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ON THE ALMOST EVERYWHERE CONVERGENCE OF BOCHNER-RIESZ MEANS OF MULTIPLE FOURIER INTEGRALS FOR RADIAL FUNCTIONS

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ABSTRACT. Let $n\geqq 2$ and $(S_*^{\delta} f)(x)=\sup_{R>0}|(S_R^{\delta} f)(x)|$, where $S_R^{\delta} f$ is the Bochner-Riesz mean of order δ of the Fourier integral for f on R^n. We show that the operator S_*^{δ} is bounded from the Lorentz space $L^{p,1}(R^n)$ into $L^{p,\infty}(R^n)$ on the critical line $\delta=n(1/p-1/2)-1/2$ for $2n/(n+2)\leqq p\leqq 2n/(n+1)$ besides $p>1$ when acting on radial functions.

§1. Introduction.

Let R^n be the $n(\geqq 2)$-dimensional Euclidean space and for any $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$ in R^n, we denote $(x,y)=x_1y_1+\cdots+x_ny_n$ and $|x|=(x,x)^{1/2}$.

For the Fourier integral of a function $f\in L^p(R^n)$ $(1\leqq p\leqq 2)$, its Bochner-Riesz mean of order $\delta\geqq 0$ is defined by

$$(1)\qquad (S_R^{\delta} f)(x)=(\sqrt{2\pi})^{-n}\int_{|y|<R}\left(1-\frac{|y|^2}{R^2}\right)^{\delta}\hat{f}(y)\,e^{i(x,y)}dy,$$

where $\hat{f}(y)$ is the Fourier transform of f, i.e.

$$\hat{f}(y)=(\sqrt{2\pi})^{-n}\int_{R^n} f(x)\, e^{-i(x,y)}dx.$$

It has been shown that for $1\leqq p\leqq 2n/(n+1)$ there exists a function $f\in L^p(R^n)$ such that $(S_R^\delta f)(x)$ diverges almost everywhere as $R\to\infty$, where $\delta=n(1/p-1/2)-1/2$ ([Ko]). On the other hand, acting on radial functions, it is known that for $1\leqq p\leqq 2n/(n+1)$ and $0\leqq\delta\leqq n(1/p-1/2)-1/2$, S_R^δ is unbounded on radial $L^p(R^n)$ ([W]). But for $1\leqq p<2n/(n+1)$ and $\delta=n(1/p-1/2)-1/2$, S_R^δ is weakly bounded on radial $L^p(R^n)$ ([CM]), and for $p=2n/(n+1)$ and $\delta=0$, S_R^δ is not weakly bounded ([KT]) but restricted weakly bounded on radial $L^p(R^n)$ ([C]).

We put

(2) $\quad (S_*^\delta f)(x)=\sup_{R>0}|(S_R^\delta f)(x)|.$

Then, it is known that for $2n/(n+1)<p\leqq 2$ and $\delta=0$, S_*^δ is bounded on radial $L^p(R^n)$ ([Ka], [P]), and for $p=2n/(n+1)$ and $\delta=0$, S_*^δ is weakly bounded on radial Lorentz space $L^{p,1}(R^n)$ ([RS]).

In this paper we shall prove the following theorem.

THEOREM. For $p>1$ and $2n/(n+2)\leqq p\leqq 2n/(n+1)$ and for the critical line $\delta=n(1/p-1/2)-1/2$, S_*^δ is weakly bounded on radial Lorentz space $L^{p,1}(R^n)$. That is, for any $\lambda>0$ and for any radial $f\in L^{p,1}(R^n)$,

$$|\{x\in R^n;\ (S_*^\delta f)(x)>\lambda\}|\leqq C_p\left(\frac{1}{\lambda}\|f\|_{p,1}\right)^p$$

is valid, where C_p is some constant independent of λ and f. Consequently, for any radial function $f\in L^{p,1}(R^n)$, $(S_R^\delta f)(x)$ converges to $f(x)$ almost everywhere as $R\to\infty$.

Here we remember the Lorentz space $L^{p,q}(R^n)$ briefly. For any measurable function f on R^n, we denote the distribution function and the non-increasing rearrangement by $\lambda_f(y)$ and $f^*(t)$ respectively, i.e.

$$\lambda_f(y)=|\{x\in R^n;\ |f(x)|>y\}| \qquad \text{for } y>0,$$

$$f^*(t)=\inf\{y>0;\ \lambda_f(y)\leqq t\} \qquad \text{for } t>0.$$

When $1\leqq p<\infty$, we put

$$\|f\|_{p,q}=\{\int_0^\infty (f^*(t)t^{1/p})^q\ t^{-1}dt\}^{1/q}$$

for $1\leqq q<\infty$ and

$$\|f\|_{p,\infty}=\sup_{t>0}\{f^*(t)t^{1/p}\}=\sup_{y>0}\{y(\lambda_f(y))^{1/p}\}.$$

We denote the set of all functions on R^n such as $\|f\|_{p,q}<\infty$ by $L^{p,q}(R^n)$. Then, $L^{p,r}\subset L^{p,s}$ for $1\leqq r\leqq s\leqq\infty$ and $L^{p,p}(R^n)=L^p(R^n)$. Also, if $1<p<\infty$ and $1<q<\infty$, then $(L^{p,q})^*=L^{p',q'}$, where p' and q' are conjugate exponents of p and q respectively, and if $1\leqq p<\infty$, then $(L^{p,1})^*=L^{p',\infty}$.

If f is a radial function on R^n, we write $f(x)=f_0(|x|)$. Then we can express $\hat{f}(\xi)$ as

$$\hat{f}(\xi)=|\xi|^{-(n-2)/2}\int_0^\infty f_0(r)J_{(n-2)/2}(|\xi|r)\ r^{n/2}\ dr,$$

where $J_\mu(t)$ is the Bessel function of order $\mu\geqq 0$. So, setting

$$(3)\qquad K_R(r,s)=\sqrt{rs}\int_0^R (1-\frac{t^2}{R^2})^\delta J_{(n-2)/2}(rt)J_{(n-2)/2}(st)\ t\ dt$$

for $r>0$, $s>0$, we rewrite (1) as

(4) $$(S_R^{\delta} f)(x)=|x|^{-(n-1)/2} \int_0^{\infty} f_0(r) r^{(n-1)/2} K_R(r,|x|)\, dr.$$

We note that

(5) $$K_R(r,s)=R\, K_1(Rr,Rs).$$

§2. Lemmas.

The following lemma is wellknown.

Lemma 1([SW]). If $f\in L^p(R^n)$ $(1\leqq p\leqq 2)$ and $\alpha\geqq 0$, then for any $\varepsilon>0$ the following formulas are valid ;

(i) $$(S_R^{\alpha+\varepsilon} f)(x)=\frac{2\Gamma(\alpha+1+\varepsilon)}{\Gamma(\alpha+1)\Gamma(\varepsilon)} R^{-2(\alpha+\varepsilon)} \int_0^R (R^2-t^2)^{\varepsilon-1} t^{2\alpha+1} (S_t^{\alpha} f)(x)\, dt,$$

(ii) $$\frac{2\Gamma(\alpha+1+\varepsilon)}{\Gamma(\alpha+1)\Gamma(\varepsilon)} R^{-2(\alpha+\varepsilon)} \int_0^R (R^2-t^2)^{\varepsilon-1} t^{2\alpha+1}\, dt=1.$$

Therefore we get

(iii) $$(S_*^{\alpha+\varepsilon} f)(x)\leqq (S_*^{\alpha} f)(x).$$

We use the fact (iii) with $\alpha=0$ and $\varepsilon=\delta$ later on.

Let M, H, H^* be the Hardy-Littlewood maximal operator, the Hilbert transform, the maximal Hilbert transform respectively, and C^* be the Carleson's operator

$$(C^*f)(x)= \sup_{R>0} \left|\int_{-\infty}^{\infty} \frac{e^{-iRt}}{s-t} f(t)\, dt\right|.$$

The following fact is known ([P]).

Lemma 2. For any radial function $f(x)=f_0(|x|)$, the inequality

$$(S_*^0 f)(x)\leqq C\, |x|^{-(n-1)/2} (M+|H|+H^*+C^*)(|f_0(r)|r^{(n-1)/2})(|x|)$$

is valid. Furthermore, denoting $M+|H|+H^*+C^*$ by T, we have

$$\int_{-\infty}^{\infty}|(Tg)(t)|^p\, w(t)\, dt \leqq C_p \int_{-\infty}^{\infty}|g(t)|^p\, w(t)\, dt$$

for $w \in A_p(R)$ $(1<p<\infty)$.

It is known that $|t|^{\gamma} \in A_p(R)$ for $-1<\gamma<p-1$.

For (3) with R=1, i.e. for $K_1(r,s)$, the following estimates are valid.

Lemma 3([CM]). For $\delta>-1$,

(i) $|K_1(r,s)| \leqq C \dfrac{\sqrt{r}}{\sqrt{1+r}} \dfrac{\sqrt{s}}{\sqrt{1+s}} |r-s|^{-\delta-1}$,

(ii) $|K_1(r,s)| \leqq C \dfrac{\sqrt{r}}{\sqrt{1+r}} \dfrac{\sqrt{s}}{\sqrt{1+s}}$,

(iii) $|K_1(r,s)| \leqq C\, (rs)^{(n-1)/2}$.

By the lemma and (5), we get

$$(6)\quad |K_R(r,|x|)| \leqq C\, R \frac{\sqrt{Rr}}{\sqrt{1+Rr}} \frac{\sqrt{R|x|}}{\sqrt{1+R|x|}} \left|Rr-R|x|\right|^{-\delta-1},$$

$$(7)\quad |K_R(r,|x|)| \leqq C\, R \frac{\sqrt{Rr}}{\sqrt{1+Rr}} \frac{\sqrt{R|x|}}{\sqrt{1+R|x|}},$$

$$(8)\quad |K_R(r,|x|)| \leqq C\, R^n\, r^{(n-1)/2}\, |x|^{(n-1)/2}.$$

The following lemma is shown easily.

Lemma 4. For $1<p\leqq\infty$, we have for radial function $f(x)=f_0(|x|)$,

$$\int_0^{\infty} |f_0(r)|\, r^{n/p-1}\, dr \leqq C_p\, \|f\|_{p,1}.$$

Proof. We put $\phi(x)=|x|^{-\mu}$ $(x\in R^n)$ for $\mu>0$. Since

$$\lambda_{\phi}(y)=|\{x\in R^n;\ |\phi(x)|>y\}|=|\{x\in R^n;\ |x|<y^{-1/\mu}\}| = \Omega_n\, y^{-n/\mu},$$

we have

$$\|\phi\|_{p',\infty}= \Omega_n^{1/p'} \sup_{y>0} y^{1-n(1-1/p)/\mu}$$

and so $\phi\in L^{p',\infty}(R^n)$ for $\mu=n(1-1/p)$. Therefore for such the μ we get

$$\int_0^{\infty} |f_0(r)|\, r^{n-1}\, r^{-\mu} dr=C\int_{R^n}|f(x)|\phi(x)dx\leq C_p\|f\|_{p,1}.$$

Since $n-1-\mu=n/p-1$, we get the conclusion.

§3. Proof of the theorem.

We put $I_k=[2^k,\ 2^{k+1})$ and $I_k{}^*=[2^{k-1},\ 2^{k+1})$, $k=0,\ \pm1,\ \pm2,\ldots$ For $1\leq p<2n/(n+1)$ we put $\delta=n(1/p-1/2)-1/2$. Since $(0,\infty)=\bigcup_{k=-\infty}^{\infty} I_k$, we have

$$\{x\in R^n;\ (S_*^{\delta} f)(x)>\lambda\}=\bigcup_{k=-\infty}^{\infty}\{x\in R^n;\ |x|\in I_k,\ (S_*^{\delta} f)(x)>\lambda\}$$

for $\lambda>0$. For each k we decompose f into

$$f=f\chi_{I_k{}^*}+ f\chi_{(I_k{}^*)^c}= f_k{}^{(1)} + f_k{}^{(2)}$$

say. Then we have

$$(9)\quad |\{x\in R^n;\ (S_*^{\delta} f)(x)>\lambda\}|$$

$$\leq\sum_{k=-\infty}^{\infty}|\{x\in R^n;\ |x|\in I_k,\ (S_*^{\delta} f_k{}^{(1)})(x)>\lambda/2\}|$$

$$+ \sum_{k=-\infty}^{\infty}|\{x\in R^n;\ |x|\in I_k,\ (S_*^{\delta} f_k{}^{(2)})(x)>\lambda/2\}|.$$

First we estimate $S_*^{\delta} f_k{}^{(1)}$. Simplifying notations, we denote $f_k{}^{(i)}(x)=f_k{}^{(i)}(|x|)$ $(i=1,2)$. By Lemma 1 and Lemma 2, we have

$$(S_*^{\delta} f_k^{(1)})(x) \leqq (S_*^{0} f_k^{(1)})(x) \leqq C|x|^{-(n-1)/2}\ T(|f_k^{(1)}(r)|r^{(n-1)/2})(|x|)$$

and so

$$\sum_{k=-\infty}^{\infty} |\{x \in R^n;\ |x| \in I_k,\ (S_*^{\delta} f_k^{(1)})(x) > \lambda/2\}|$$

$$\leqq \sum_{k=-\infty}^{\infty} |\{x \in R^n;\ |x| \in I_k,\ T(|f_k^{(1)}(r)|r^{(n-1)/2})(|x|) > C_1 |x|^{(n-1)/2} \lambda\}|$$

$$\leqq \sum_{k=-\infty}^{\infty} |\{x \in R^n;\ |x| \in I_k,\ T(|f_k^{(1)}(r)|r^{(n-1)/2})(|x|) > C_2\ 2^{k(n-1)/2} \lambda\}|$$

$$\leqq C_3 \sum_{k=-\infty}^{\infty} (2^{k(n-1)/2} \lambda)^{-p} \int_{I_k} \{T(|f_k^{(1)}(r)|r^{(n-1)/2})(s)\}^p\ s^{n-1}\ ds$$

$$\leqq C_4 \sum_{k=-\infty}^{\infty} (2^{k(n-1)/2} \lambda)^{-p} 2^{k(n-1)} \int_{I_k} \{T(|f_k^{(1)}(r)|r^{(n-1)/2})(s)\}^p\ ds$$

$$\leqq C_4 \lambda^{-p} \sum_{k=-\infty}^{\infty} 2^{k(n-1)(1-p/2)} \int_0^{\infty} \{T(|f_k^{(1)}(r)|r^{(n-1)/2})(s)\}^p\ ds.$$

If $p>1$, we apply Lemma 2 with $w(t)=1=|t|^0 \in A_p(R)$. Then the last term is

bounded by

$$\leqq C_5 \lambda^{-p} \sum_{k=-\infty}^{\infty} 2^{k(n-1)(1-p/2)} \int_0^{\infty} (|f_k^{(1)}(r)|r^{(n-1)/2})^p\ dr$$

$$= C_5 \lambda^{-p} \sum_{k=-\infty}^{\infty} 2^{k(n-1)(1-p/2)} \int_{I_k^*} |f_0(r)|^p\ r^{(n-1)p/2}\ dr$$

$$\leqq C_6 \lambda^{-p} \sum_{k=-\infty}^{\infty} 2^{k(n-1)(1-p/2)}\ 2^{k(n-1)(p/2-1)} \int_{I_k^*} |f_0(r)|^p\ r^{n-1}\ dr$$

$$= C_6 \lambda^{-p} \sum_{k=-\infty}^{\infty} \int_{I_k^*} |f_0(r)|^p\ r^{n-1}\ dr \leqq C_7 \lambda^{-p} \int_0^{\infty} |f_0(r)|^p\ r^{n-1}\ dr$$

$$(10) \quad = C_8 \lambda^{-p} \int_{R^n} |f(x)|^p\ dx = C_8 (\|f\|_p/\lambda)^p \leqq C_8 (\|f\|_{p,1}/\lambda)^p.$$

Thus we get the desired estimate for $f_k^{(1)}$.

Next we estimate $(S_*^{\delta} f_k^{(2)})(x)$ on the set $\{x \in R^n;\ |x| \in I_k\}$. We want to prove

the estimate

(11) $\quad (S_*^{\delta} f_k^{(2)})(x) \leqq C_p |x|^{-n/p} \int_0^{\infty} |f_0(r)|\ r^{n/p-1}\, dr$

for (p, δ) in the theorem.

If we can get (11), then by Lemma 4 we will have

$$(S_*^{\delta} f_k^{(2)})(x) \leqq C_p' |x|^{-n/p} \| f \|_{p,1}$$

and so

$$\sum_{k=-\infty}^{\infty} |\{x \in R^n;\ |x| \in I_k,\ (S_*^{\delta} f_k^{(2)})(x) > \lambda/2\}|$$

$$\leqq \sum_{k=-\infty}^{\infty} |\{x \in R^n;\ |x| \in I_k,\ |x| < (2C_p' \| f \|_{p,1}/\lambda)^{p/n}\}|$$

$$= |\{x \in R^n;\ |x| < (2C_p' \| f \|_{p,1}/\lambda)^{p/n}\}|$$

$$= \Omega_n (2C_p' \| f \|_{p,1}/\lambda)^p = C_p'' (\| f \|_{p,1}/\lambda)^p.$$

Thus we will have the desired estimate for $f_k^{(2)}$ and so, by (9) and (10), the theorem will be obtained.

For the sake of the proof of (11), we fix any $x \in R^n$ with $|x| \in I_k$. So $|x| \sim 2^k$. Since supp $f_k^{(2)}(r) \subset (2^{k-1}, 2^{k+2})^c$, by (4) we decompose $(S_*^{\delta} f_k^{(2)})(x)$ as

(12) $\quad (S_*^{\delta} f_k^{(2)})(x) = |x|^{-(n-1)/2} \int_0^{\infty} f_k^{(2)}(r)\ r^{(n-1)/2} K_R(r, |x|)\, dr$

$$= \int_0^{2^{k-1}} + \int_{2^{k+2}}^{\infty} = I_R^{(1)}(x) + I_R^{(2)}(x)$$

say. We shall estimate $I_R^{(j)}(x), j=1,2,$ in the case of $0 < R \leqq 1/|x|$ and $R \geqq 1/|x|$ separately.

(I) Estimates for $I_R^{(2)}(x)$.

In the case of $0<R\leqq 1/|x|$, by (7) we have

$$|K_R(r,|x|)|\leqq C\ R^{3/2}|x|^{1/2}$$

and so

$$|I_R^{(2)}(x)|\leqq C\ |x|^{-(n-2)/2}\ R^{3/2}\int_{2^{k+2}}^{\infty}|f_0(r)|\ r^{(n-1)/2}\,dr.$$

Since $(n-1)/2-(n/p-1)=-\delta\leqq 0$, we have

$$|I_R^{(2)}(x)|\leqq C|x|^{-(n-2)/2}R^{3/2}(2^{k+2})^{(n-1)/2-(n/p-1)}\int_{2^{k+2}}^{\infty}|f_0(r)|r^{n/p-1}dr$$

$$\leqq C_1R^{3/2}|x|^{-n/p+3/2}\int_0^{\infty}|f_0(r)|r^{n/p-1}dr$$

$$\leqq C_1|x|^{-n/p}\int_0^{\infty}|f_0(r)|r^{n/p-1}dr.$$

In the case of $R\geqq 1/|x|$, noting that $r-|x|\geqq r/2$ for $r\geqq 2^{k+2}$, we have by (6)

$$|K_R(r,|x|)|\leqq C\ R^{-\delta}(r-|x|)^{-(\delta+1)}\leqq C_1\ R^{-\delta}\ r^{-(\delta+1)},$$

and so we get

$$|I_R^{(2)}(x)|\leqq C\ |x|^{-(n-1)/2}\ R^{-\delta}\int_{2^{k+2}}^{\infty}|f_0(r)|\ r^{(n-1)/2-(\delta+1)}\ dr.$$

Since $(n-1)/2-(\delta+1)-(n/p-1)=-n(1/p-1/2)-1/2-\delta<0$, we have

$$|I_R^{(2)}(x)|$$

$$\leqq C_1|x|^{-(n-1)/2}\ R^{-\delta}(\ 2^{k+2}\ )^{-n(1/p-1/2)-1/2-\delta}\int_{2^{k+2}}^{\infty}|f_0(r)|r^{n/p-1}\ dr$$

$$\leqq C_2\ R^{-\delta}\ |x|^{-n/p-\delta}\int_0^{\infty}|f_0(r)|r^{n/p-1}\ dr$$

$$\leqq C_2\ |x|^{-n/p}\int_0^{\infty}|f_0(r)|r^{n/p-1}\ dr.$$

Therefore for $1\leqq p\leqq 2n/(n+1)$ we get the estimate

(13) $$\sup_{R>0}|I_R^{(2)}(x)|\leqq C_p|x|^{-n/p}\int_0^\infty |f_0(r)|r^{n/p-1}dr.$$

(II) Estimates for $I_R^{(1)}(x)$.

In the case of $0<R\leqq 1/|x|$, noting that $Rr\leqq 1/2$ for $0\leqq r\leqq 2^{k-1}$, we have by (8)

$$|I_R^{(1)}(x)|\leqq C|x|^{-(n-1)/2}\int_0^{2^{k-1}}|f_0(r)|r^{(n-1)/2}R^n r^{(n-1)/2}|x|^{(n-1)/2}dr$$
$$=CR^n\int_0^{2^{k-1}}|f_0(r)|r^{n-1}dr.$$

Since $(n-1)-(n/p-1)=n(1-1/p)\geqq 0$, we have

$$|I_R^{(1)}(x)|\leqq CR^n(2^{k-1})^{n(1-1/p)}\int_0^{2^{k-1}}|f_0(r)|r^{n/p-1}dr$$
$$\leqq C_1R^n|x|^{n(1-1/p)}\int_0^\infty|f_0(r)|r^{n/p-1}dr$$
$$\leqq C_1|x|^{-n/p}\int_0^\infty|f_0(r)|r^{n/p-1}dr.$$

In the case of $R\geqq 1/|x|$, we decompose $I_R^{(1)}(x)$ as

$$|I_R^{(1)}(x)|\leqq|x|^{-(n-1)/2}\int_0^{2^{k-1}}|f_0(r)|r^{(n-1)/2}|K_R(r,\ |x|)|dr$$
$$\leqq|x|^{-(n-1)/2}\int_0^{|x|/2}|f_0(r)|r^{(n-1)/2}|K_R(r,\ |x|)|dr$$
$$=\int_0^{1/(2R)}+\int_{1/(2R)}^{|x|/2}=J_R^{(1)}(x)+J_R^{(2)}(x)$$

say.

For $J_R^{(2)}(x)$, noting that $Rr\geqq 1/2$ and $|x|-r\geqq|x|/2$ for $1/(2R)\leqq r\leqq|x|/2$, we have by (6)

$$|K_R(r,\ |x|)|\leqq CR^{-\delta}(|x|-r)^{-(\delta+1)}\leqq C_1R^{-\delta}|x|^{-(\delta+1)}$$

and so

$$J_R^{(2)}(x) \leqq C_1 |x|^{-(n-1)/2-(\delta+1)} R^{-\delta} \int_{1/(2R)}^{|x|/2} |f_0(r)|\, r^{(n-1)/2}\, dr.$$

Since $(n-1)/2-(n/p-1)=-\delta \leqq 0$, we have

$$J_R^{(2)}(x) \leqq C_1 |x|^{-(n+1)/2-\delta} R^{-\delta} (1/(2R))^{-\delta} \int_{1/(2R)}^{|x|/2} |f_0(r)|\, r^{n/p-1}\, dr$$

$$\leqq C_2 |x|^{-(n+1)/2-\delta} \int_0^{\infty} |f_0(r)|\, r^{n/p-1}\, dr$$

$$= C_2 |x|^{-n/p} \int_0^{\infty} |f_0(r)|\, r^{n/p-1}\, dr.$$

Lastly, we estimate $J_R^{(1)}(x)$. Noting that $Rr \leqq 1/2$ and $|x|-r \geqq |x|/2$ for $0 \leqq r \leqq 1/(2R)$, we have by (6)

$$|K_R(r, |x|)| \leqq C R^{3/2-(\delta+1)} r^{1/2} (|x|-r)^{-(\delta+1)}$$

$$\leqq C_1 R^{1/2-\delta} |x|^{-(\delta+1)} r^{1/2}$$

and so we get

$$J_R^{(1)}(x) \leqq C_1 |x|^{-(n-1)/2-(\delta+1)} R^{1/2-\delta} \int_0^{1/(2R)} |f_0(r)|\, r^{n/2}\, dr.$$

Since $n/2-(n/p-1) \geqq 0$ is valid only for $p \geqq 2n/(n+2)$, for $p \geqq 2n/(n+2)$ we have

$$J_R^{(1)}(x)$$

$$\leqq C_1 |x|^{-(n+1)/2-\delta} R^{1/2-\delta} (1/(2R))^{n/2-(n/p-1)} \int_0^{1/(2R)} |f_0(r)|\, r^{n/p-1}\, dr$$

$$\leqq C_2 |x|^{-(n+1)/2-\delta} \int_0^{\infty} |f_0(r)|\, r^{n/p-1}\, dr$$

$$= C_2 |x|^{-n/p} \int_0^{\infty} |f_0(r)|\, r^{n/p-1}\, dr.$$

Thus for $2n/(n+2) \leqq p \leqq 2n/(n+1)$ we get

$$(14) \qquad \sup_{R>0} |I_R{}^{(1)}(x)| \leqq C_p |x|^{-n/p} \int_0^\infty |f_0(r)| \, r^{n/p-1} \, dr.$$

By (12), (13) and (14) we get (11) and our proof is complete.

Remark. I don't know whether the statement of the theorem is true when $1 \leqq p < 2n/(n+2)$ for general n.

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