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Regularity of the solutions of some hypoelliptic operators

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Abstract. Let $P(D)$ be an hypoelliptic operator with constant coefficients, having a fundamental solution that is locally integrable in $\mathbb{R}^n$. Let u be a distribution defined on an open set Ω in $\mathbb{R}^n$ such that $Pu = f$. It's proved that if $f \in L^1_{loc}(\Omega)$ then $u \in L^1_{loc}(\Omega)$ and if f is in $C^m(\Omega)$ so is u.

1. Introduction.

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 1$. Let $A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ be a differential operator of order m with $a_\alpha \in C^m(\Omega)$. Let A^* denote the adjoint operator of A. Let $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\Omega)$, it is proved in [1], there exists a weak solution of $Au = f, u \in L^p(\Omega)$ and $\|u\|_p \leq c$ if and only if $| < f, \phi > | \leq c\|A^*\phi\|_q$ for all $\phi \in C_0^\infty(\Omega)$. In this note we discuss the possibility of finding an $L^1_{loc}(\Omega)$ solution u for the equation $Au = f$, if it is known that $f \in L^1_{loc}(\Omega)$.

Now it is known that if $P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ is a hypoelliptic differential operator of order m with constant coefficients and Ω is a convex open set of $\mathbb{R}^n$, then for any $T \in D'(\Omega)$, there exists a distribution $u \in D'(\Omega)$ such that $Pu = T$ (see [2]). If we suppose moreover that P is elliptic, then the above result is true even if Ω is

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assumed to be just open in $\mathbb{R}^n$. Consequently, for such operators P, given $f \in L^1_{loc}(\Omega)$ there always exists a distribution solution $u \in D'(\Omega)$. Our interest here is to find out whether u also is in $L^1_{loc}(\Omega)$. It turns out that this is true if the fundamental solution of P is locally integrable. As for this condition, it's known (see (see [3]) that if P is an elliptic differential operator with constant coefficients in $\mathbb{R}^n$, then there exists a locally integrable function E such that $PE = \delta$. Also for some of the non-elliptic but hypoelliptic type operators (for example, the heat operator $P = \dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}$), we have fundamental solutions that are locally integrable.

Our main result is that if P is an hypoelliptic differential operator with constant coefficients in $\mathbb{R}^n$, having a fundamental solution that is locally integrable, and if for a given $f \in L^1_{loc}(\Omega)$, Ω an open set, $Pu = f$ has a solution $u \in D'(\Omega)$, then u should also be in $L^1_{loc}(\Omega)$. Moreover in the special case when $f \in C^m(\Omega)$, we prove that $u \in C^m(\Omega)$ for any $m \geq 0$.

2. Locally integrable solutions

Lemma 2.1. Let P be an hypoelliptic differential operator with constant coefficients having a fundamental solution that is locally integrable. Let Ω and Ω_0 be open sets of $\mathbb{R}^n$ such that $\bar{\Omega}_0 \subset \Omega$. If $f \in L^1_{loc}(\Omega)$ and $u \in D'(\Omega)$ is a solution of $Pu = f$ then $u \in L^1_{loc}(\Omega_0)$.

Proof. Let dist $(\partial\Omega_0, \partial\Omega) = \eta > 0$ and E be a locally integrable fundamental solution of P. We suppose that $o \in \Omega_0$ for the convenience of writing. We choose $\mu > 0$ such that $\mu < \eta$ and $\varphi \in D(\Omega)$ such that supp $\varphi \subset \bar{B}(0, \mu)$ and $\varphi = 1$ on $\bar{B}(0, \frac{\mu}{2})$, where $\bar{B}(0, \mu) = \{x \in \Omega; |x| \leq \mu\} \subset \Omega_0$.

We'll prove that $P(\varphi E) - \delta \in D(\mathbb{R}^n)$. For, by using Leibnitz formula in $D'(\Omega)$ we have

$$\begin{aligned} P(\varphi E) &= \sum_{|\alpha|\le m} a_\alpha \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha-\beta} E \\ &= \varphi P E + \sum_{|\alpha|\le m} a_\alpha \sum_{\substack{\beta\le\alpha \\ \beta\ne 0}} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha-\beta} E \end{aligned}$$

Since $PE = \delta$, $\varphi PE = \varphi\delta = \varphi(0)\delta = \delta$,

Let $\psi = P(\varphi E) - \delta = \sum_{|\alpha|\le m} a_\alpha \sum_{\substack{\beta\le\alpha \\ \beta\ne 0}} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha-\beta} E.$

In $\Omega\backslash\{0\}$, E is C^∞ and $\psi \in C^\infty(\mathbb{R}^n\backslash\{0\})$; and since $\varphi = 1$ on $\bar{B}(0, \frac{\mu}{2})$, $D^\beta\varphi = 0$ in $B(0, \frac{\mu}{2})$ for all $\beta \ne 0$. So $\psi = 0$ on $\bar{B}(0, \frac{\mu}{2})$, hence $\psi \in C^\infty(\mathbb{R}^n)$.

Since the supp ψ is contained in the supp φ, $\psi \in D(\mathbb{R}^n)$. Let $u \in D'(\Omega)$ be a solution of $Pu = f$, we have then,

$$\begin{aligned} u = \delta * u &= [P(\varphi E) - \psi] * u \\ &= P(\varphi E) * u - \psi * u \\ &= \varphi E * Pu - \psi * u \\ u &= \varphi E * f - \psi * u \qquad (1) \end{aligned}$$

φE is a function on Ω such that supp $(\varphi E) \subset \bar{B}(0,\mu)$ and $\varphi E \in L^1_{loc}(\Omega)$; also since $f \in L^1_{loc}(\Omega)$, $\varphi E * f \in L^1_{loc}(\Omega_0)$. As $\psi * u \in C^\infty(\Omega_0)$, $\psi * u \in L^1_{loc}(\Omega_0)$. Hence $u \in L^1_{loc}(\Omega_0)$.

Theorem 2.2. With the same hypotheses as in Lemma 2.1, if $f \in L^1_{loc}(\Omega)$ and if $u \in D'(\Omega)$ is a solution of $Pu = f$ then $u \in L^1_{loc}(\Omega)$.

Proof. Let K be any compact set of Ω. Let dist $(K, \Omega) = \eta_1 > 0$ and we take $\Omega_0 = \bigcup_{x\in K} B(x, \eta_1/2)$, so $\bar{\Omega}_0 \subset \Omega$. By using the lemma 2.1, we have $u \in L^1_{loc}(\Omega_0)$, hence $u \in L^1(K)$. Thus $u \in L^1_{loc}(\Omega)$.

Theorem 2.3. With the same hypotheses as in Lemma 2.1, if f is a continuous function in Ω and $u \in D'(\Omega)$ is a solution of $Pu = f$, then u also is continuous.

Proof. From the equation (1) in the proof of lemma 2.1, we have $u = \varphi E * f - \psi * u$. Since supp $(\varphi E) \subset \bar{B}(0, \mu)$ and f is continuous on Ω, $\varphi E * f$ is continuous on Ω. For, if $x_0 \in \Omega_0$, we have

$$\varphi E * f(x) - \varphi E * f(x_0) = \int_{\bar{B}(0,\mu)} (\varphi E)(y)(f(x-y) - f(x_0 - y))dy.$$

We put

$$M = \int_{\bar{B}(0,\mu)} |(\varphi E)(y)|dy.$$

Let $y_1 \in \bar{B}(0, \mu)$, so $x_0 - y_1 \in \Omega$.

Given $\epsilon > 0$, since f is continuous at $x_0 - y_1$ there exists an open neighbourhood $V_1 \subset \Omega_0$ of x_0 and an open neighbourhood U_1 of y_1 such that: $|f(x-y) - f(x_0 - y_1)| < \frac{\epsilon}{M}$ for all $x \in V_1$, and all $y \in U_1$. Since such neigbourhoods U_1 cover the compact set $\bar{B}(0, \mu)$, there exists $U_1, U_2, \ldots, U_n$ s.t. $\bar{B}(0, \mu) \subset \bigcup_{i=1}^{n} U_i$. Let $V = \bigcap_{i=1}^{n} V_i$. Then we have $|f(x-y) - f(x_0 - y)| < \frac{\epsilon}{M}$ for all $x \in V$ and all $y \in \bar{B}(0, \mu)$.

Hence,

$$\begin{aligned} |\varphi E * f(x) - \varphi E * f(x_0)| &\leq \int_{\bar{B}(0,\mu)} |(\varphi E)(y)| f(x-y) - f(x_0 - y)|dy \\ &< \frac{\epsilon}{M} \int_{\bar{B}(0,\mu)} |(\varphi E)(y)|dy = \epsilon \end{aligned}$$

for all $x \in V$. So $\varphi E * f$ is continuous at any point $x_0 \in \Omega_0$. As $\psi * u \in C^\infty(\Omega_0)$, we deduce from (1) that u is continuous on Ω_0, and hence u is continuous on Ω.

Corollary 2.4. With the same hypotheses as in lemma 2.1, if u is a distribution such that $Pu = f \in C^m(\Omega)$, $m \geq 0$, then u also belongs to $C^m(\Omega)$.

Proof. From the equation (1) in lemma 2.1 $D^\alpha u = \varphi E * D^\alpha f - D^\alpha(\psi * u)$. This proves the corollary.

References

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