ..., A_n) \equiv \frac{A_1 + \cdots + A_n}{n}.$$

Now we have a variation of Theorems 1 and 2:

Theorem 4. *Let M and L be n-ary operator means with $M \leq L$. If every A_k satisfies* (1) *for $k = 1, \ldots, n$, then*

$$M(\|A_1\|, ..., \|A_n\|)R \leq L(\operatorname{Re} A_1, ..., \operatorname{Re} A_n).$$

Proof. By (6),(7) and monotonity, we have

$$\begin{aligned} M(\|A_1\|, ..., \|A_n\|)R &\leq L(\|A_1\|, ..., \|A_n\|)R \\ &= L(\|A_1\|R, ..., \|A_n\|R) \leq L(\operatorname{Re} A_1, ..., \operatorname{Re} A_n). \end{aligned}$$

On the other hand, Bhagwat and Subramanian [4] introduced the *power mean*

$$P_t(A_1, ..., A_n) = \left(\frac{A_1^t + \cdots + A_n^t}{n}\right)^{1/t}$$

Then, P_1 (resp. P_{-1}) is the arithmetic (resp. harmonic) (n-ary) operator mean M_a (resp. M_h). However, P_t is not an n-ary operator mean in general. As a matter of fact, we see the monotonity does not hold for $P_{1/2}$: Putting $A = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \leq 1$, we have

$$P_{1/2}(A,1) = \left\{\frac{1}{2}\left(\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + 1\right)\right\}^2 = \frac{1}{4}\begin{pmatrix} 13 & 12 \\ 12 & 13 \end{pmatrix},$$

$$P_{1/2}(A,P) = \left\{\frac{1}{2}\left(\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + P\right)\right\}^2 = \frac{1}{4}\begin{pmatrix} 13 & 10 \\ 10 & 8 \end{pmatrix},$$

so that $1 \geq P$ does not ensure $P_{1/2}(A,1) \not\geq P_{1/2}(A,P)$ by $\begin{pmatrix} 13 & 12 \\ 12 & 13 \end{pmatrix} \not\geq \begin{pmatrix} 13 & 10 \\ 10 & 8 \end{pmatrix}$.

Nevertheless, $P_t \leq P_s$ holds for $t \leq s$ and $t, s \notin (-1,1)$, so that monotonity for scalars shows:

Theorem 5. *Let P_t be the power mean. If every A_k satisfies*

$$0 \leq r \leq \frac{\operatorname{Re} A_k}{\|A_k\|}$$

for $k = 1, \ldots, n$, then, for $t \leq s$ and $t, s \notin (-1,1)$,

$$rP_t(\|A_1\|, \ldots, \|A_n\|) \leq P_s(\operatorname{Re} A_1, \ldots, \operatorname{Re} A_n).$$

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