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On $*$-Representations of Partial $*$-Algebras

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Abstract

The first purpose of this paper is to study $*$-subrepresentations of a $*$-representation of a partial $*$-algebra. The second purpose is to characterize invariant positive sesquilinear forms of type I,II,III.

1. Introduction.

In this paper we shall investigate the fundamental properties of $*$-representations of partial $*$-algebras. The study of $*$-representations of partial $*$-algebras and partial O^*-algebras were began by Antoine and Karwouski [1], and have been continued by Antoine, Inoue and Trapani [2], from the situation of pure mathmatical and the physical applications. But, the studies of $*$-subrepresentations and invariant positive sesquilinear forms on partial $*$-algebras seem to be insufficient, and so we shall study these points in this paper.

In partial $*$-algebras, the multiplication is defined only partially and it dose not have the associative low. And so, to extend arguments that are considerd in the case of $*$-algebras, we need to reconsider some conditions. For example, the quasi-weak commutant $C_{\mathrm{qw}}(\pi)$ is considered instead of the usual weak commutant $C_{\mathrm{w}}(\pi)$ of π.

Let π be a $*$-representation of a partial $*$-algebra $\mathcal{A}$. For each projection E in $C_{\mathrm{qw}}(\pi)$, we can define the $*$-representation π_E of $\mathcal{A}$ by

$$\mathcal{D}(\pi_E) := \mathrm{E}\mathcal{D}(\pi), \ \ \pi_E(x) := \pi(x)E \ \ (x \in \mathcal{A}).$$

To define another $*$-subrepresentation of π, we define the notion of representable subspaces of $\mathcal{D}(\pi)$ as follows: A subspace $\mathfrak{M}$ of $\mathcal{D}(\pi)$ is said to be *representable* if $\pi(\mathcal{A})\mathfrak{M} \subset \overline{\mathfrak{M}}$. Of course, $\mathrm{E}\mathcal{D}(\pi)$ is a representable subspace of $\mathcal{D}(\pi)$ for each $\mathrm{E} \in C_{\mathrm{qw}}(\pi)$. For a representable subspace $\mathfrak{M}$ of $\mathcal{D}(\pi)$ we put

$$\mathcal{D}(\pi\lceil_{\mathfrak{M}}) := \mathfrak{M}, \quad \pi\lceil_{\mathfrak{M}}(x) := \pi(x)\lceil_{\mathfrak{M}} \quad (x \in \mathcal{A}).$$

Then $\pi\lceil_{\mathfrak{M}}$ is a $*$-representation of $\mathcal{A}$ on the Hilbelt space $\overline{\mathfrak{M}}$ whose full closure is denoted by $\pi_{\mathfrak{M}}$. It is natural to consider the following questions: Let $\mathfrak{M}$ be a representable subspace of $\mathcal{D}(\pi)$.

[Q1] When does $\mathrm{E}_{\overline{\mathfrak{M}}}$ (:= proj $\overline{\mathfrak{M}}$) belong to $C_{\mathrm{qw}}(\pi)$?
[Q2] When does the equation $\pi_{E_{\overline{\mathfrak{M}}}} = \pi_{\mathfrak{M}}$ hold ?

In Section 3 we shall solve the avobe questions.

Each bounded $*$-representation is decomposed into the direct sum of cyclic $*$-representations. We shall consider whether this result holds for fully closed $*$-repreaentations of partial $*$-algebras or not. In case of $*$-algebras, using the arguments of π-invariant subspaces, we investigated this problem [7]. But, in case of partial $*$-algebras, it's a problem that for each $\xi \in \mathcal{D}(\pi)$, even if $\mathfrak{M}_\xi$ is representable, $\mathrm{E}_{\overline{\mathfrak{M}_\xi}} \notin C_{\mathrm{qw}}(\pi)$ in general, where $\mathfrak{M}_\xi := \{\pi(y)\xi;\ y$ is the right multiplier of all elements of $\mathcal{A}\}$. So, in Section 4, we define the notion of self-adjoint vectors and obtain the decomposition theorem: Every self-adjoint representation π of a partial $*$-algebra is decomposed into

$$\pi = \pi_1 \oplus \pi_2,$$

where π_1 is a direct sum of self-adjoint cyclic representations of $\mathcal{A}$ and π_2 is a fully closed $*$-representation of $\mathcal{A}$ which dose not have any non-zero self-adjoint vector in $\mathcal{D}(\pi)$.

In Section 5, we shall define the types of self-adjoint representations π of partial $*$-algebras by the types of the von Neumann algebra $C_{\mathrm{qw}}(\pi)$.

In Section 6, we shall obtain the results about the characterization of primary Riesz forms of type I (II, III) using some order relation in the space of all Riesz forms on partial $*$-algebras.

2. Preliminaries

In this section we state the definitions and the basic properties about $*$-representations and invariant positive sesquilinear forms of partial $*$-algebras. For more details refer to [2].

$\mathcal{A}$ is called a *partial $*$-algebra* if the following conditions are satisfied:

(1) $\mathcal{A}$ is a linear space over $\mathbf{C}$ with an involution $*$.

(2) There is a subset Γ of $\mathcal{A} \times \mathcal{A}$ such that

  (i) $(x, y) \in \Gamma$ if and only if $(y^*, x^*) \in \Gamma$,

  (ii) if $(x, y), (x, z) \in \Gamma$, then $(x, \lambda y + \mu z) \in \Gamma$, for each $\lambda, \mu \in \mathbf{C}$,

  (iii) for each $(x, y), (x, z) \in \Gamma$ there is a $x \cdot y, x \cdot z \in \mathcal{A}$ such that $(x \cdot y)^* = y^* \cdot x^*$ and $x \cdot (\lambda y + \mu z) = \lambda(x \cdot y) + \mu(x \cdot z)$ for each $\lambda, \mu \in \mathbf{C}$.

If $(x, y) \in \Gamma$, x (resp. y) is called the *left multiplier* of y (resp. the *right multiplier* of x) and denoted by $x \in L(y)$ (resp. $y \in R(x)$). And we write

$$L(\mathcal{A}) := \bigcap_{x \in \mathcal{A}} L(x), \quad R(\mathcal{A}) := \bigcap_{x \in \mathcal{A}} R(x).$$

As usual, $\mathcal{D}$ denotes a dense subspace in a Hilbert space $\mathcal{H}$, and $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is the set of all linear operators X such that $\mathcal{D}(X) = \mathcal{D}$ and $\mathcal{D}(X^*) \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial $*$-algebra when equipped with the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $\dagger : X \to X^\dagger := X^* \restriction_{\mathcal{D}}$, and the partial multiplication $\Box$: for $X_1, X_2 \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, such that $X_2\mathcal{D} \subset \mathcal{D}(X_1^{\dagger *})$ and $X_1^\dagger \mathcal{D} \subset \mathcal{D}(X_2^*)$, $X_1 \Box X_2 := X_1^{\dagger *} X_2$. A *partial O^*-algebra* on $\mathcal{D}$ is a partial $*$-subalgebra of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$.

A *$*$-representation* of a partial $*$-algebra $\mathcal{A}$ is a $*$-homomorphism of $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ for some pair $\mathcal{D} \subset \mathcal{H}$, that is, a linear map $\pi : \mathcal{A} \to \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ such that,

(i) $\pi(x^*) = \pi(x)^\dagger$ for every $x \in \mathcal{A}$;

(ii) for each $y \in \mathcal{A}$, if $x \in L(y)$ then $\pi(x) \in L(\pi(y))$ and $\pi(x) \Box \pi(y) = \pi(xy)$.

The extension of $*$-representations is defined in the natural way. Let π_1 and π_2 be two $*$-representations of a partial $*$-algebra $\mathcal{A}$ in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$.

If $\pi_1(x) \subset \pi_2(x)$ for all $x \in \mathcal{A}$, then π_2 is said to be an *extention* of π_1 and this is denoted by $\pi_1 \subset \pi_2$.

As in the case of $*$-algebras, we can consider some representations for a given representation. For a $*$-representation π of a partial $*$-algebra $\mathcal{A}$, we define the *adjoint* π^* and the *closuer* $\hat{\pi}$ of π :

$$\mathcal{D}(\pi^*) = \bigcap_{x \in \mathcal{A}} \mathcal{D}(\pi(x)^*), \quad \pi^*(x) = \pi(x^*)^* \lceil_{\mathcal{D}(\pi^*)}, \ x \in \mathcal{A}.$$

$$\mathcal{D}(\hat{\pi}) = \bigcap_{x \in \mathcal{A}} \mathcal{D}(\overline{\pi(x)}), \quad \hat{\pi}(x) = \overline{\pi(x)} \lceil_{\mathcal{D}(\hat{\pi})}, \ x \in \mathcal{A}.$$

If $\pi = \pi^*$(resp. $\pi = \hat{\pi}$) then π is called *self-adjoint* (resp. *fully closed*) .

For the partial O*-algebra $\pi(\mathcal{A})$ with domain $\mathcal{D}(\pi) \subset \mathcal{H}$, we can define some commutants. As stated in Introduction, we deal with the *quasi-weak commutant* $C_{\rm qw}(\pi)$ defined as follows:

$$\begin{aligned} C_{\rm qw}(\pi) := \{C \in C_{\rm w}(\pi); \ & (C\pi(x^*)\xi \mid \pi(y)\eta) = (C\xi \mid \pi(xy)\eta) \\ & (C^*\pi(x^*)\xi \mid \pi(y)\eta) = (C^*\xi \mid \pi(xy)\eta) \\ & \text{for each } x, y \in \mathcal{A}\ , \xi, \eta \in \mathcal{D}(\pi)\}, \end{aligned}$$

where $C_{\rm w}(\pi) = \pi(\mathcal{A})'_{\rm w} = \{C \in B(\mathcal{H}); (C\pi(x)\xi \mid \eta) = (C\xi \mid \pi(x^*)\eta)$, for each $x \in \mathcal{A}$ and $\xi, \eta \in \mathcal{D}(\pi)\}$.

For two $*$-representations π_1, π_2 of a partial $*$-algebra $\mathcal{A}$, we define the direct sum as follows:

$$\begin{aligned} \mathcal{D}(\pi_1 \oplus \pi_2) \quad & := \{(\xi_1, \xi_2); \xi_1 \in \mathcal{D}(\pi_1), \ \ \xi_2 \in \mathcal{D}(\pi_2)\} \\ (\pi_1 \oplus \pi_2)(x)(\xi_1, \xi_2) \ & := (\pi_1(x)\xi_1, \pi_2(x)\xi_2). \end{aligned}$$

A vector $\xi \in \mathcal{D}(\pi)$ is said to be *cyclic* for π if $\pi(R(\mathcal{A}))\xi$ is dense in $\mathcal{D}(\pi)$ with respect to the graph topology.

For positive linear functionals of $*$-algebras, the GNS-construction generates the $*$-representation. In order to extend it to partial $*$-algebras, we introduce the notion of invariant positive sesquilinear form for which the GNS-construction is always possible.

A *sesquilinear form* on $\mathcal{A} \times \mathcal{A}$ is a mapping of $\mathcal{A} \times \mathcal{A}$ into $\mathbf{C}$ which is linear in the first and conjugate linear in the second variable. If $\varphi(x, x) \geq 0$ for all $x \in \mathcal{A}$, then φ is said to be *positive*. For each positive sesquilinear form φ on $\mathcal{A} \times \mathcal{A}$, we have

$$\varphi(x, y) = \overline{\varphi(y, x)}, \quad x, y \in \mathcal{A}\ ;$$
$$| \varphi(x, y) |^2 \leq \varphi(x, x)\varphi(y, y) \quad x, y \in \mathcal{A}\ ,$$

and hence we have the subspace $\mathcal{N}_\varphi$ of $\mathcal{A}$, where

$$\begin{aligned}\mathcal{N}_\varphi &:= \{x \in \mathcal{A} \,; \varphi(x,x) = 0\} \\ &= \{x \in \mathcal{A} \,; \varphi(x,y) = 0, \text{ for each } \ y \in \mathcal{A}\}.\end{aligned}$$

For each $x \in \mathcal{A}$, we denote by $\lambda_\varphi(x)$ the coset of the quotient space $\mathcal{A} / \mathcal{N}_\varphi$ which contains x, and define an inner product on $\lambda_\varphi(\mathcal{A})$ by:

$$(\lambda_\varphi(x) \mid \lambda_\varphi(y)) := \varphi(x,y), \quad x,y \in \mathcal{A} .$$

We denote by $\mathcal{H}_\varphi$ the Hilbert space obtained by the completion of the pre-Hilbert space $\lambda_\varphi(\mathcal{A})$. A positive sesquilinear form φ on $\mathcal{A} \times \mathcal{A}$ is called *invariant* if

(i) $\lambda_\varphi(R(\mathcal{A}))$ is dense in $\mathcal{H}_\varphi$;
(ii) $\varphi(xy_1, y_2) = \varphi(y_1, x^* y_2)$ for each $x \in \mathcal{A}$ and $y_1, y_2 \in R(\mathcal{A})$;
(iii) $\varphi({x_1}^* y_1, x_2 y_2) = \varphi(y_1, (x_1 x_2) y_2)$ for each $x_1 \in L(\mathcal{A})$ and $y_1, y_2 \in R(\mathcal{A})$.

Let φ is an invariant positive sesquilinear form on $\mathcal{A} \times \mathcal{A}$. We put

$$\pi_\varphi(x)\lambda_\varphi(a) := \lambda_\varphi(xa), \quad x \in \mathcal{A}\, , a \in R(\mathcal{A}).$$

Then π_φ is a $*$-representation of $\mathcal{A}$ on $\mathcal{H}_\varphi$ [3]. We call the triple $(\widehat{\pi_\varphi}, \lambda_\varphi, \mathcal{H}_\varphi)$ the GNS-construction for φ. If $\widehat{\pi_\varphi}$ is self-adjoint then φ is said to be a *Riesz form.*

3. Subrepresentations

In this section we consider the questions [Q1] and [Q2] in Introduction.

Let π be a fully closed $*$-representation of a partial $*$-algebra $\mathcal{A}$. For a projection E in $C_{\mathrm{qw}}(\pi)$, we define

$$\begin{aligned}\mathcal{D}(\pi_E) &:= \mathrm{E}\mathcal{D}(\pi), \\ \pi_E(x)\mathrm{E}\xi &:= \mathrm{E}\pi(x)\xi, \quad \text{for } x \in \mathcal{A}\, , \xi \in \mathcal{D}(\pi).\end{aligned}$$

Then we have the following property:

LEMMA 3.1. Let E be a projection in $C_{qw}(\pi)$. Then the following statements hold.

(i) π_E is a $*$-representation of $\mathcal{A}$ on $E\mathcal{H}$ satisfying

$$\mathcal{D}(\pi_E) = E\mathcal{D}(\pi) \subset \mathcal{D}(\pi_E{}^*) \subset E\mathcal{D}(\pi^*),$$
$$\pi_E{}^*(x)\xi = \pi^*(x)\xi, \quad \text{for } x \in \mathcal{A}, \xi \in \mathcal{D}(\pi_E{}^*).$$

(ii) Suppose $E\mathcal{D}(\pi) \subset \mathcal{D}(\pi)$. Then π_E is fully closed and

$$E\mathcal{D}(\pi^*) = \mathcal{D}(\pi_E{}^*).$$

Furthermore, π_E is self-adjoint if and only if $E\mathcal{D}(\pi) = E\mathcal{D}(\pi^*)$.

Proof. (i)Take an arbitrary $y \in \mathcal{A}$ and $x \in L(y)$. For each $\xi, \eta \in \mathcal{D}(\pi)$ we have

$$\begin{aligned}(\pi_E(x^*)E\xi \mid \pi_E(y)E\eta) &= (E\pi(x^*)\xi \mid \pi(y)\eta)\\ &= (E\xi \mid E\pi(xy)\eta)\\ &= (E\xi \mid \pi_E(xy)E\eta).\end{aligned}$$

Similarly, we have

$$(\pi_E(y)E\xi \mid \pi_E(x^*)E\eta) = (E\xi \mid \pi_E(y^*x^*)E\eta).$$

Hence we have $\pi_E(y) \in L(\pi_E(x))$ and, $\pi_E(x)\Box\pi_E(y) = \pi_E(xy)$.

Since

$$\begin{aligned}(\pi_E(x)E\xi \mid E\eta) &= (E\pi(x)\xi \mid \eta)\\ &= (E\xi \mid \pi(x^*)\eta)\\ &= (E\xi \mid \pi_E(x^*)E\eta)\end{aligned}$$

for each $x \in \mathcal{A}$ and $\xi, \eta \in \mathcal{D}(\pi)$, we have $\pi_E(x^*) \subset \pi_E(x)^*$ for each $x \in \mathcal{A}$. Therefore π_E is a $*$-representation of $\mathcal{A}$ on $E\mathcal{H}$.

Let $E\xi = \xi \in \mathcal{D}(\pi_E{}^*)$, then for each $x \in \mathcal{A}$ and $\eta \in \mathcal{D}(\pi)$, we have

$$(\xi \mid \pi(x^*)\eta) = (E\xi \mid \pi(x^*)\eta) = (\xi \mid \pi_E(x^*)E\eta) = (\pi_E(x^*)^*\xi \mid \eta).$$

So, $\xi \in \mathcal{D}(\pi^*)$ and $\pi^*(x)\xi = \pi_E{}^*(x)\xi$.

(ii) Suppose $E\mathcal{D}(\pi) \subset \mathcal{D}(\pi)$. Take an arbitrary $E\xi = \xi \in \mathcal{D}(\widehat{\pi_E})$ $(= \bigcap\{\mathcal{D}(\overline{\pi_E(x)}), x \in \mathcal{A}\})$. Let $x \in \mathcal{A}$. Then there is a sequence $\{E\xi_n\} \subset E\mathcal{D}(\pi)$ such that $E\xi_n \longrightarrow \xi$ and $\pi_E(x)E\xi_n \longrightarrow \overline{\pi_E(x)}\xi$. Since

$$\pi_E(x)E\xi_n = \pi^*(x)E\xi_n = \pi(x)E\xi_n, \quad E\xi_n \in \mathcal{D}(\pi(x)),$$

we have $\xi \in \mathcal{D}(\overline{\pi(x)})$. Since π is fully closed, we have $\xi \in \mathcal{D}(\pi)$. Then $\xi = E\xi \in E\mathcal{D}(\pi) = \mathcal{D}(\pi_E)$, and $\mathcal{D}(\widehat{\pi_E}) = \mathcal{D}(\pi_E)$. Therefore π_E is fully closed.

For each $\xi \in \mathcal{D}(\pi^*)$, it is easy to show that $E\xi \in \mathcal{D}(\pi_E{}^*)$ and $\pi_E{}^*(x)E\xi = E\pi^*(x)\xi$ for $x \in \mathcal{A}$, $\xi \in \mathcal{D}(\pi^*)$. Therefore it follows from (i) that π_E is self-adjoint if and only if $E\mathcal{D}(\pi) = E\mathcal{D}(\pi^*)$. □

Remark. The condition of $E \in C_{qw}(\pi)$ is that $E \in C_w = \pi(\mathcal{A})'_w$ and $E\mathcal{D}(\pi) \subset \mathcal{D}(\pi)$.

In case of a $*$-representation τ of a $*$-algebra $\mathcal{A}$, for a τ-invariant subspace $\mathfrak{M}$(i.e. $\tau(\mathcal{A})\mathfrak{M} \subset \mathfrak{M}$), we define the restriction of τ to $\mathfrak{M}$. But in this case, the condition "π-invariant" is too strict. So, if a subspace $\mathfrak{M}$ of the domain $\mathcal{D}(\pi)$ satisfies the condition that $\pi(\mathcal{A})\mathfrak{M} \subset \overline{\mathfrak{M}}$, $\mathfrak{M}$ is called *representable* and we consider such representable subspaces $\mathfrak{M}$ instead of π-invariant subspaces.

Let $\mathfrak{M}$ be a representable subspace. We put

$$\mathcal{D}(\pi\lceil_{\mathfrak{M}}) := \mathfrak{M},$$
$$\pi\lceil_{\mathfrak{M}}(x)\xi := \pi(x)\xi, \quad \xi \in \mathfrak{M}, \quad x \in \mathcal{A} .$$

Then $\pi\lceil_{\mathfrak{M}}$ is a $*$-representation of $\mathcal{A}$ on the Hilbelt space $\overline{\mathfrak{M}}$ whose full closure is denoted by $\pi_{\mathfrak{M}}$. Then π is the extention of $\pi_{\mathfrak{M}}$ with $\mathcal{H}(\pi_{\mathfrak{M}}) \subset \mathcal{H}(\pi)$, i.e.

$$\text{(3.1)} \qquad \pi_{\mathfrak{M}}(x)\xi = \pi(x)\xi \ \text{ for} x \in \mathcal{A} \, , \xi \in \mathcal{D}(\pi_{\mathfrak{M}}).$$

And we have the following properties:

LEMMA 3.2. Let $\mathfrak{M}$ be a representable subspace of $\mathcal{D}(\pi)$ and $E_{\overline{\mathfrak{M}}}$ the projection of $\mathcal{H}$ onto $\overline{\mathfrak{M}}$. Then,

(i) $\mathcal{D}(\pi_{\mathfrak{M}}) \subset E_{\overline{\mathfrak{M}}}\mathcal{D}(\pi) \subset E_{\overline{\mathfrak{M}}}\mathcal{D}(\pi^*) \subset \mathcal{D}(\pi_{\mathfrak{M}}{}^*)$,
(ii) $\pi_{\mathfrak{M}}{}^*(x)E_{\overline{\mathfrak{M}}}\xi = E_{\overline{\mathfrak{M}}}\pi^*(x)\xi$ for $x \in \mathcal{A}$, $\xi \in \mathcal{D}(\pi^*)$.

Proof. We prove the last relation of inclusion in (i) and (ii) at once. Let $\xi \in \mathcal{D}(\pi^*)$. For each $x \in \mathcal{A}$ and $\eta \in \mathfrak{M}$, we have

$$\begin{aligned}(E_{\overline{\mathfrak{M}}}\xi \mid \pi_{\mathfrak{M}}(x^*)\eta) &= (\xi \mid E_{\overline{\mathfrak{M}}}\pi(x^*)\eta) &&= (\xi \mid \pi(x^*)\eta) \\ &= (\pi(x^*)^*\xi \mid E_{\overline{\mathfrak{M}}}\eta) &&= (E_{\overline{\mathfrak{M}}}\pi^*(x)\xi \mid \eta).\end{aligned}$$

Hence $E_{\overline{\mathfrak{M}}}\xi \in \mathcal{D}(\pi_{\mathfrak{M}}{}^*)$ and $\pi_{\mathfrak{M}}{}^*(x)E_{\overline{\mathfrak{M}}}\xi = E_{\overline{\mathfrak{M}}}\pi^*(x)\xi$.□

THEOREM 3.3. Let $\mathfrak{M}$ be a representable subspace of $\mathcal{D}(\pi)$.
I. Consider the following statements.

(i) $\pi_{\mathfrak{M}}$ is self-adjoint.
(ii) $E_{\overline{\mathfrak{M}}}\mathcal{D}(\pi^*) = \mathcal{D}(\pi_{\mathfrak{M}})$.
(iii) $E_{\overline{\mathfrak{M}}}\mathcal{D}(\pi) = \mathcal{D}(\pi_{\mathfrak{M}})$.
(iv) $E_{\overline{\mathfrak{M}}}\mathcal{D}(\pi) \subset \mathcal{D}(\pi)$.
(v) $E_{\overline{\mathfrak{M}}} \in C_{qw}(\pi)$.

Then the following inplications hold :

$$\begin{array}{c}(i)\\ \Updownarrow\\ (ii)\end{array} \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)\ .$$

II. Suppose $E_{\overline{\mathfrak{M}}} \in C_{qw}(\pi)$. Then,

$$\pi_{\mathfrak{M}} \subset \pi_{E_{\overline{\mathfrak{M}}}} \subset \pi_{E_{\overline{\mathfrak{M}}}}{}^* \subset \pi_{\mathfrak{M}}{}^*, \pi \subset \pi_{E_{\overline{\mathfrak{M}}}} \oplus \pi_{I-E_{\overline{\mathfrak{M}}}} \subset \pi^*,$$

and

$$\pi = \pi_{E_{\overline{\mathfrak{M}}}} \oplus \pi_{I-E_{\overline{\mathfrak{M}}}} \quad \text{if and only if } E_{\overline{\mathfrak{M}}}\mathcal{D}(\pi) \subset \mathcal{D}(\pi).$$

III. In particular, if π is self-adjoint and $E_{\overline{\mathfrak{M}}} \in C_{qw}(\pi)$, then $\pi_{E_{\overline{\mathfrak{M}}}}$ and $\pi_{I-E_{\overline{\mathfrak{M}}}}$ are self-adjoint and $\pi = \pi_{E_{\overline{\mathfrak{M}}}} \oplus \pi_{I-E_{\overline{\mathfrak{M}}}}$. Furthermore, $\pi_{\mathfrak{M}}$ is self-adjoint if and only if $\pi_{\mathfrak{M}} = \pi_{E_{\overline{\mathfrak{M}}}}$.

Proof. I. (i) $\Leftrightarrow$(ii) Using Lenmma 3.2, it's easy to show this.
(ii) $\Rightarrow$(iii) $\Rightarrow$(iv) These follow from Lenmma 3.2(i) and $\mathcal{D}(\pi_{\mathfrak{M}}) \subset \mathcal{D}(\pi)$.
(iv) $\Rightarrow$(v) We can prove $E_{\overline{\mathfrak{M}}} \in C_w(\pi)$ by the same way in ([7] Theorem 3.3(ii)). And for each $y \in \mathcal{A}$, $x \in L(y)$ and $\xi, \eta \in \mathcal{D}(\pi)$, we have

$$\begin{aligned}(E_{\overline{\mathfrak{M}}}\pi(x^*)\xi \mid \pi(y)\eta) &= (E_{\overline{\mathfrak{M}}}\pi^*(x^*)\xi \mid \pi(y)\eta)\\ &= (\pi_{\mathfrak{M}}{}^*(x^*)E_{\overline{\mathfrak{M}}}\xi \mid \pi(y)\eta)\\ &= (\pi(x^*)E_{\overline{\mathfrak{M}}}\xi \mid \pi(y)\eta)\\ &= (E_{\overline{\mathfrak{M}}}\pi(x^*)\xi \mid \pi(y)\eta)\\ &= (E_{\overline{\mathfrak{M}}}\xi \mid \pi(xy)\eta).\end{aligned}$$

Hence $E_{\overline{\mathfrak{M}}} \in C_{qw}(\pi)$.

II and III are proved by the analogue with ([7] Corollary 3.4 and Theorem 3.3 (i′). □

REMARK 3.4. The converse of Theorem 3.3, I and the equations in II do not hold. The counter-examples for them are in ([7] EXAMPLE 3.6 and REMARK 3.5(2)).

4. Self-adjoint vectors.

In case of a $*$-algebra $\mathcal{A}$, if π is a closed $*$-representation of $\mathcal{A}$, then for each $\xi \in \mathcal{D}(\pi)$, $\mathfrak{M}_\xi := \pi(\mathcal{A})\xi$ is a π-invariant subspace of $\mathcal{D}(\pi)$, and so we can define the $\pi_{\mathfrak{M}_\xi}$. But this is not true in case of a partial $*$-algebra $\mathcal{A}$. So we introduce some notions for a vector $\xi \in \mathcal{D}(\pi)$. In this section, we deal with a self-adjoint representation π of a partial $*$-algebra $\mathcal{A}$ to avoid the complicated arguments.

DEFINITION 4.1. For a vector $\xi \in \mathcal{D}(\pi)$, we put $\mathfrak{M}_\xi := \pi(R(\mathcal{A}))\xi$.
ξ is said to be a *cyclically representable* if $\pi(\mathcal{A})\mathfrak{M}_\xi \subset \overline{\mathfrak{M}_\xi}$.
ξ is said to be a *self-adjoint vector for* π if ξ is cyclically representable and $\pi_{\mathfrak{M}_\xi}$ is self-adjoint.

By THEOREM 3.3, we characterize the self-adjointness of vectors in $\mathcal{D}(\pi)$ as follows:

COROLLARY 4.2. For each $\xi \in \mathcal{D}(\pi)$, E_ξ denotes the projection on $\mathcal{H}(\pi)$ onto $\overline{\mathfrak{M}_\xi}$. Then $\xi \in \mathcal{D}(\pi)$ is a self-adjoint vector for π if and only if $E_\xi \in C_{\mathrm{w}}(\pi)$ and

$$E_\xi \mathcal{D}(\pi) = \bigcap_{x \in \mathcal{A}} \mathcal{D}(\overline{\pi(x)\lceil_{\mathfrak{M}_\xi}}).$$

Proof. If $\xi \in \mathcal{D}(\pi)$ is a self-adjoint vector for π, then by THEOREM 3.3 we have $E_\xi \mathcal{D}(\pi) = \mathcal{D}(\pi_{\mathfrak{M}_\xi})$ and $E_\xi \in C_{\mathrm{qw}}(\pi) = C_{\mathrm{w}}(\pi)$.

We show the converse. From the assumptions,

$$\pi(\mathcal{A})\mathfrak{M}_\xi \subset \pi(\mathcal{A})E_\xi \mathcal{D}(\pi) = E_\xi \pi(\mathcal{A})\mathcal{D}(\pi) \subset \overline{\mathfrak{M}_\xi}.$$

Hence ξ is cyclically representable and furthermore $\pi_{\mathfrak{M}_\xi}$ is self-adjoint, becouse of $E_\xi \mathcal{D}(\pi) = \mathcal{D}(\pi_{\mathfrak{M}_\xi})$ and THEOREM 3.3. III. □

Using this result, we have the following property and its proof is much same as ([7] THEOREM 4.2).

THEOREM 4.3. For any self-adjoint representation of $\mathcal{A}$, we have the following decomposition:

$$\pi = \pi_1 \oplus \pi_2$$

where π_1 is a direct sum of self-adjoint cyclic representations of $\mathcal{A}$, and π_2 is a fully closed $*$-representation of $\mathcal{A}$ which dose not have any non-zero self-adjoint vector in $\mathcal{D}(\pi)$.

EXAMPLE 4.4. Even if $\mathcal{A}$ is a $*$-algebra, there exist a self-adjoint representation of $\mathcal{A}$ such that any non-zero vector of $\mathcal{D}(\pi)$ is not a self-adjoint vector for π ([7] EXAMPLE 4.4.).

EXAMPLE 4.5. Let $\mathcal{D}$ be a dence subspace of a Hilbert space $\mathcal{H}$. If the maximal partial O^*-algebra $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is self-adjoint, then every non-zero vector in $\mathcal{D}$ is a self-adjoint vector for $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$.

EXAMPLE 4.6. Let $\mathcal{D}$ be a dence subspace of a separable Hilbert space $\mathcal{H}$, $\mathcal{H}\otimes\overline{\mathcal{H}}$ the Hilbert space of Hilbert- Schmidt operators on $\mathcal{H}$, and

$$\mathcal{D}\otimes\overline{\mathcal{H}} := \{T \in \mathcal{H}\otimes\overline{\mathcal{H}}; T\mathcal{H} \subset \mathcal{D}\}.$$

Let $\mathcal{M}$ be a partial O^*-algebra on $\mathcal{D}$ with the identity operator I. We put

$$\sigma_2(\mathcal{M}) := \{T \in \mathcal{H}\otimes\overline{\mathcal{H}}; XT \in \mathcal{D}\otimes\overline{\mathcal{H}}, \text{ for each } X \in \mathcal{M}\},$$
$$\pi(X)T := XT, \qquad \text{for each } X \in \mathcal{M}, T \in \sigma_2(\mathcal{M}).$$

Then π is a $*$-representation of $\mathcal{M}$ and in particular, if $\mathcal{M}$ is self-adjoint, then so is π. Furthermore, if $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is self-adjoint, then every $\Omega \in$

$\sigma_2(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H}))$ is a self-adjoint vector for the self-adjoint representation π of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$.

5. Type of $*$-representations.

In this section we define the type of fully closed $*$-representations of a partial $*$-algebra $\mathcal{A}$, and mention a decomposition of them. We begin with some definitions.

Let $\mathcal{A}$ be a partial $*$-algebra with identity e and π_1, π_2 a fully closed $*$-representations of $\mathcal{A}$. π_1 is said to be a *$*$-subrepresentation of* π_2 if $\pi_1 = (\pi_2)_{\mathfrak{M}}$ for some π_2-representable subspace $\mathfrak{M}$ of $\mathcal{D}(\pi_2)$. π_1 is *contained in* π_2 if π_1 is unitarily equivalent to some $*$-subrepresentation π of π_2, and it is denoted by $\pi_1 \leq \pi_2$.

To make clear the essential part of argument, we treat with suitable $*$-representations. We denote by $Rep\mathcal{A}$ the set of all fully-closed $*$-representations π of $\mathcal{A}$ such that $C_{\rm qw}(\pi)\mathcal{D}(\pi) \subset \mathcal{D}(\pi)$, and denote by $Rep^s\mathcal{A}$ the set of all self-adjoint representations of $\mathcal{A}$. It is clear that $Rep^s\mathcal{A} \subset Rep\mathcal{A}$. For $\pi \in Rep\mathcal{A}$, we denote by $Rep\pi$ the set of all fully-closed $*$-subrepresentations τ of π such that $C_{\rm qw}(\tau)\mathcal{D}(\tau) \subset \mathcal{D}(\tau)$ and denote by $Rep^s\pi$ the set of all self-adjoint subrepresentations of π. Then $\{\pi_E; {\rm E} \in C_{\rm qw}(\pi)\} \subset Rep\pi$. In fact, let π be in $Rep\mathcal{A}$ and E a projection in $C_{\rm qw}(\pi)$, then by LEMMA 3.1. (ii), π_E is a fully-closed $*$-representation of $\mathcal{A}$ and furthermore

$$(4.1) \qquad C_{\rm qw}(\pi_E) = C_{\rm qw}(\pi)_E,$$

and so $C_{\rm qw}(\pi_E)\mathcal{D}(\pi_E) \subset \mathcal{D}(\pi_E)$.

By THEOREM 3.3, we have the following result for the relation of $Rep\pi$ and $Rep^s\pi$.

PROPOSITION 5.1. For each $\pi \in Rep\mathcal{A}$, $Rep^s\pi \subset Rep\pi$. In particular, if π is self-adjoint then $Rep^s\pi = Rep\pi$.

DEFINITION 5.2. Let π, π_1 and π_2 in $Rep\mathcal{A}$. If each non-trivial $\tau_1 \in Rep\pi_1$ and $\tau_2 \in Rep\pi_2$ are inequivalent, then π_1 and π_2 are said to be *disjoint* and denoted by $\pi_1 \ \delta \ \pi_2$.

If the von Neumann algebra $(C_{\text{qw}}(\pi))'$ is a factor (resp. of type I, of type II, of type III) then π is said to be a *factor representation (resp. of type I, of type II, of type III).*

By ([8] THEOREM 3.4), (4.1) and PROPOSITION 4.1, we have the following

PROPOSITION 5.3. Let π be in $Rep\mathcal{A}$. Then there uniqly exist mutually orthogonal projections E_I, E_{II}, E_{III} in $C_{\text{qw}}(\pi) \cap (C_{\text{qw}}(\pi))'$ such that $E_I + E_{II} + E_{III} = I$, π_{E_I} (resp. $\pi_{E_{II}}, \pi_{E_{III}}$) is in $Rep\pi$ and it is of type I (resp. type II, type III).

6. Type of invariant positive sesquilinear forms.

Let $\mathcal{A}$ be a partial $*$-algebra with an identity e and $\mathcal{R}(\mathcal{A} \times \mathcal{A})$ the set of all Riesz forms on $\mathcal{A} \times \mathcal{A}$. For $\varphi \in \mathcal{R}(\mathcal{A} \times \mathcal{A})$ and $a \in R(\mathcal{A})$ we put

$$\varphi_a(x, y) := \varphi(xa, ya) \quad \text{for} \quad x, y \in \mathcal{A}.$$

Then it is easily shown that φ_a is an invariant positive sesquilinear form on $\mathcal{A} \times \mathcal{A}$. We denote by $\mathcal{R}_\varphi$ the set of all Riesz forms ψ on $\mathcal{A} \times \mathcal{A}$ for which there exists a net $\{a_\alpha\}$ in $R(\mathcal{A})$ such that

$$\lim_\alpha \varphi_{a_\alpha}(x, y) = \psi(x, y) \qquad \text{and} \qquad \lim_{\alpha,\beta} \varphi_{a_\alpha - a_\beta}(x, x) = 0$$

for each $x, y \in \mathcal{A}$.

DEFINITION 6.1. Let $\varphi, \psi \in \mathcal{R}(\mathcal{A} \times \mathcal{A})$. We write $\psi \prec \varphi$ when $\mathcal{R}_\psi \subset \mathcal{R}_\varphi$, and $\psi \sim \varphi$ when $\mathcal{R}_\psi = \mathcal{R}_\varphi$.

It is clear that $(\mathcal{R}(\mathcal{A} \times \mathcal{A}), \prec)$ is an orderd set.

PROPOSITION 6.2. Let $\varphi, \psi \in \mathcal{R}(\mathcal{A} \times \mathcal{A})$. Then the following statements are equivalent.

(i) $\psi \prec \varphi$.
(ii) $\psi \in \mathcal{R}_{\varphi}$.
(iii) $\pi_{\psi} \leq \pi_{\varphi}$.
(iv) There exists an element ξ of $\mathcal{D}(\pi_{\varphi})$ such that $\psi(x,y) = (\pi_{\varphi}(x)\xi \mid \pi_{\varphi}(y)\xi)$ for all $x, y \in \mathcal{A}$.

Proof. (i) $\Rightarrow$(ii) $\Rightarrow$(iv). This is trivial.

(iv)$\Rightarrow$(iii). Since π_{φ} is self-adjoint, it follows that $\mathfrak{M}_{\xi} := \pi_{\varphi}(R(\mathcal{A}))\xi$ is π_{φ}-representable. By the assumption (iv), $(\pi_{\varphi})_{\mathfrak{M}_{\xi}} \sim \pi_{\psi}$ and so $(\pi_{\varphi})_{\mathfrak{M}_{\xi}} \in Rep^{s}\pi_{\varphi}$. Therefore $\pi_{\psi} \leq \pi_{\varphi}$.
(iii) $\Rightarrow$(i). Take an arbitrary $\psi' \in R_{\psi}$. Since the implication (ii) $\Rightarrow$(iii) holds and $\pi_{\psi} \leq \pi_{\varphi}$, we have $\pi_{\psi'} \leq \pi_{\varphi}$. Hence there exists a π_{φ}-representable subspace $\mathfrak{M}$ in $\mathcal{D}(\pi_{\varphi})$ such that $\pi_{\psi'} \sim (\pi_{\varphi})_{\mathfrak{M}}$, that is, there existes an isometry U of $\mathcal{H}_{\psi'}$ onto $\overline{\mathfrak{M}}$ such that $U\mathcal{D}(\pi_{\psi'}) = \overline{\mathfrak{M}}$ and $\pi_{\psi'}(x)\xi = U^{*}\pi_{\varphi}(x)U\xi$ for each $x \in \mathcal{A}$ and $\xi \in \mathcal{D}(\pi_{\psi'})$. Hence we have

$$\psi'(x,y) = (\pi_{\varphi}(x)U\lambda_{\varphi}(e) \mid \pi_{\varphi}(y)U\lambda_{\varphi}(e))$$

for each $x, y \in \mathcal{A}$, which impleis $\psi' \in \mathcal{R}_{\varphi}$. Therefore $\mathcal{R}_{\psi} \subset \mathcal{R}_{\varphi}$. $\square$

DEFINITION 6.3. Let $\varphi, \psi \in R(\mathcal{A} \times \mathcal{A})$. If $R_{\psi} \cap R_{\varphi} = \{0\}$, then φ and ψ are said to be *disjoint* and denoted by $\varphi \,\delta\, \psi$.

If for some $\gamma > 0$, $\psi(x,x) \leq \gamma\varphi(x,x)$ for each $x \in \mathcal{A}$, then ψ is said to be *dominated by* φ, and denoted by $\psi \leq \gamma\varphi$. If each ψ in $\mathcal{R}(\mathcal{A} \times \mathcal{A})$ with $\psi \leq \varphi$ has the form $\psi = \gamma\varphi$ for some scalar γ, then φ is said to be *pure*. If π_{φ} is a $*$-representation of typeI (resp. II,III), then φ is said to be of typeI (resp. II,III).

PROPOSITION 6.4. Let $\varphi, \psi \in \mathcal{R}(\mathcal{A} \times \mathcal{A})$. Then the following statements hold.

(i) $\varphi \; ó \; \psi$ if and only if $\pi_\varphi \; ó \; \pi_\psi$.
(ii) φ is pure if and only if $\pi_\varphi(\mathcal{A})'_{\mathrm{w}} = CI$.
(iii) φ is uniqly decomposed into $\varphi = \varphi_I + \varphi_{II} + \varphi_{III}$, where φ_I (resp. $\varphi_{II}, \varphi_{III}$) is a Riesz form on $\mathcal{A} \times \mathcal{A}$ of type I (resp. II,III) .

Proof. (i). This follows from PROPOSITION 6.2.
(ii). It is easily shown that $\psi \leq \gamma\varphi$ if and only if $\psi = \varphi_C$ for some $C \in C_{\mathrm{w}}(\pi_\varphi)$, that is, $\psi(x,y) = \varphi_C(x,y) := (C\lambda_\varphi(x) \mid \lambda_\varphi(y))$ for each $x, y \in \mathcal{A}$, which implies the statement (ii).
(iii). By PROPOSITION 5.3, there uniqly exists a projection E_I (resp. E_{II}, E_{III}) in $C_{\mathrm{w}}(\pi)$ such that $(\pi_\varphi)_{E_I}$ (resp. $(\pi_\varphi)_{E_{II}}, (\pi_\varphi)_{E_{III}}$) of type I (resp. II, III) and $E_I + E_{II} + E_{III} = I$. We now put

$$\varphi_I := \varphi_{E_I}, \quad \varphi_{II} := \varphi_{E_{II}}, \quad \varphi_{III} := \varphi_{E_{III}},$$

then the statement (iii) holds for this $\varphi_I, \varphi_{II}, \varphi_{III}$. □

DEFINITION 6.5. A Riesz form φ on $\mathcal{A} \times \mathcal{A}$ is said to be *primary* if π_φ is a factor. Let $\mathcal{R}_p(\mathcal{A} \times \mathcal{A})$ denote the set of all primary Riesz forms on $\mathcal{A} \times \mathcal{A}$.

PROPOSITION 6.6. Let $\varphi \in \mathcal{R}(\mathcal{A} \times \mathcal{A})$. Then the following statements are equivalent.

(i) φ is primary.
(ii) $\mathcal{R}_\varphi$ does not have any non-zero disjoint form.
(iii) $(\mathcal{R}_\varphi, \prec)$ is a totally orderd set.

Proof. (i) $\Rightarrow$(iii) $\Rightarrow$(ii). This follows from PROPOSITION 6.2 and ([3] COROLLARY 5.1.4, 5.1.5).

(ii) $\Rightarrow$(i). Suppose φ is not primary. Then there exists a projection $E \in C_{\mathrm{w}}(\pi_\varphi)$ such that $E \neq 0$ and $E \neq I$. For $x, y \in \mathcal{A}$, we put

$$\psi_1(x,y) = (\pi_\varphi(x)E\lambda_\varphi(e) \mid \pi_\varphi(y)E\lambda_\varphi(e)),$$
$$\psi_2(x,y) = (\pi_\varphi(x)(I-E)\lambda_\varphi(e) \mid \pi_\varphi(y)(I-E)\lambda_\varphi(e)).$$

Then it follows from PROPOSITION @@@ that $\psi_1, \psi_2 \in \mathcal{R}(\mathcal{A} \times \mathcal{A})$ and

$$\pi_{\psi_1} \sim (\pi_\varphi)_E \ \delta \ (\pi_\varphi)_{I-E} \sim \pi_{\psi_2},$$

which implies $\psi_1 \ \delta \ \psi_2$ by PROPOSITION 6.4. This is a contradiction. □

Using PROPOSITION 6.2, 6.4, 6.6, we can state the characterization of primary Riesz form of type I (II,III) and the proofs are similar to those of ([8] THEOREM 4.10).

THEOREM 6.7. Let $\varphi \in \mathcal{R}_p(\mathcal{A} \times \mathcal{A})$. Then the following statements hold.

(i) φ is of type I if and only if there exists a pure Riesz form ψ on $\mathcal{A} \times \mathcal{A}$ such that $\psi \prec \varphi$.
(ii) φ is of type II if and only if any Riesz form ψ on $\mathcal{A} \times \mathcal{A}$ with $\psi \prec \varphi$ is not minimal.
(iii) φ is of type III if and only if it is maxmal and minimal in $(\mathcal{R}_p(\mathcal{A} \times \mathcal{A}), \prec)$ and it is not pure.

These results are an extention of those to the case of partial $*$-algebras.

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