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$O(p) \times O(q)$-INVARIANT MINIMAL HYPERSURFACES IN HYPERBOLIC SPACE

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0. Introduction.

One of the most classical examples among minimal surfaces in $\mathbb{R}^3$ is a catenoid, and it is the only non-flat rotational minimal surface. Levitt and Rosenberg [4] gave a characterization of the catenoid (i.e, minimal rotational hypersurface) in a hyperbolic space as follows: *Let M be a connected minimal hypersurface immersed in H^n and regular at ∞ (cf. §1). Suppose the asymptotic boundary of M is the union of disjoint round hyperspheres S_1 and S_2. Then M is a catenoid.*

The orthogonal group $O(n)$ acts on H^n ($\cong$ the interior of the unit ball in $\mathbb{R}^n$) as a matrix multiplication, so the subgroup $O(p) \times O(q)$ $(p+q=n)$ also acts on H^n. In this paper, we consider a hypersurface in H^n which is invariant under the action of $O(p) \times O(q)$ $(p, q \geq 2)$ (say *$O(p) \times O(q)$-invariant hypersurface*). A hypersurface M in H^n is $O(p) \times O(q)$-invariant if and only if there is a codimension 1 foliation of M such that each leaf is congruent to the product of round spheres $S^{p-1}(d_1) \times S^{q-1}(d_2) \subset S^{n-1}(d) \subset H^n$. Note that the catenoid is $O(1) \times O(n-1)$-invariant hypersurface ($O(1) \cong \mathbb{Z}_2$). In §2, we will construct complete minimal embeddings of M diffeomorphic to $S^{p-1} \times \mathbb{R}$ into H^n such that M is $O(p) \times O(q)$ -invariant and its asymptotic boundary $= S^{p-1}(c_1) \times S^{q-1}(c_2)$ (modulo conformal transformation of $S^{n-1} =$ the asymptotic boundary of H^n). The method of construction is due to Ferus and Karcher [3]. In §3, we will give a characterization of $O(p) \times O(q)$-invariant complete minimal hypersurfaces in H^n in terms of the asymptotic boundary.

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1. Notations and preliminaries.

In this paper, we denote by $H^n(-c)$ a hyperbolic space with constant curvature $-c$, $H^n = H^n(-1)$ and by $S^n(c)$ a round sphere of constant curvature $c > 0$. According to [4], we refer to plane, distance, line, etc. as the hyperbolic object in H^n. First we work with *Poincaré model* of H^n (the interior of the unit ball in $\mathbb{R}^n$). The asymptotic boundary of H^n is identified with the boundary of the unit ball and denoted by $S(\infty)$. Given $A \subset H^n$, we denote by

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$\partial_\infty A$ the set of accumulation points of A in $S(\infty)$ and call it the *asymptotic boundary* of A.

We shall use the *latitude-longitude system* as the coordinate of H^n. Fix a hyperplane P_0 in H^n. Choose coordinates in P_0 and let γ be the geodesic orthogonal to P_0 at a origin $o \in P_0$. Let γ_t be the 1-parameter group of isometries of H^n which along γ is translation by a distance t and such that the curves $t \to \gamma_t(x)$ are orthogonal to P_0 for each $x \in P_0$ (a positive sense along γ is chosen once and for all). Then each point of H^n has coordinates (x, t) where $x \in P_0$ and $\gamma_t(x) = (x, t)$.

Denote by P_t the plane $\gamma_t(P_0)$. We refer to P_t as a *holizontal plane* and the curve $t \to \gamma_t(x)$ as the *vertical curve* through x. Notice that for each s the reflection of H^n through the plane P_s is given by the formula $(x, t) \to (x, 2s-t)$.

Let $S_t = \partial_\infty P_t$. Then the coordinate system (x, t) extends to a coordinate system on $S(\infty)$ where each point (except the two limits points of γ) has a unique coordinate (x, t), $x \in S_0$, $t \in \mathbb{R}$. By a Möbius transformation we can send γ to the north pole-south pole geodesic and P_0 to the equatorial plane. Then the coordinates on $S(\infty)$ are the usual latitude-longitude coordinates.

We say that $A \subset H^n$ is a *graph* over P_s if the vertical projection of A to P_s is injective, and A has *locally bounded slope* if the vertical field $v = (0, 1)$ is not tangent to A at any interior point of A.

We say that A is above B, $A \geq B$, if whenever a vertical curve meets both A and B, then every point of A (on this vertical) is above every point of B. These notations extend directly to $S(\infty)$ with respect to the horizontals S_t and the vertical curves.

For $A \subset H^n \cup S(\infty)$ and $s \in \mathbb{R}$, let $A_{s^+} = \{(x.t) \in A; t \geq s\}$ and similarly let A_{s^-} be the set of points of A below P_s. Let $A^*_{s^+} = \{(x, 2s-t); (x, t) \in A_{s^+}\}$. Also let H_{s^+} (resp. H_{s^-}) be the set of all points above P_s (resp. below P_s).

Let M be a complete hypersurface of H^n. We say that M is *regular at* ∞ if the asymptotic boundary B of M is a C^2 codimension one submanifold of $S(\infty)$ and $\overline{M} = M \cup B$ is of class C^1 on B.

We also use polar coordinates $[0, \infty) \times S^{n-1}(1)$ of H^n given by

$$g = dr^2 + \sinh^2 r \cdot d\omega^2$$

where $d\omega^2$ denotes the standard metric of $S^{n-1}(1)$. Then natural correspondence between $[0, \infty) \times S^{n-1}(1)$ and H^n is the following:

$$[0, \infty) \times S^{n-1} \ni (r, \xi) \longrightarrow (\tanh r)\xi \in H^n.$$

When we consider the Poincaré model, the orthogonal group $O(n)$ and its subgroup $O(p) \times O(q)$ $(p+q = n)$ act on H^n and $S(\infty) = S^{n-1}$ naturally. The orbit space of the action of $O(p) \times O(q)$ on H^n (resp. $S(\infty)$) is identified with the subset of H^2 given by $\{(r, \varphi) \in [0, \infty) \times [0, \pi/2]\}$ (resp. the subset of S^1 given by $\{\varphi \in [0, \pi/2]\}$.

2. Construction.

In this section, we construct minimal embeddings of M diffeomorphic to $S^{p-1} \times \mathbb{R}^q$ ($p+q=n$ and $p,q \geq 2$) into a hyperbolic space H^n such that M is complete, $O(p) \times O(q)$-invariant and its asymptotic boundary $\partial_\infty M$ is the product of round spheres $S^{p-1}(c_1) \times S^{q-1}(c_2)$ (modulo conformal transformation of $S(\infty)$). The construction is essentially due to Ferus and Karcher, so see [3] for more detailed description.

Let F be a quadratic polynomial on $\mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^n$, defined by $F(x,y) = \langle x,x\rangle - \langle y,y\rangle$ where $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$. We restrict F to unit sphere $S^{n-1}(1)$ in $\mathbb{R}^n$. Then the levels $F^{-1}(\{\cos 2\varphi\}) \cap S^{n-1}(1)$ $(0 < \varphi < \pi/2)$ form an isoparametric family

$$\cos\varphi \cdot S^{p-1}(1) \times \sin\varphi \cdot S^{q-1}(1) \subset S^{n-1}(1) \tag{2.1}$$

with 2 distinct constant principal curvatures.

We consider all distance spheres $\{r\} \times S^{n-1}$ in H^n admit the isoparametric family (2.1). Let $(r(s), \varphi(s))$, $s \in J$, be a differential curve in H^2 with $0 \leq r(s)$, $0 \leq \varphi(s) \leq \pi/2$, where J is an open interval of $\mathbb{R}$ and s is an arc length parameter (i.e. $r'(s)^2 + \sinh^2 r(s) \cdot \varphi'(s)^2 \equiv 1$). Then we obtained a hypersurface M in H^n given by the mapping $f : J \times S^{p-1} \times S^{q-1} \to H^n$

$$f(s,u,v) = (\tanh\frac{1}{2}r(s)\cdot\cos\varphi(s)\cdot u, \tanh\frac{1}{2}r(s)\cdot\sin\varphi(s)\cdot v), \tag{2.2}$$

for $s \in J$, $u \in S^{p-1}$, $v \in S^{q-1}$. We note that M is $O(p) \times O(q)$-invariant. Topological type of M is the following: M is immersed except that it may have conical singularities over the focal manifold $\varphi = 0$, $\varphi = \pi/2$. It is immersed, if $\varphi(J) \subset (0, \pi/2)$, or if

$$\begin{aligned} &r(s_0 - s) \equiv r(s_0 + s), \varphi(s_0 - s) \equiv -\varphi(s_0+s) \text{ for } 0 \leq s \ll 1 \\ &\quad \text{whenever } r(s_0) > 0, r'(s_0) = 0 \text{ and } \varphi(s_0) = 0 \text{ for } s_0 \in J. \end{aligned} \tag{2.3}$$

It is embedded, if moreover the curve (r, φ) is injective. M is diffeomorphic to $S^{p-1} \times \mathbb{R}^q$ (resp. $\mathbb{R}^p \times S^{q-1}$), if just one end of the curve reaches $\varphi = 0$ (resp $\varphi = \pi/2$) with $r' = 0$. M is diffeomorphic to $S^{p-1} \times S^{q-1} \times \mathbb{R}$, if $\varphi(J) \subset (0, \pi/2)$.

Note that when (2.3) is satisfied, the regularity of the hypersurface M yields that M admits a reparametrization: $(u,y) \in S^{p-1} \times B^q(\delta) \mapsto (k(|y|^2)\cdot u, y)$ at a sufficiently small neighborhood $S^{p-1} \times B^q(\delta)$ of the point $f(s_0,u,v)$, where $B^q(\delta)$ denotes an open disk of radius δ in $\mathbb{R}^q$ and $|y|$ is a norm of y. Outline of the proof is as follows: Let $l(s) := \tanh\frac{1}{2}r(s_0+s)\cdot\sin\varphi(s_0+s)$, and $k(s) := \tanh\frac{1}{2}r(s_0+s)\cdot\cos\varphi(s_0+s)$. Then $l(s)$ is odd, $k(s)$ is even and $l'(0) = \tanh\frac{1}{2}r(s_0)/\{\pm\sinh r(s_0)\} \neq 0$. Hence $^\exists\epsilon > 0$, $^\exists\delta > 0$ such that $l : (-\epsilon, \epsilon) \to (-\delta, \delta)$ is a diffeomorphism. Let $s = h(\sigma)$ be the inverse function of $\sigma = l(s)$. Then the function $k(h(\sigma))$ is even. By Whitney' theorem [4], there exists a C^∞-function ρ such that $k(h(\sigma)) = \rho(\sigma^2)$, for $|\sigma| < \delta$. From this, the above statement holds (cf. [2, pp.269-270]).

By curvature computations, any solution of the following 3-dimensional first-order differential equation produces an $O(p) \times O(q)$-invariant minimal hypersurface in H^n:

$$
\begin{aligned}
&r' = \sin\alpha \\
(2.4) \qquad &\varphi' = \cos\alpha/\sinh r \\
&\alpha' = (n-1)\cos\alpha/\sinh r + h(\varphi)\sin\alpha/\sinh r
\end{aligned}
$$

where $h(\varphi) = (p-1)\tan\varphi - (q-1)\cot\varphi$.

As in §4 of [3], we can find solutions of the differential equation (2.4), for which $r' \to 0$ as $\varphi \to 0$ or $\pi/2$. By studying qualitative description of the solution curves of

$$
(2.5) \quad \begin{cases} \dot{r} &= \sin\alpha\sinh r\sin 2\varphi, \\ \dot{\varphi} &= \cos\alpha\sin 2\varphi, \\ \dot{\alpha} &= (n-1)\cos\alpha\sin 2\varphi + 2\sin\alpha((p-1)\sin^2\varphi - (q-1)\cos^2\varphi), \end{cases}
$$

instead of (2.4), and of the cylindrical levels of

$$
L(\varphi,\alpha) = \sin^{q-1}\varphi \cdot \cos^{p-1}\varphi \cdot \sin\alpha,
$$

we obtain complete minimal hypersurfaces M which are embeddings of $S^{p-1} \times \mathbb{R}^q$ (or $\mathbb{R}^p \times S^{q-1}$) into H^n (cf. §5 and §6 of [3]). Note that if a solution of (2.5) satisfies $r(t_0) > 0$, $r'(t_0) = 0$ and $\varphi(t_0) = 0$ at a point t_0, then we can see that the solution also satisfies $r(t_0 - t) \equiv r(t_0 + t)$, $\varphi(t_0 - t) \equiv -\varphi(t_0 + t)$ and $\alpha(t_0 - t) \equiv -\alpha(t_0 + t) + \pi$ for $0 \leq s \ll 1$ by the uniqueness of the solution of ODE. Since $r(s)$ increases monotonically to $+\infty$ as $s \to \infty$ [3, p.258], $\varphi'(s) \to 0$ as $s \to \infty$. So $\varphi(s)$ converges to some constant c with $0 < c < \pi/2$ [3, §5, (g)] and the curve $(r(s), \varphi(s))$ meets the orbit space of $S(\infty)$ at one point $c \in (0, \pi/2)$. Consequently the asymptotic boundary of M is the product of round spheres $S^{p-1}(c_1) \times S^{q-1}(c_2)$ (modulo conformal transformation of $S(\infty)$).

Remark. Similarly we can construct complete minimal immersions of M diffeomorphic to $S^{p-1} \times S^{q-1} \times \mathbb{R}$ into H^n such that M is $O(p) \times O(q)$-invariant. Note that $O(p) \times O(q)$-invariant complete minimal hypersurface in H^n is either (a) embedded $S^{p-1} \times \mathbb{R}^q$, or (b) (immersed) $S^{p-1} \times S^{q-1} \times \mathbb{R}$. In fact, by [3, §5, (a)] we can see that the solution curves of (2.5) satisfy $\sharp\{s \in J; \varphi(s) = 0$ or $\pi/2\} = 1$ (case (a)) or 0 (case (b)), when M obtained by (2.2) and (2.4) is complete.

3. Characterization.

In this section we prove the following

Theorem 3.1. *Let M be a connected complete immersed minimal hypersurface in H^n such that M is regular at ∞ and its asymptotic boundary $\partial_\infty M$ is the product of round spheres $S^{p-1}(c_1) \times S^{q-1}(c_2)$ where $p+q=n$ and $p,q \geq 2$ (modulo conformal transformation of $S(\infty)$). Then M is $O(p)\times O(q)$-invariant.*

For the proof, we use the following result of Levitt and Rosenberg.

Proposition 3.2. [4] *Let $B \subset S(\infty)$ be a C^2 codimension one immersed boundary, not necessarily connected. Assume B_0^+ is a graph of locally bounded slope and $B_0^{*+} \geq B_0^-$. Let M be a minimal hypersurface immersed in H^n with $\partial_\infty M = B$ and regular at ∞. Then M_0^+ is a graph of locally bounded slope and $M_0^{*+} \geq M_0^-$.*

Proof of Theorem 3.1. We can assume that $\partial_\infty M = S^{p-1}(c_1) \times S^{q-1}(c_2) \subset \mathbb{R}^p \times \mathbb{R}^q$. Let P be a hyperplane of H^n defined by $(\mathbb{R}^{p-1} \times \mathbb{R}^q) \cap H^n$, where $\mathbb{R}^{p-1}$ is a hyperplane through the origin of $S^{p-1}(c_1)$ in $\mathbb{R}^p$. Then $B = S^{p-1}(c_1) \times S^{q-1}(c_2)$ satisfies the hypothesis of Proposition 3.2 from above and below P so M is invariant by reflection through P. By replacing $\mathbb{R}^p$ and $\mathbb{R}^q$, we can see that M is $O(p) \times O(q)$-invariant. □

It seems to worthwhile to consider the following problem: *Under the same situation as Theorem 3.1, if the asymptotic boundary $\partial_\infty M$ is an isoparametric hypersurface in $S(\infty)$ with 3, 4 or 6 distinct principal curvatures, then does M admits codimension 1 foliation such that each leaf is an isoparametric hypersurface of some round hypersphere of H^n?*

With respect to the asymptotic boundary of minimal varieties in H^n, Anderson [1] showed the following theorem: *If B^{p-1} is a closed submanifold of $S(\infty)$, then there exists a complete absolutely area-minimizing locally integral p-current Σ in H^n and B is the asymptotic boundary of Σ. More over, if $p \leq 6$, then Σ is smooth.*

So if $p > 6$, then Σ may have a singularity. Theorem 3.1 implies that if $B = S^{p-1}(1/\cos^2\theta) \times S^{q-1}(1/\sin^2\theta)$, then Σ with $\partial_\infty M = B$ is smooth if and only if there is a solution of (2.4) such that $\varphi(s) \to \theta$ as $s \to \infty$ provided that Σ is regular at infinity. So if the above problem is true, then the regularity of minimal varieties Σ in H^n with $\partial_\infty M =$"isoparametric hypersurface" can be seen by studying the behavior of solutions of the corresponding ODE (cf. §2) at infinity.

Finally we see that $O(p)\times O(q)$-invariant hypersurface is a generalization of tubes of constant radius over totally geodesic H^p $(2 \leq p \leq n-2)$ in H^n. Let u be a non-negative smooth function on $\Omega \subset H^p$ and suppose that u depends only on the distance from some point in H^p. Let $M = \{\exp_x u(x)\xi_x; x \in \Omega \text{ and } \xi_x \text{ is a unit normal vector at } x\}$. Then M is $O(p) \times O(q)$-invariant. Moreover if u is a positive constant, then M is a tube of radius u over H^p and M is a Riemannian product of $H^p(-1/\cosh^2 u)$ and $S^{n-p-1}(1/\sinh^2 u)$.

Theorem 3.1 states that some "converse" of the above fact holds as: Fix a totally geodesic submanifold H^p of H^n, and choose coordinates in H^p. Let γ_ξ be the geodesic of H^n through the origin $o \in H^p$ with the initial vector

$\xi \in UN_oH^p = \{\text{unit normal vectors at } o \in H^p \text{ in } H^n\} \cong S^{n-p-1}$. Denote by $\gamma_{\xi,s}$ the 1-parameter group of isometries of H^n which along $\gamma_\xi(s)$ $(s \geq 0)$ is a translation by a distance s and such that the curves $t \mapsto \gamma_{\xi,t}(x)$ are orthogonal to H^p for each $x \in H^p$. Let $M = \{\gamma_{\xi,u}(x);\ x \in H^p,\ \xi \in UN_oH^p\}$, where $u = u(x,\xi) \in C^\infty(H^p \times S^{n-p-1})$ and $u \geq 0$. Suppose M is a connected complete minimal hypersurface immersed in H^n such that M is regular at ∞ and its asymptotic boundary $\partial_\infty M = \{\gamma_{\xi,r}(r);\ x \in \partial_\infty H^p,\ \xi \in UN_oH^p\}$ for some $r > 0$ (hence $\partial_\infty M = S^{p-1} \times S^{n-p-1}$). Then $u(x,\xi) = u(x)$ (i.e., M is $O(n-p)$-invariant), and moreover u depends only on the distance from some point of H^p (i.e., M is $O(p)$-invariant).

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