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Convergent Theorems and L^p–selections for Banach–valued Multifunctions*)

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Abstract

Some convergence theorems for multifunctions are established. Egorov type theorem in Hausdorff limit is proved. The convergence theorems between L^p–integrably bounded multifunctions and L^p–selection families are also established. The result for generalized selection theorem is applied to solve a one–stage decision problem.

1. Introduction

In recent years the multifunction theory has been developed extensively by many authors with applications to mathematical economics, optimization and optimal control. (See for example, [1–5, 10–12]). The analysis of multifunction theory is based on the concepts of convergence of sets, the measurability, continuity and integrability for multifunctions. In this paper we will give some fundamental theorems for Kuratowski and Hausdorff limits of sets and for convergent sequence of measurable selections as well as

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L^p–selection families for multifunctions. Finally we apply the measurable theorem selections to a decision problem.

We mention in section 2 for some basic properties of Hausdorff and Kuratowski convergences on sets. Some representations for convergences and some counter examples are discussed. Sections 3 and 4 are the main parts of the paper. In section 3 we establish a generalized selection theorem as well as some results in real analysis. It is proved that a sequence of multifunctions convergent in Hausdorff limit is convergent in measure, Egorov type theorem is extended to the case of multifunctions in Hausdorff limit. In Section 4 we disucss the L^p–selections for Banach valued multifunctions. It is proved that if a sequence of L^p–integrably bounded multifunctons converges in Hausdorff limit, then the L^p–selections will converge in Hausdorff limit. While for the case of the Kuratowski limit, one can consult Aubin and Frankowska [2]. Finally in section 5 we give an application of measurable selection for multifunctions to optimization problem of a decision system. In Lai and Tanaka [8] (cf also [9]), they considered an infinite–stage Markovian decision system for vector–valued payoff function, and showed the existence of optimal strategy of average criterion in a vector–valued decision system under discrete average–time. For the average–time of one–stage decision problem in the determinestic case, one can consult Schal [12].

2. Hausdorff convergence and Kuratowski convergence for sets.

Throughout the paper let (T, Σ, μ), or simply T, be a complete σ–finite measure space, and let X be a separable Banach space. Denote by $P_f(X)$ the collection of all nonempty closed subsets of X.

The Hausdorff distance between E and F in $P_f(X)$ is defined by

$$(2.1) \qquad h(E, F) = \max \{ \sup_{y \in E} d(y, F), \sup_{z \in F} d(z, E)\}$$

Alternatively, it can be easily characterized by:

$$(2.2) \qquad h(E, F) = \sup_{x \in X} |d(x, E) - d(x, F)| = \inf \{r: E \subset F_r \text{ and } F \subset E_r\}$$

where $d(x, A) = \inf \{\|x - a\|;\ a \in A\}$ for any nonempty subset A of X, and $A_r = \{x\ ;\ d(x, A) < r\}$. $(P_f(X), h)$ is a metric space. As X is complete, $(P_f(X), h)$ is also complete. Let $P_k(X)$ be the set of all nonempty compact subsets of X. Then $P_k(X)$ is a closed and separable subset in $(P_f(X), h)$ provided X is separable.

A sequence $\{E_n\}$ of elements in $P_f(X)$ is said to be **Hausdorff convergent** to E in $P_f(X)$, and is written by

$$(2.3) \qquad E = h\text{-}\lim E_n \quad \text{or} \quad E_n \xrightarrow{h} E$$

if $h(E_n, E) \longrightarrow 0\ (n \to +\infty)$.

We have that, if $E_n \xrightarrow{h} E$, then

$$(2.4) \qquad E = \bigcap_{n=1}^{\infty} \mathrm{cl}\,[\bigcup_{k=n}^{\infty} E_k].$$

For $E, E_n \subset X$ $(n = 1, 2, \cdots)$, the sequence $\{E_n\}$ is said to be **Kuratowski convergent** (or **K–convergent**) to E if

$$(2.5) \qquad \tau\text{-}\lim\sup E_n \subset E \subset \tau\text{-}\lim\inf E_n$$

where τ stands for the norm–topology of X,

$$(2.6) \qquad \tau\text{-}\lim\sup E_n = \bigcap_{I \in \mathfrak{N}} \mathrm{cl}\,[\bigcup_{k \in I} E_k]$$

and

$$(2.7) \qquad \tau\text{-}\lim\inf E_n = \bigcap_{J \in \ddot{\mathfrak{N}}} \mathrm{cl}\,[\bigcup_{k \in J} E_k].$$

Here $\mathfrak{N}$ is the **Frechet filter** (at $+\infty$) defined by

$$(2.8) \qquad \mathfrak{N} = \{I \subset \mathbb{N} \mid \{k \in \mathbb{N}: k \geq n\} \subset I \text{ for some } n \in \mathbb{N}\}$$

where $\mathbb{N}$ is the set of all natural numbers; and $\ddot{\mathfrak{N}}$ is the **grill** of $\mathfrak{N}$, defined by

(2.9) $\quad \ddot{\mathfrak{N}} = \{J \subset \mathbb{N} \mid J \cap I \neq \emptyset \text{ for all } I \in \mathfrak{N}\}$.

Evidently $\mathfrak{N} \subset \ddot{\mathfrak{N}}$. It follows that τ–lim inf $E_n \subset \tau$–lim sup E_n and that $E_n \xrightarrow{k} E$ if and only if τ–lim inf $E_n = E = \tau$–lim sup E_n.

Some characterizations of (2.6) and (2.7) are given by:

Proposition 2.1. (cf. Aubin and Frankowska [2])

$$\text{(a)} \quad \tau\text{–}\lim\sup E_n = \bigcap_{n=1}^{\infty} \mathrm{cl}\,[\bigcup_{k=n}^{\infty} E_k]$$

$$\text{(b)} \quad \tau\text{–}\lim\inf E_n = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} (E_k)_{m^{-1}},$$

$$\text{where } (E_k)_{m^{-1}} = \{x: d(x, E_k) < m^{-1}\}.$$

Attouch showed in [1, p.92, Prop 1.32] that τ–lim inf $E_n = \mathrm{cl}[\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \mathrm{cl}\, E_k]$. But this is not true. For examples,

(1) If $X = \mathbb{R}$, $\tau =$ the usual topology on $\mathbb{R}$, and $E_n = \{\frac{1}{n}\}$, $n = 1, 2, \cdots$, then

$$\tau\text{–}\lim\inf E_n = \{0\} \neq \emptyset = \mathrm{cl}\,[\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \mathrm{cl}\, E_k]$$

(2) If $X = \mathbb{R}^2$, $\tau =$ the usual topology on $\mathbb{R}^2$, and $E_n = \{(x, y) \in \mathbb{R}^2 : y = (\frac{1}{n})x\}$, $n = 1, 2, \cdots$, then

$$\mathrm{cl}\,[\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \mathrm{cl}\, E_k] = \{(0, 0)\}.$$

But τ–lim inf E_n is the whole x–axis, and hence

$$\tau\text{–}\lim\inf E_n \underset{\neq}{\supset} \mathrm{cl}\,[\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \mathrm{cl}\, E_k]. \qquad \square$$

For monotone sequence of subsets in X, we have

Proposition 2.2. (cf. Aubin and Frankowska [2])

Let $\{E_n\}$ be a sequence of subsets in X.

(a) If $E_n \subset E_{n+1}$ $(n = 1, 2, \cdots)$ and $E = \mathrm{cl}\,[\bigcup_{n=1}^{\infty} E_n]$, then $E = K-\lim E_n$.

(b) If $E_n \supset E_{n+1}$ $(n = 1, 2, \cdots)$ and $E = \bigcap_{k=1}^{\infty} \mathrm{cl}\, E_n$, then $E = K-\lim E_n$.

However a sequence of sets convergent in Hausdorff metric is also convergent in Kuratowski limit but not the converse. For example,

(1) (Finite dimensional case). If $X = \mathbb{R}^2$ with usual Euclidean metric and

$$E_n = \{(x, y) \in \mathbb{R}^2 : y \geq \frac{1}{n} x \geq 0\},$$

then $E_n \subset E_{n+1}$, $n = 1, 2, \cdots$. It follows from Proposition 2.2 (a) that

$$E_n \xrightarrow{k} E = \mathrm{cl}\,[\bigcup_{n=1}^{\infty} E_n] = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}.$$

But $h(E_n, E) = +\infty$ for each $n \in \mathbb{N}$, so that $\{E_n\}$ is not convergent in Hausdorff limit.

(2) (Infinite dimensional case). If $X = C_0(\mathbb{R})$, the space of all continuous functions on $\mathbb{R}$ vanishing at infinity with the norm $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$, and define $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f_n(x) = \begin{cases} x - n + 1 & \text{if} \quad n-1 < x \leq n \\ -x + n + 1 & \text{if} \quad n < x \leq n + 1 \\ 0 & \text{otherwise}, \end{cases}$$

then the sequence of subsets $E_n = \{0, f_n\}$, $n = 1, 2, \cdots$ converges to $\{0\}$ in Kuratowski sense. But $h(\{E_n\}, \{0\}) = \| f_n \|_\infty = 1$ for all $n \in \mathbb{N}$. Hence

$$E_n \xrightarrow{k} E \not\Rightarrow E_n \xrightarrow{h} E \quad \text{with} \quad E = \{0\}.$$ □

3. Sequences of measurable multifunctions.

Recall that (T, Σ, μ) is a complete σ–finite measure space, and that X is a separable Banach space. Let $\Gamma : T \longrightarrow P_f(X)$ be a multifunction. Denote by

$$\mathrm{Grp}\,\Gamma = \{(t, x) \in T \times X \mid x \in \Gamma(t)\} \text{ and } \Gamma^{-1}(x) = \{t \in T \mid x \in \Gamma(t)\}.$$

For a subset $A \subset X$ we denote by

$$\Gamma^{-1}(A) = \bigcup_{x \in A} \Gamma^{-1}(x) = \{t \in T;\ \Gamma(x) \cap A \neq \emptyset\}.$$

A mapping $\sigma : T \longrightarrow X$ is said to be a selection for a multifunction Γ on T if,

$$\sigma(t) \in \Gamma(t) \quad \text{for all} \quad t \in T.$$

The following characterization is known (cf. Castaing and Valadier [3]).

Proposition 3.1. Let $\Gamma : T \longrightarrow P_f(X)$ be a multifunction. Then the following statements are equivalent:

(i) $\Gamma^{-1}(B) \in \Sigma$ for any Borel set B in X.

(ii) $\Gamma^{-1}(F) \in \Sigma$ for any closed set F in X.

(iii) $\Gamma^{-1}(U) \in \Sigma$ for any open set U in X.

(iv) For each $x \in X$, the mapping $t \longrightarrow d(x, \Gamma(t))$ is Σ – measurable.

(v) The mapping $(t, x) \longrightarrow d(x, \Gamma(t))$ is $\Sigma \otimes \mathfrak{B}(X)$ – measurable, where $\mathfrak{B}(X)$ stands for the Borel σ–algebra of X.

(vi) There is a sequence $\{\sigma_n\}$ of measurable selections for Γ such that $\Gamma(t) = \mathrm{cl}\,\{\sigma_n(t) \mid n \in \mathbb{N}\}$ for all $t \in T$.

(vii) $\mathrm{Grp}\,\Gamma \in \Sigma \otimes \mathfrak{B}(X)$.

A multifunction $\Gamma : T \longrightarrow P_f(X)$ is said to be measurable if any one of (i) ~ (vii) in Proposition 3.1 holds. We denote by $\mathfrak{M}(T, X)$ the set of all measurable multifunctions from T into $P_f(X)$.

Assume that $f: T \times X \longrightarrow R$ is a function satisfying:

(A1) for each $x \in X$, $t \longrightarrow f(t, x)$ is measurable,

(A2) for each $t \in T$, $x \longrightarrow f(t, x)$ is l.s.c..

Then it follows that if $\sigma : T \longrightarrow X$ is measurable, the mapping $t \longrightarrow f(t, \sigma(t))$ is also measurable. For any compact set $B \subset X$ Ekeland and Temam [4] proved that there is a measurable mapping $\sigma : T \longrightarrow B$ such that

$$f(t, \sigma(t)) = \min \{f(t, x) \mid x \in B\}.$$

The following measurable selection theorem is a slight genralization of Ekeland–Teman's result. In section 5, we shall apply this selection theorem to a decision problem.

Theorem 3.2 (generalized selection theorem). Let $B : T \longrightarrow P_k(X)$ be a measurable multifunction on T to the family of all nonempty compact subsets of X. If $f : T \times X \longrightarrow \mathbb{R}$ satisfies (A1) and (A2), then there exists a measurable selection σ for B such that

$$f(t, \sigma(t)) = \min \{f(t, x) \mid x \in B(t)\}.$$

Proof. For any $t \in T$, the set: $\Gamma(t) = \{x \in B(t) \mid f(t, x) = \min_{y \in B(t)} f(t, y)\}$ is a nonempty closed subset of X since $B(t)$ is compact, $f(t, y)$ ℓ.s.c. in y. By Proposition 3.1 (vi), there is a measurable sequence $\{\sigma_n\}$ for the measurable multifunction B such that

$$B(t) = \mathrm{cl}\, \{\sigma_n(t) \mid n \in \mathbb{N}\} \qquad \text{for all} \quad t \in T.$$

It follows that $t \longrightarrow f(t, \sigma_n(t))$ is measurable, and the graph of Γ :

$$\begin{aligned} \mathrm{Grp}\, \Gamma &= \{(t, x) \in T \times X \mid x \in \Gamma(t)\} \\ &= \{(t, x) \mid f(t, x) = \inf_{y \in B(t)} f(t, y)\} \\ &= \{(t, x) \mid f(t, x) = \inf_{n \in \mathbb{N}} f(t, \sigma_n(t))\} \end{aligned}$$

is $\Sigma \otimes \mathfrak{B}(X)$ – measurable. Therefore from Proposition 3.1 (vii), Γ is measurable and, hence, it admits a measurable selection σ. □

Now we turn to the convergence sequence of multifunctions. At first we see that if $\Gamma_n : T \longrightarrow P_f(X)$ $(n = 1, 2, \cdots)$ are measurable multifunctions such that either

(1) $\Gamma_n(t) \xrightarrow{h} \Gamma(t)$ or (2) $\Gamma_n(t) \xrightarrow{k} \Gamma(t)$, then $\Gamma \in \mathfrak{M}(T, X)$.

We mainly consider the Hausdorff limit in the rest part of this section.

Like single–valued functions, we say that a sequence of measurable multifunctions Γ_n $(n = 1, 2, \cdots)$ is **convergent in measure** to Γ if for any $\epsilon > 0$,

$$\mu\{t \in T \mid h(\Gamma_n(t), \Gamma(t)) > \epsilon\} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

$\{\Gamma_n\}$ is called convergent to Γ **almost uniformly** if for every $\epsilon > 0$, there exists a measurable set $T_0 \subset T$ such that $\mu(T \setminus T_0) < \epsilon$ and

$$\Gamma_n(t) \xrightarrow{h} \Gamma(t) \text{ uniformly on } T_0.$$

Like in real analysis, we will prove the following theorems for multifunctions.

Theorem 3.3. Assume that $\mu(T) < +\infty$ and let $\{\Gamma_n\}$ be a sequence in $\mathfrak{M}(T, X)$ such that $\Gamma_n(t) \xrightarrow{h} \Gamma(t)$ a.e.. Then $\{\Gamma_n\}$ converges to Γ in measure.

Proof. Note that $\Gamma \in \mathfrak{M}(T, X)$ and the mapping $t \longrightarrow h(\Gamma_n(t), \Gamma(t))$ is measurable. If Γ_n was not convergent to Γ in measure, then there is an $\epsilon > 0$ such that

$$\mu\{t \in T \mid h(\Gamma_n(t), \Gamma(t)) > \epsilon\} \longrightarrow 0 \ (\text{as } n \longrightarrow \infty).$$

Thus there exist $\delta > 0$, $n_1 < n_2 < \cdots$ such that

$$\mu\{t \in T \mid h(\Gamma_{n_k}(t), \Gamma(t)) > \epsilon\} > \delta \text{ for } k = 1, 2, \cdots.$$

Let $T_n = \{t \in T \mid h(\Gamma_n(t), \Gamma(t)) > \epsilon\}$ and $T_0 = \bigcap_{p=1}^{\infty} \bigcup_{k=p}^{\infty} T_{n_k}$. Since $\{\bigcup_{k=p}^{\infty} T_{n_k}\}$, $p = 1, 2, \cdots$, is a decreasing sequence and tends to T_0, we have

$$\mu(T_0) = \lim_{p\to\infty} \mu(\bigcup_{k=p}^{\infty} T_{n_k}) \text{ since } \mu(T) < +\infty.$$

Since $\mu(\bigcup_{k=p}^{\infty} T_{n_k}) \geq \mu(T_{n_p}) > \delta$, $\mu(T_0) \geq \delta$. But $\Gamma_n(t) \xrightarrow{h} \Gamma(t)$ a.e., there exists a $t_1 \in T_0$ such that

$$\Gamma_n(t_1) \longrightarrow \Gamma(t_1).$$

As $t_1 \in T_0$, we can find a subsequence $\{n_{k_\ell}\}$ of $\{n_k\}$ such that $t_1 \in T_{n_{k_\ell}}$, $\ell = 1, 2, \cdots$.

Hence

$$h(\Gamma_{n_{k_\ell}}(t_1), \Gamma(t_1)) > \epsilon .$$

This contradicts to $\Gamma_n(t_1) \xrightarrow{h} \Gamma(t_1)$. □

It seems never to be seen the Egorov Theorem for multifunctions so that the following theorem represented here in new.

Theorem 3.4 (Egorov type theorem). Let $\mu(T) < +\infty$ and $\{\Gamma_n\} \subset \mathfrak{M}(T, X)$ such that $\Gamma_n(t) \xrightarrow{h} \Gamma(t)$ a.e.. Then Γ_n converges to Γ almost uniformly.

Proof. Without loss of generality, we may assume that $\Gamma_n(t) \xrightarrow{h} \Gamma(t)$ for all $t \in T$. Then $\Gamma \in \mathfrak{M}(T, X)$ and for fixed $m \in \mathbb{N}$,

$$T_n(m) = \bigcup_{k=n}^{\infty} \{t \in T : h(\Gamma_k(t), \Gamma(t)) \geq \tfrac{1}{m}\},\ n = 1, 2, \cdots,$$

is a decreasing sequence of sets. Since $\Gamma_n(t) \xrightarrow{h} \Gamma(t)$ there is a $k_0 \in \mathbb{N}$ such that

$$h(\Gamma_k(t), \Gamma(t)) < \tfrac{1}{m} \quad \text{for} \quad k \geq k_o ,$$

that is, $t \in T_{k_o}(m)$. This shows that

$$\bigcap_{n=1}^{\infty} T_n(m) = \emptyset \quad \text{and} \quad T_n(m) \downarrow \emptyset, \text{ as } n \to \infty .$$

It follows that $\mu(T_n(m)) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence for any $\epsilon > 0$ and $m \in \mathbb{N}$, there is an $n(m) \in \mathbb{N}$ such that

$$\mu(T_{n(m)}(m)) < {}^{\epsilon}/2^m.$$

Let $T_0 = T \setminus \bigcup_{m=1}^{\infty} T_{n(m)}(m)$. Then

$$\mu(T \setminus T_0) = \mu(\bigcup_{m=1}^{\infty} T_{n(m)}(m)) \leq \sum_{m=1}^{\infty} \mu(T_{n(m)}(m)) < \sum_{m=1}^{\infty} \frac{\epsilon}{2^m} = \epsilon.$$

For any $\lambda > 0$, let M be a positive integer such that $\frac{1}{M} < \lambda$, we write $n(M) = N$. Since

$$T_0 = T \setminus \bigcup_{m=1}^{\infty} T_{n(m)}(m) = \bigcap_{m=1}^{\infty} [T_{n(m)}(m)]^C,$$

and

$$[T_{n(m)}(m)]^C = (\bigcup_{k=n(m)}^{\infty} \{t \in T \mid h(\Gamma_k(t), \Gamma(t) \geq \frac{1}{m}\})^C$$

$$= \bigcap_{k=n(m)}^{\infty} \{t \in T \mid h(\Gamma_k(t), \Gamma(t)) < \frac{1}{m}\}$$

where $A^C = T \setminus A$ stands the complement of A, it follows that

$$T_0 = \bigcap_{m=1}^{\infty} \bigcap_{k=n(m)}^{\infty} \{t \in T \mid h(\Gamma_k(t), \Gamma(t)) < \frac{1}{m}\}$$

$$\subset \bigcap_{k=N}^{\infty} \{t \in T \mid h(\Gamma_k(t), \Gamma(t)) < \lambda\}.$$

This inclusion shows that for any $t \in T_0$, the inequality $h(\Gamma_k(t), \Gamma(t)) < \lambda$ holds for all $k \geq N$. Since λ is arbitrary, $\Gamma_n(t) \xrightarrow{h} \Gamma(t)$ is uniformly on T_0. □

It is known that if a sequence of measurable multifunctions is convergent either in Hausdorff limit or in Kuratowski limit, the limit multifunction is measurable. The question arises that whether we can find a convergent measurable selections from a convergent measurable multifunctions. The following theorem will be a key to this question. The following result is similar to a theorem given in Salinati and Wets [10].

Theorem 3.5. Let $\{\Gamma_n\} \subset \mathfrak{M}(T, X)$ be a sequence satisfying τ–lim inf $\Gamma_n(t) = \Gamma(t) \in \mathfrak{M}(T, X)$ for each $t \in T$. If σ is a measurable selection for Γ, then there exists a sequence of measurable selections σ_n for Γ_n $(n = 1, 2, \cdots)$ such that

$\lim \sigma_n(t) = \sigma(t)$ for each $t \in T$.

Proof. Since $\sigma(t) \in \Gamma(t) = \tau$–lim inf $\Gamma_n(t)$, there are $\rho_n(t) \in \Gamma_n(t)$, $n = 1, 2, \cdots$, such that $\rho_n(t) \longrightarrow \sigma(t)$. It follows that

$$d(\sigma(t), \Gamma_n(t)) \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ for each } t \in T.$$

Define

$$\Lambda_n(t) = \{x \in \Gamma_n(t) \mid d(x, \sigma(t)) \leq d(\sigma(t), \Gamma_n(t)) + \tfrac{1}{n}\}, \quad n = 1, 2, \cdots.$$

Clearly, $\Lambda_n(t)$ is a nonempty closed subset of X. Since the function $d(x, \sigma(t))$ is continuous in x and measurable in t, it follows that $(t, x) \longrightarrow d(x, \sigma(t))$ is $\Sigma \otimes \mathfrak{B}(X)$ – measurable. Since $\Gamma_n \in \mathfrak{M}(T, X)$, the mapping $(t, x) \longrightarrow d(x, \Gamma_n(t))$ is a Caratheodory function so that $(t, x) \longrightarrow d(x, \sigma(t)) - d(x, \Gamma_n(t))$ is $\Sigma \otimes \mathfrak{B}(X)$ – measurable. It follows that

$$\operatorname{Grp} \Lambda_n = \{(t, x) \in T \times X \mid d(x, \sigma(t) \leq d(\sigma(t), \Gamma_n(t)) + \tfrac{1}{n}\} \cap \operatorname{Grp} \Gamma_n$$

is also $\Sigma \otimes \mathfrak{B}(X)$ – measurable. Hence $\Lambda_n \in \mathfrak{M}(T, X)$. By Proposition 3.1 (vi), $\Lambda_n(t) \subset \Gamma_n(t)$ has a measurable selection $\sigma_n(t)$ such that $d(\sigma(t), \sigma_n(t)) \longrightarrow 0$. □

From Theorem 3.5, we easily get the following theorem.

Theorem 3.6. Let $\{\Gamma_n\} \subset \mathfrak{M}(T, X)$ be a sequence such that for each $t \in T$, $\Gamma_n(t) \xrightarrow{k} \Gamma(t) \neq \emptyset$ or $\Gamma_n(t) \xrightarrow{h} \Gamma(t)$. If σ is a measurable selection for Γ, then there exists a sequence $\{\sigma_n\}$ of measurable selections for $\{\Gamma_n\}$ such that $\sigma_n(t) \longrightarrow \sigma(t)$ $(n \longrightarrow \infty)$ for each $t \in T$.

4. Convergence Theorems for L^p–selections

For $\Gamma \in \mathfrak{M}(T, X)$, we define

$$(4.1) \qquad \mathscr{S}^p_\Gamma = \{f \in L^p(T, X) \mid f(t) \in \Gamma(t) \text{ a.e.}\}, \quad 1 \leq p \leq +\infty$$

namely the family of L^p–selections for Γ. Here $L^p(T, X)$ stands for the Bochner integral space of measurable functions $f : T \longrightarrow X$ such that $\|f(\cdot)\|_X \in L^p(T, \mathbb{R}) = L^p(T)$.

Then $\mathscr{S}^p_\Gamma$ is a closed subset of $L^p(T, X)$ (See Hiai and Umegaki [5, Theorem 1.4]).

For $\Gamma \in \mathfrak{M}(T, X)$, $t \in T$ we define

(4.2) $\quad |\Gamma(t)| = \sup \{\|x\| \mid x \in \Gamma(t)\}.$

In this section we will study the relationship between the sequence $\{\Gamma_n\}$ in $\mathfrak{M}(T, X)$ and the sequence $\{\mathscr{S}^p_{\Gamma_n}\}$ for L^p–selections. Papageorgiou [9] proved that:

(i) if $\Gamma_n \xrightarrow{h} \Gamma$ with convex condition on Γ_n, then $\mathscr{S}^p_{\Gamma_n} \xrightarrow{h} \mathscr{S}^p_\Gamma$;

(ii) if $\Gamma_n \xrightarrow{k} \Gamma$, then the inclusion $\lim \mathscr{S}^p_{\Gamma_n} \subset \mathscr{S}^p_\Gamma$ holds.

(See [10] Theorem 4.2 for weak topology on $L^p(T, X)$ and Theorem 4.1 for $p = 1$). For (i), we shall delete the convex condition on Γ_n and establish the same result. For (ii) the equality instead of the inclusion for Kuratowski convergence is given by Aubin and Frankowska [2, Theorem 8.4.1]. Although the following result can be reduced form [2, Theorem 8.4.1], we will give a direct proof for Hausdorff convergence by no medium of using Kuratowski convergence.

Theorem 4.1. Let $\{\Gamma_n\}$ be a sequence of multifunctions in $\mathfrak{M}(T, X)$ such that

$$\Gamma(t) = \text{h–lim}\ \Gamma_n(t), \qquad \text{for } t \in T.$$

Suppose that there is a nonnegative function $\rho \in L^p(T)$, $1 \le p < +\infty$ such that

(4.2) $\quad |\Gamma_n(t)| \le \rho(t)$ for $t \in T$ and $n \in \mathbb{N}$.

Then

$$\mathscr{S}^p_\Gamma = \text{h–lim}\ \mathscr{S}^p_{\Gamma_n}$$

To prove this theorem we need the following lemma.

Lemma. Let Γ_1 and Γ_2 in $\mathfrak{M}(T, X)$ be such that

$$|\Gamma_j(t)| \le \rho(t) \qquad \text{for some } \rho \in L^p(T).$$

Then

$$(4.3) \qquad h(\mathscr{S}^p_{\Gamma_1}, \mathscr{S}^p_{\Gamma_2}) \le \{\int_T [h(\Gamma_1(t), \Gamma_2(t)]^p \, d\mu\}^{1/p}, \qquad 1 \le p < \infty.$$

Proof of Lemma. We claim that

$$(c) \qquad \sup_{f_1 \in \mathscr{S}^p_{\Gamma_1}} d(f_1, \mathscr{S}^p_{\Gamma_2}) \le \{\int_T [h(\Gamma_1(t), \Gamma_2(t)]^p \, d\mu\}^{1/p}$$

as well as

$$(d) \qquad \sup_{f_2 \in \mathscr{S}^p_{\Gamma_2}} d(f_2, \mathscr{S}^p_{\Gamma_1}) \le \{\int_T [h(\Gamma_1(t), \Gamma_2(t)]^p \, d\mu\}^{1/p} .$$

Let $f_1 \in \mathscr{S}^p_{\Gamma_1}$. Then for any positive function α on T there exists an $x \in \Gamma_2(t)$ such that

$$d(f_1(t), \Gamma_2(t)) = \inf_{y \in \Gamma_2(t)} \|f_1(t) - y\| \ge \|f_1(t) - x\| - \alpha(t).$$

Without loss of generality, we let $\alpha \in L^p(T)$. Then the multifunction

$$\Gamma(t) = \{x \in \Gamma_2(t) \mid \|f_1(t) - x\| \le d(f_1(t), \Gamma_2(t)) + \alpha(t)\}$$

has closed ranges in $P_f(X)$ and

$$\text{Grp } \Gamma = \text{Grp } \Gamma_2 \cap \{(t, x) \in T \times X \mid \|f_1(t) - x\| \le d(f_1(t), \Gamma_2(t)) + \alpha(t)\}$$

is $\Sigma \otimes \mathfrak{B}(X)$ – measurable. From Proposition 3.1 (vii), we see that $\Gamma \in \mathfrak{M}(T, X)$. So there is a measurable selection γ for Γ as well as for Γ_2 such that

$$\|f_1(t) - \gamma(t)\| \le d(f_1(t), \Gamma_2(t)) + \alpha(t)$$

$$\le h(\Gamma_1(t), \Gamma_2(t) + \alpha(t)$$

or

$$\|\gamma(t)\| \le \|f_1(t)\| + |\Gamma_1(t)| + |\Gamma_2(t)| + \alpha(t)$$

$$\leq \|f_1(t)\| + 2\rho(t) + \alpha(t) \in L^p(T).$$

Hence $\gamma \in \mathscr{S}^p_{\Gamma_2}$ and $\|f_1 - \gamma\|_p < \infty$. It follows from Heai and Umegaki [5, Theorem 2.2] that

$$\inf_{g \in \mathscr{S}^p_{\Gamma_2}} \|f_1 - g\|_p = \left(\int_T \inf_{x \in \Gamma_2(t)} \|f_1(t) - x\|^p \, d\mu\right)^{1/p}$$

$$= \left(\int_T [d(f_1(t), \Gamma_2(t))]^p \, d\mu\right)^{1/p}$$

$$\leq \left(\int_T h[\Gamma_1(t), \Gamma_2(t)]^p \, d\mu\right)^{1/p}.$$

So $d(f_1, \mathscr{S}^p_{\Gamma_2}) \leq \left(\int_T h[\Gamma_1(t), \Gamma_2(t)]^p \, d\mu\right)^{1/p}$ for all $f_1 \in \mathscr{S}^p_{\Gamma_1}$.

By interchanging the roles of $\mathscr{S}^p_{\Gamma_2}$ and $\mathscr{S}^p_{\Gamma_1}$, we then get (c) and (d). Hence (4.3) holds.□

Proof of Theorem 4.1. From the above Lemma we have

$$h(\mathscr{S}^p_{\Gamma}, \mathscr{S}^p_{\Gamma_n}) = \max\left(\sup_{f \in \mathscr{S}^p_{\Gamma}} d(f, \mathscr{S}^p_{\Gamma_n}), \sup_{f_n \in \mathscr{S}^p_{\Gamma_n}} d(f_n, \mathscr{S}^p_{\Gamma})\right)$$

$$\leq \left\{\int_T h[\Gamma(t), \Gamma_n(t)]^p \, d\mu\right\}^{1/p}$$

As the condition $|\Gamma_n(t)| \leq \rho(t)$ holds for $\rho \in L^p(T)$, $n = 1, 2, \cdots$, we have

$$|\Gamma(t)| = h(\Gamma(t), \{0\}) \leq h(\Gamma(t), \Gamma_n(t)) + |\Gamma_n(t)|$$

$$\leq h(\Gamma(t), \Gamma_n(t)) + \rho(t)$$

$$\longrightarrow \rho(t) \quad \text{as} \quad n \longrightarrow \infty.$$

Hence

$$h(\Gamma(t), \Gamma_n(t)) \leq |\Gamma(t)| + |\Gamma_n(t)| \leq 2\,\rho(t) \; (\in L^p(T)).$$

By Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_T h(\Gamma(t), \Gamma_n(t)^p \, d\mu = \int_T \lim h(\Gamma(t), \Gamma_n(t))^p \, d\mu = 0.$$

Therefore $h(\mathscr{S}^p_{\Gamma_n}, \mathscr{S}^p_{\Gamma}) \longrightarrow 0$ and the theorem is proved. □

Remark 4.1. Theorem 4.1 is not true for $p = +\infty$. Even in Kuratowski limit, it is also not true. For example,

(i) If $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f_n(t) = \begin{cases} 1, & t > n \\ 0, & t \le n, \end{cases}$

then $f_n(t) \longrightarrow 0 \ (n \longrightarrow \infty)$ for all $t \in \mathbb{R}$. But $\|f_n - f\|_\infty = 1 \longrightarrow 0$. This shows that $\mathscr{S}^\infty_\Gamma \ne \text{k–lim } \mathscr{S}^\infty_{\Gamma_n}$.

(ii) The limit $h(\Gamma(t), \Gamma_n(t)) \longrightarrow 0$, for any $t \in T$, can not imply that $\operatorname*{ess\,sup}_{t \in T} h(\Gamma(t), \Gamma_n(t)) \longrightarrow 0$.

5. A Simple Application of Theorem 3.2.

Consider a one–stage decision problem which we figure as the following four objects:

(i) (T, Σ, μ) represents the **state space** which is a complete σ–finite measure space.

(ii) (X, d) stands the **action space** which is a complete separable metric space, in particular X is a separable Banach space.

(iii) $\Gamma : T \longrightarrow P_f(X)$ is specified as the set of **admissible actions** $\Gamma(t)$ if the system is at the state t.

(iv) $f : T \times X \longrightarrow \mathbb{R}$ is the **payoff function.**

Then an optimization problem will be formulated as the form:

(P) $\quad \operatorname*{Minimize}_{x \in \Gamma(t)} f(t, x).$

A **plan / strategy** σ in problem (P) is a measurable selection for Γ. A plan σ is said to be **optimal** if it is a minimizer to the payoff function at $t \in T$. That is,

$$(5.1) \qquad f(t, \sigma(t)) = \inf_{x \in \Gamma(t)} f(t, x) \qquad \text{for } t \in T.$$

From Theorem 3.2, the following theorem is immediate.

Theorem 5.1. For the problem (P), suppose that

(A1) $f(t, x)$ is measurable in t and lower semicontinuous in x.

(A2) $\Gamma \in \mathfrak{M}(T, X)$ and $\Gamma(t)$ is compact for each $t \in T$.

Then problem (P) has a solution (= optimal plan).

Note that the compactness of $\Gamma(t)$ in (A2) was used only to ensure that the set

$$(5.2) \qquad \Lambda(t) = \{x \in \Gamma(t) : f(t, x) = \inf_{y \in \Gamma(t)} f(t, y)\} \neq \emptyset.$$

If we relax the condition that $f(t, \cdot)$ attains its infimum on $\Gamma(t)$, then we still have the optimal solution for (P). That is,

Corollary 5.2. If f satisfies (A1) in Theorem 5.1 and $\Gamma \in \mathfrak{M}(T, X)$ such that $f(t, \cdot)$ attains its infimum on $\Gamma(t)$ for $t \in T$, then there is an optimal plan to problem (P).

Remark 5.1. Suppose that for each $t \in T$, $\Gamma(t)$ is a closed convex subset in a reflexive Banach space X, and $f(t, \cdot)$ is a convex ℓ.s.c. function on X. If $\Gamma(t)$ is bounded or $\lim_{\|x\| \to \infty} f(t, x) = +\infty$, then the set $\Lambda(t) \neq \emptyset$ in (5.2). So problem (P) still has an optimal plan.

If in (5.2), $\Lambda(t) = \emptyset$, we introduce the **ϵ–optimal plan** for problem (P) as follows: For any measurable function $\epsilon : T \to (0, +\infty)$, a plan σ is said to be **ϵ–optimal** if

$$(5.3) \qquad f(t, \sigma(t)) \leq \inf_{x \in \Gamma(t)} f(t, x) + \epsilon(t).$$

Then we have the following theorem.

Theorem 5.3. Suppose that the payoff function f satisfies the condition (A1) in Theorem 5.1 and the range set $\{f(t, x) \, ; \, t \in T, x \in X\}$ is bounded below. Then for $\Gamma \in \mathfrak{M}(T, X)$, there exists an ϵ–optimal plan to problem (P).

Proof. Let $\epsilon : T \longrightarrow (0, +\infty)$ be any measurable function, and let

$$\Lambda(t) = \{x \in \Gamma(t) \mid f(t, x) \leq [\inf_{y \in \Gamma(t)} f(t, y)] + \epsilon(t)\}.$$

By the assumption on f, the set $\Lambda(t)$ is nonempty and closed for each $t \in T$. Since $\Gamma \in \mathfrak{M}(T, X)$, there is a sequence $\{\sigma_n\}$ of measurable selections for Γ such that

$$\Gamma(t) = \mathrm{cl}\,\{\sigma_n(t) \mid n \in \mathbb{N}\} \qquad t \in T \ \text{(Proposition 3.1 (vi))}.$$

It follows that the graph

$$\mathrm{Grp}\,\Lambda = \{(t, x) \in T \times X \mid f(t, x) \leq \inf_{n \in \mathbb{N}} f(t, \sigma_n(t)) + \epsilon(t)\} \cap \mathrm{Grp}\,\Gamma$$

is $\Sigma \otimes \mathfrak{B}(X)$ measurable. Therefore Λ has a measurable selection σ which is an ϵ–optimal plan. □

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