Nihonkai Math. J.
Vol.6 (1995), 153-163

ON REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE WITH η-RECURRENT SECOND FUNDAMENTAL TENSOR

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0. Introduction.

Let M be an m-dimensional manifold with a linear connection Γ. A non zero tensor field K of type (r,s) on M is said to be *recurrent* if there exists a 1-form α such that $\nabla K = K \otimes \alpha$, where ∇ is covariant derivative with respect to Γ. We know the recurrent condition has a close relation to holonomy group in the sense of the following theorem (cf. [5] and [10]).

Theorem W. *œWe denote $L(M)$ be a bundle of frames of M and $T^r_s(\mathbf{R}^m)$ be a tensor bundle of type (r,s) over $\mathbf{R}^m$. Let $f : L(M) \to T^r_s(\mathbf{R}^m)$ be the mapping which corresponds to a given tensor field K of type (r,s). Then K is recurrent if and only if, for the holonomy bundle $P(u_0)$ through any $u_0 \in L(M)$, there exists a differentiable function $\psi(u)$ with no zero on $P(u_0)$ such that*

$$f(u) = \psi(u) f(u_0) \quad \text{for} \quad u \in P(u_0).$$

As a special case, K is parallel if and only if $f(u)$ is constant on $P(u_0)$.

We consider a real hypersurface M of real dimension $m = 2n-1$ in a complex projective space $P_n(\mathbf{C})$, $n \geqq 2$ with Fubini-Study metric of constant holomorphic sectional curvature 4. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure of $P_n(\mathbf{C})$. Many differential geometers have studied M by using the almost contact structure, for example [1], [2], [3], [4], [6] and [8]. It is well-known that there does not exist a real hypersurface M of $P_n(\mathbf{C})$ satisfying the condition that second fundamental tensor A of M is parallel. We have the following result under the weaker condition that the second fundamental tensor A is recurrent (cf. [7] and [9]).

Theorem 1. *There are no real hypersurfaces with recurrent second fundamental tensor of $P_n(\mathbf{C})$ on which ξ is a principal curvature vector.*

On the other hand Kimura and Maeda ([4]) introduced the notion of an η-parallel second fundamental tensor, which is defined by $g((\nabla_X A)Y, Z) = 0$

for any tangent vector field X, Y and Z orthogonal to ξ. In this paper we consider the notion that the second fundamental tensor is *η-recurrent* i.e. there exists a 1-form α such that the second fundamental tensor A of M satisfies $g((\nabla_X A)Y, Z) = \alpha(X)g(AY, Z)$ for any X, Y and Z which are orthogonal to ξ. We get the following:

Theorem 2. *Let M be a real hypersurface of $P_n(\mathbf{C})$. Then the second fundamental tensor of M is η-recurrent and ξ is a principal curvature vector if and only if M is locally congruent to a tube of some radius r over one of the following Kähler submanifolds:*

(A_1) *hyperplane $P_{n-1}(\mathbf{C})$, where $0 < r < \pi/2$,*
(A_2) *totally geodesic $P_k(\mathbf{C})$ $(1 \leqq k \leqq n-2)$, where $0 < r < \pi/2$,*
(B) *complex quadric Q_{n-1}, where $0 < r < \pi/4$.*

The author would like to express his sincere gratitude to Professors Y. Matsuyama and K. Ogiue for their valuable suggestions and continuous encouragement during the preparation of this paper.

1. Preliminaries.

Let M be a real hypersurface of $P_n(\mathbf{C})$. In a neighborhood of each point, we choose a unit normal vector field N in $P_n(\mathbf{C})$. The Riemannian connections $\tilde{\nabla}$ in $P_n(\mathbf{C})$ and ∇ in M are related the following formulas for arbitrary vector fields X and Y on M.

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{1.1}$$

$$\tilde{\nabla}_X N = -AX, \tag{1.2}$$

where g denotes the Riemannian metric of M induced from the Fubini-Study metric G of $P_n(\mathbf{C})$ and A is the second fundamental tensor of M in $P_n(\mathbf{C})$. We denote by TM tangent vector bundle of M. An eigenvector X of the second fundamental tensor A is called a *principal curvature vector*. Also an eigenvalue λ of A is called a *principal curvature*. In what follows, we denote by V_λ the eigenspace of A associated with eigenvalue λ. We know that M has an almost contact metric structure induced from the Kähler structure J on $P_n(\mathbf{C})$, that is, we define a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η on M by $g(\phi X, Y) = G(JX, Y)$ and $g(\xi, X) = \eta(X) = G(JX, N)$. Then we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0. \tag{1.3}$$

It follows from (1.1) that

$$\nabla_X \xi = \phi AX, \tag{1.4}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{1.5}$$

Let $\widetilde{R}$ and R be the curvature tensors of $P_n(\mathbf{C})$ and M, respectively. From the expression of the curvature tensor $\widetilde{R}$ of $P_n(\mathbf{C})$, we have the following Gauss and Codazzi equations:

$$
\begin{aligned}
(1.6) \qquad R(X,Y)Z = g(Y,Z)X - g(X,Z)Y \\
+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\
+ g(AY,Z)AX - g(AX,Z)AY,
\end{aligned}
$$

$$
(1.7) \qquad (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi.
$$

Now we prepare without proof the following in order to prove our results.

Lemma 1.1. ([6]) *If ξ is a principal curvature vector, then the corresponding principal curvature a is locally constant.*

Lemma 1.2. ([6]) *Assume that ξ is a principal curvature vector and the corresponding principal curvature is a. If $AX = \lambda X$ for $X \perp \xi$, then we have $A\phi X = ((a\lambda + 2)/(2\lambda - a))\phi X$.*

Lemma 1.3. ([4]) *We assume that ξ is a principal curvature vector. If $AX = \lambda X$ for $X \perp \xi$, then we have $\xi\lambda = 0$, that is, λ is locally constant along the direction ξ.*

Lemma 1.4. ([4]) *Let M be a real hypersurface of $P_n(\mathbf{C})$. Then the following are equivalent:*

(i) *The holomorphic distribution $T^0M (= \{X \in T_xM : X \perp \xi\}$ for $x \in M)$ is integrable.*
(ii) *$g((\phi A + A\phi)X, Y) = 0$ for any $X, Y \in T^0M$.*

Theorem T. ([8])*Let M be a homogeneous real hypersurface of $P_n(\mathbf{C})$. Then M is a tube of some radius r over one of the following Kähler submanifolds:*

(A_1) *hyperplane $P_{n-1}(\mathbf{C})$, where $0 < r < \pi/2$,*
(A_2) *totally geodesic $P_k(\mathbf{C})$ $(1 \leqq k \leqq n-2)$, where $0 < r < \pi/2$,*
(B) *complex quadric Q_{n-1}, where $0 < r < \pi/4$,*
(C) *$P_1(\mathbf{C}) \times P_{(n-1)/2}(\mathbf{C})$, where $0 < r < \pi/4$, and $n (\geqq 5)$ is odd,*
(D) *complex Grassmann $G_{2,5}(\mathbf{C})$, where $0 < r < \pi/4$ and $n = 9$,*
(E) *Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$ and $n = 15$.*

Theorem C-R. ([1]) *Let M be a real hypersurface of $P_n(\mathbf{C})$. Then M has at most two distinct principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface of type (A_1).*

Remark. They showed this theorem without the condition that ξ is a principal curvature vector in case of dimension $n \geqq 3$.

Theorem K1. ([3]) *Let M be a real hypersurface of $P_n(\mathbf{C})$. Then M has constant principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface.*

2. The recurrent real hypersurfaces of $P_n(\mathbf{C})$.

We prepare the lemma to prove Theorem 1.

Lemma 2.1. *Let M be a real hypersurface of $P_n(\mathbf{C})$ with recurrent second fundamental tensor A. If all principal curvatures of M are constant then the second fundamental tensor of M is parallel.*

Proof. We choose a unit principal curvature vector Y with a principal curvature λ. Then we have

$$\begin{aligned} g((\nabla_X A)Y, Y) &= g(\nabla_X(AY), Y) - g(A\nabla_X Y, Y) \\ &= X\lambda \end{aligned}$$

for any $X \in TM$. On the other hand, from the assumption we obtain

$$\begin{aligned} g((\nabla_X A)Y, Y) &= \alpha(X)g(AY, Y) \\ &= \alpha(X)\lambda. \end{aligned}$$

Since all principal curvatures of M are constant we get $\alpha(X)\lambda = 0$ for any $X \in TM$. So the second fundamental tensor A of M is parallel. □

Proof of Theorem 1. We may assume that $A\xi = a\xi$, then by Lemma 1.1. the principal curvature a of ξ is locally constant. From (1.4) we calculate the following:

$$\begin{aligned} (\nabla_X A)\xi &= \nabla_X(A\xi) - A\nabla_X\xi \\ &= a\nabla_X\xi - A\nabla_X\xi \\ &= a\phi AX - A\phi AX \end{aligned}$$

for arbitrary tangent vector field X on M. On the other hand, by the assumption that the second fundamental tensor A of M is recurrent, there exists a 1-form α and we have

$$\begin{aligned} (\nabla_X A)\xi &= \alpha(X)A\xi \\ &= \alpha(X)a\xi \end{aligned}$$

for any $X \in TM$. Consequently we get

$$a\phi AX - A\phi AX - \alpha(X)a\xi = 0.$$

We choose X as a principal curvature vector of M such that $AX = \lambda X$ and X is orthogonal to ξ, by Lemma 1.2. we have the following:

$$(a\lambda - \lambda\frac{a\lambda + 2}{2\lambda - a})\phi X + \alpha(X)a\xi = 0.$$

Using (1.3), ϕX is orthogonal to ξ , so

$$a\lambda - \lambda\frac{a\lambda + 2}{2\lambda - a} = 0.$$

Since a is constant, we know that M has at most three distinct constant principal curvatures. By Lemma 2.1. the second fundamental tensor A of M is parallel but it is well-known that there does not exists such a real hypersurface in $P_n(\mathbf{C})$. □

3. The η-recurrent real hypersurfaces of $P_n(\mathbf{C})$.

In [4], Kimura and Maeda introduced the notion of an η-parallel, which is defined by $g((\nabla_X A)Y, Z) = 0$ for any tangent vector field X, Y and Z orthogonal to ξ.

Theorem K-M1. *Let M be a real hypersurface of $P_n(\mathbf{C})$. Then the second fundamental tensor of M is η-parallel and ξ is a principal curvature vector if and only if M is locally congruent to a tube of some radius r over one of the following Kähler submanifolds:*

(A_1) *hyperplane $P_{n-1}(\mathbf{C})$, where $0 < r < \pi/2$,*
(A_2) *totally geodesic $P_k(\mathbf{C})$ $(1 \leqq k \leqq n-2)$, where $0 < r < \pi/2$,*
(B) *complex quadric Q_{n-1}, where $0 < r < \pi/4$.*

Let M be a real hypersurface of $P_n(\mathbf{C})$ with η-recurrent second fundamental tensor, that is, there exists an 1-form α such that $g((\nabla_X A)Y, Z) = \alpha(X)g(AY, Z)$ for any tangent vector fields X, Y and Z which are orthogonal to ξ. In what follows if M has η-recurrent second fundamental tensor then we call it M is η-recurrent. It is easily seen that if the second fundamental tensor A of M is η-parallel then M is η-recurrent. By Theorem K-M1 we know that the homogeneous real hypersurfaces of type (A_1), (A_2) and (B) is η-recurrent. We show that if ξ is principal curvature vector then (A_1), (A_2) and (B) are the only η-recurrent real hypersurfaces of $P_n(\mathbf{C})$. Now we define the holomorphic distribution T^0M by $T^0_xM = \{X \in T_xM : X \perp \xi\}$.

Proof of Theorem 2. Let Y be a unit principal curvature vector orthogonal to ξ with principal curvature μ, we calculate the following:

$$\begin{aligned} g((\nabla_X A)Y, Y) &= g(\nabla_X(AY) - A\nabla_X Y, Y) \\ &= X\mu. \end{aligned}$$

By hypothesis that the second fundamental tensor A is η-recurrent we have

$$\begin{aligned} g((\nabla_X A)Y, Y) &= \alpha(X)g(AY, Y) \\ &= \alpha(X)\mu \end{aligned}$$

Therefore we obtain

$$X\mu = \alpha(X)\mu \tag{3.1}$$

for arbitrary $X \in T^0M$. On the other hand using (1.3) and Codazzi equation (1.7) we note that

$$g((\nabla_X A)Y - (\nabla_Y A)X, Z) = 0 \tag{3.2}$$

for arbitrary tangent vector fields X, Y and $Z \in T^0M$. By hypothesis there is a 1-form α such that

$$g((\nabla_X A)Y - (\nabla_Y A)X, Z) = \alpha(X)g(AY, Z) - \alpha(Y)g(AX, Z)$$

for any X, Y and $Z \in T^0M$. Therefore by (3.2) there is a function b on M, we have

$$\alpha(X)AY - \alpha(Y)AX = b\xi.$$

If we choose $X \in V_\lambda$ and $Y \in V_\mu$, $\lambda \neq \mu$, such that $X, Y \perp \xi$ then we have

$$\alpha(X)\mu Y - \alpha(Y)\lambda X = 0. \tag{3.3}$$

If we can't choose these principal curvature vectors X, Y, i.e. in the case $T^0M = V_\lambda$, then by Theorem C-R we know that M is a homogeneous real hypersurface of type (A_1). Consequently we may assume $\lambda \neq \mu$ then we have

$$\alpha(X)\mu = 0 \quad \text{and} \quad \alpha(Y)\lambda = 0 \tag{3.4}$$

for any $X \in V_\lambda$ and $Y \in V_\mu$. Using (3.1) we obtain

$$X\mu = 0 \tag{3.5}$$

for any $X \in T^0M$ orthogonal to $Y \in V_\mu$.

If all principal curvatures of M are nonzero, then by (3.4) we conclude that

$$Y\mu = 0 \tag{3.6}$$

for any $Y \in V_\mu$.

We remark that we are not able to choose two distinct principal curvatures $\lambda \neq 0$ and $\mu \neq 0$, i.e. $T^0M = V_{\lambda=0} \oplus V_{\mu\neq 0}$. By Lemma 1.1 and Lemma 1.2. we conclude that μ is constant.

Now we decompose holomorphic distribution that $T^0M = V_{\lambda=0} \oplus V_{\mu_1\neq 0} \oplus \cdots \oplus V_{\mu_k \neq 0}$. Then we have a choice of two distinct principal curvatures $\mu_i \neq 0$ and $\mu_j \neq 0, (i \neq j)$. By (3.4) we obtain

$$\alpha(Y_i) = 0$$

for any principal curvature vector $Y_i \in T^0M$ such that it has nonzero principal curvature $\mu_i, (1 \leqq i \leqq k)$. Using (3.1) we have

$$Y_i\mu_i = 0. \tag{3.6$'$}$$

Therefore by Lemma 1.3., (3.5), (3.6) and (3.6)$'$, we know that all principal curvatures of T^0M is constant. Together with Lemma 1.1. we conclude that all principal curvatures are constant. So by Theorem K1 M is locally congruent to homogeneous real hypersurface in $P_n(\mathbf{C})$. So the rest of proof is to show the second fundamental tensor A of M, which is congruent to a homogeneous real hypersurface of type $(C), (D)$ and (E), is not η-recurrent. Suppose that the second fundamental tensor of M is η-recurrent. Here we review the following: Our real hypersurface M has five distinct constant principal curvatures (say $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and α), so that $TM = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus \{\xi\}_{\mathbf{R}}$. Let $x = \cot\theta (0 < \theta < \pi/4)$. Then we may write ([8])

$$\lambda_1 = x,\ \lambda_2 = -\frac{1}{x},\ \lambda_3 = \frac{1+x}{1-x},\ \lambda_4 = \frac{x-1}{x+1} \quad \text{and} \quad \alpha = x - \frac{1}{x}.$$

Since all principal curvatures are nonzero, using (3.4) we obtain

$$\alpha(X) = 0$$

for any $X \in T^0M$. Therefore the second fundamental tensor of M is η-parallel. By Theorem K-M1, the homogeneous real hypersurfaces of type $(C), (D)$ and (E) are not η-recurrent. $\square$

We know the example of non-homogeneous real hypersurface in $P_n(\mathbf{C})$. Kimura and Maeda constructed a ruled real hypersurface of $P_n(\mathbf{C})$. Let $\gamma(t)$ $(t \in I)$ be an arbitrary regular curve in $P_n(\mathbf{C})$. Then for every $t (\in I)$ there exists a totally geodesic submanifold $P_{n-1}(\mathbf{C})$ (in $P_n(\mathbf{C})$) which is orthogonal to the plane τ_t spanned by $\{\gamma'(t), J\gamma'(t)\}$. Here we denote by $P^{(t)}_{n-1}(\mathbf{C})$ such a totally geodesic submanifold $P_{n-1}(\mathbf{C})$. Let $M = \{x \in P^{(t)}_{n-1}(\mathbf{C}) : t \in I\}$. Then the construction of M asserts that M is a ruled real hypersurface in $P_n(\mathbf{C})$. The distribution T^0M is integrable and its integral manifold is a totally geodesic submanifold $P_{n-1}(\mathbf{C})$.

Let $H(X)$ be the sectional curvature of the holomorphic 2-plane spanned by a unit tangent vector X which is orthogonal to ξ , that is, $H(X) =$ the sectional curvature of span$\{X, \phi X\}$. They showed the followings:

Theorem K2. ([2]) *Let M be a real hypersurface of $P_n(\mathbf{C})$ on which H is constant and T^0M is integrable then M is locally congruent to a ruled real hypersurface ($H = 4$).*

Remark. They completely classified the real hypersurface of $P_n(\mathbf{C})$ on which H is constant.

Theorem K-M2. ([4]) *Let M be a real hypersurface of $P_n(\mathbf{C})$. Then the second fundamental tensor of M is η-parallel and T^0M is integrable if and only if M is locally congruent to a ruled real hypersurface of $P_n(\mathbf{C})$.*

First we remark that ruled real hypersurfaces of $P_n(\mathbf{C})$ don't admit the recurrent second fundamental tensor.

Proposition 3. *There are no ruled real hypersurfaces of $P_n(\mathbf{C})$ which has the recurrent second fundamental tensor.*

Proof of Proposition 3. We know that we may write the second fundamental tensor A of a ruled real hypersurface M in $P_n(\mathbf{C})$:

$$\begin{aligned} A\xi &= \mu\xi + \nu U \quad (\nu \neq 0), \\ AU &= \nu\xi, \\ AX &= 0 \quad (\text{for any } X \perp \xi, U), \end{aligned}$$

where U is a unit vector orthogonal to ξ, μ and ν are differential functions on M ([2] and [4]). By means of the assumption that the second fundamental tensor A of M is recurrent, we have

$$\begin{aligned} g((\nabla_\xi A)X, Y) &= \alpha(\xi)g(AX, Y) \\ &= 0. \end{aligned}$$

for any nonzero tangent vector $X, Y(\perp \xi, U)$. By Codazzi equation (1.7) we get the following:

$$\begin{aligned} g((\nabla_\xi A)X, Y) &= g((\nabla_X A)\xi + \phi X, Y) \\ &= g(\nabla_X(\mu\xi + \nu U) + A\nabla_X\xi + \phi X, Y) \\ &= g((X\mu)\xi + \mu\phi AX + (X\nu)U + \nu\nabla_X U + A\phi AX + \phi X, Y) \\ &= \nu g(\nabla_X U, Y) + g(\phi X, Y) \end{aligned}$$

Consequently we have

$$\nu g(\nabla_X U, Y) + g(\phi X, Y) = 0.$$

On the other hand, we get

$$\begin{aligned} g((\nabla_X A)\xi, Y) &= \alpha(X)g(A\xi, Y) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} g((\nabla_X A)\xi, Y) &= g(\nabla_X(\mu\xi + \nu U) - A\phi AX, Y) \\ &= g((X\mu)\xi + \mu\nabla_X\xi + (X\nu)U + \nu\nabla_X U, Y) \\ &= \mu g(\phi AX, Y) + \nu g(\nabla_X U, Y) \\ &= \nu g(\nabla_X U, Y) \end{aligned}$$

for arbitrary $X, Y(\perp \xi, U) \in TM$.

So we conclude that $\nu g(\nabla_X U, Y) = 0$ and

$$g(\phi X, Y) = 0$$

for any $X, Y(\perp \xi, U) \in TM$. If we put $Y = \phi X$, we have $g(X, X) = 0$. It is contradiction, so any ruled real hypersurface does not admit a recurrent second fundamental tensor. □

Using the idea of the proof of Theorem K-M2 we show the following theorem.

Theorem 4. *Let M be a real hypersurface of $P_n(\mathbf{C})$. Then M is η-recurrent and the holomorphic distribution $T^0M(=\{X \in T_x(M) : X \perp \xi\}$ for $x \in M)$ is integrable if and only if M is locally congruent to a ruled real hypersurface of $P_n(\mathbf{C})$.*

Proof of Theorem 4. We assume that T^0M is integrable and M is η-recurrent. We show that such a real hypersurface of $P_n(\mathbf{C})$ has a constant sectional curvature of holomorphic 2-plane, i.e. $H(X) =$ constant for arbitrary $X \in T^0M$.

It follows from Lemma 1.4. that

$$g(AY, \phi Z) = g(\phi Y, AZ) \tag{3.7}$$

for any $Y, Z \in T^0M$. We get

$$X(g(AY, \phi Z)) = X(g(\phi Y, AZ))$$

for arbitrary X, Y and $Z \in T^0M$ and we have

$$\begin{aligned} &g((\nabla_X A)Y + A\nabla_X Y, \phi Z) + g(AY, (\nabla_X \phi)Z + \phi\nabla_X Z) \\ &\quad = g((\nabla_X \phi)Y + \phi\nabla_X Y, AZ) + g(\phi Y, (\nabla_X A)Z + A\nabla_X Z). \end{aligned} \tag{3.8}$$

Now by the assumption we obtain

$$g((\nabla_X A)Y, \phi Z) = \alpha(X) g(AY, \phi Z)$$

and

$$g(\phi Y, (\nabla_X A)Z) = \alpha(X) g(\phi Y, AZ).$$

Using Lemma 1.4. we have

$$g((\nabla_X A)Y, \phi Z) = g(\phi Y, (\nabla_X A)Z). \tag{3.9}$$

It follows from (1.5), (3.8) and (3.9) that

$$\begin{aligned} &g(A\nabla_X Y, \phi Z) - g(AX, Z)\eta(AY) + g(\phi\nabla_X Z, AY) \\ &\quad = -g(AX, Y)\eta(AZ) + g(\phi\nabla_X Y, AZ) + g(A\nabla_X Z, \phi Y) \end{aligned} \tag{3.10}$$

We put

$$\nabla_X Y = (\nabla_X Y)_0 + \eta(\nabla_X Y)\xi, \tag{3.11}$$

where $(*)_0$ denotes the T^0M-component of $(*)$. Then, from (3.7) we have

$$g(A(\nabla_X Y)_0, \phi Z) = g(\phi(\nabla_X Y)_0, AZ) \tag{3.12}$$

for any X, Y and $Z \in T^0M$.

Substituting (3.11) into (3.10), by (3.12) we have

$$\eta(\nabla_X Y)g(A\xi, \phi Z) - g(AX, Z)\eta(AY) + \eta(\nabla_X Z)g(\phi\xi, AZ)$$
$$= -g(AX, Y)\eta(AZ) + \eta(\nabla_X Y)g(\phi\xi, AZ) + \eta(\nabla_X Z)g(A\xi, \phi Y).$$

Thus using (1.3) and (1.4) we obtain

$$g(Y, \phi AX)g(A\xi, \phi Z) + g(AX, Z)\eta(AY)$$
$$= g(AX, Y)\eta(AZ) + g(Z, \phi AX)g(A\xi, \phi Y)$$

for any X, Y and $Z \in T^0M$. We put

$$A\xi = \mu\xi + \nu U, \tag{3.13}$$

where ξ and U are orthonormal.

Because of the hypothesis and Lemma 1.4., we may assume that $\nu \neq 0$. By (1.3) we get

$$g(Y, \phi AX)g(U, \phi Z) + g(AX, Z)g(U, Y)$$
$$= g(AX, Y)g(U, Z) + g(Z, \phi AX)g(U, \phi Y) \tag{3.14}$$

By putting $Y = \phi U$ and $Z = U$, we see

$$g(A\phi U, X) = 0$$

for any $X \in T^0M$. On the other hand, it follows from (3.13) that

$$g(A\phi U, \xi) = g(\phi U, \mu\xi + \nu U) = 0.$$

Therefore we get

$$A\phi U = 0. \tag{3.15}$$

We put $Z = U$ in (3.14), from (3.15) we have

$$g(AX, U)g(U, Y) = g(AX, Y)$$

for arbitrary $X, Y \in T^0M$. By this equation and (3.13) we obtain

$$AX = 0 \tag{3.16}$$

for any $X(\perp U) \in T^0M$.

Now putting $Y = U$ and $Z = \phi U$ in (3.7), from (3.15) we get $g(AU, U) = 0$. By (3.16) we have $g(AU, X) = 0$ for any $X(\perp U) \in T^0M$. So it follows from (3.13) that

$$AU = \nu\xi. \tag{3.17}$$

Thus from (1.6), (3.16) and (3.17) we obtain

$$g(R(X, \phi X)\phi X, X) \equiv 4$$

for arbitrary $X \in T^0M$.

Due to Theorem K2 we conclude that M is a ruled real hypersurface of $P_n(\mathbf{C})$. □

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Received February 3, 1995