Nihonkai Math. J.
Vol.2(1991),169-175

ON A CLASS OF UNIVALENT FUNCTIONS RELATED WITH RUSCHEWEYH DERIVATIVE

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ABSTRACT

We introduce a class $K^*(\alpha,n)$ using the nth Ruscheweyh derivative $D^n f$ and investigate some of its important properties. We show that $K^*(\alpha,n)$ is a subclass of univalent functions, and we note that this class generalizes several known subclasses of univalent functions.

Key Words and Phrases: Univalent, Close-to-convex, Ruscheweyh derivative, Convolution.

1980 AMS (MOS) Subject Classification. 30C45, 30A32.

1. INTRODUCTION

Let A denote the class of functions $f: f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the unit disc $E = \{z: |z|<1\}$. By S, K, S^* and C, denote the subclasses of A which are univalent, close-to-convex, starlike and convex in E respectively. Let P be the class of functions p, analytic in E with $p(0)=1$ and satisfying $\mathrm{Re}\, p(z)>0, z\in E$.

The Hadamard product or convolution of two functions f, $g\in A$ is denoted by $f*q$. Let

$$D^n f = \frac{z}{(1-z)^{n+1}} * f, \quad n \in N_0 = \{0,1,2,3,\ldots\}$$

which implies that

$$D^n f = \{z(z^{n-1}f)^{(n)}\} / n!, \quad n \in N_0$$

$D^n f$ is called the nth Ruscheweyh derivative. Using this concept, Ahuja [1] has defined the class R_n. A function $f\in A$ is said to be in the class R_n if, and only if,

$$\frac{z(D^n f(z))'}{D^n f(z)} \in P$$

for $z \in E$. It is clear that $R_0 \equiv S^*$ and $R_1 \equiv C$. It is known [1] that $R_{n+1} \subset R_n$ for each $n \in N_0$ which implies that $f \in R_n$ is starlike in E. Also $f \in R_n$ implies that $D^n f \in S^*$.

We now define the following.

DEFINITION 1.1

Let $f \in A$. Then $f \in K_n$, $n \in N_0$ if, and only if, there exists a function $g \in R_n$ such that, for $z \in E$

$$\frac{z(D^n f(z))'}{D^n g(z)} \in P.$$

We note that $K_0 \equiv K$ and $K_1 \equiv C^*$, the class of quasi-convex univalent functions, see [4].

The class K_n has been studied, in some details, in [6]. It has been shown that $K_{n+1} \subset K_n$ for each $n \in N_0$ and hence $f \in K_n$ is a close-to-convex univalent function. In fact $f \in K_n$ if, and only if, $D^n f \in K$.

DEFINITION 1.2

Let $\alpha > 0$, $n \in N_0$ and $f \in A$. Then $f \in K^*(\alpha,n)$ if, and only if, there exists a function $g \in R_n$ such that, for $z \in E$,

$$\left\{(1-\alpha)\frac{z(D^n f(z))'}{D^n g(z)} + \frac{\alpha(z(D^n f(z))')'}{(D^n g)'}\right\} \in P. \qquad (1.1)$$

We note that

(i) $K^*(0,0) \equiv K$

(ii) $K^*(1,0) \equiv K^*$, a subclass of close-to-convex functions defined and studied in [5], and $K^*(0,n) \equiv K_n$, $n \in N_0$.

Also the class $K^*(\alpha,0)$ has been discussed in some details in [6].

2. MAIN RESULTS

THEOREM 2.1

Let $f \in K^*(\alpha,n)$, $n \in N_0$, $\alpha > 0$. Then f is close-to-convex and hence univalent.

PROOF

Since $f \in K^*(\alpha,n)$, there exists a function $g \in R_n$ such that, for $z \in E$, (1.1) holds. Since $D^n g \in S^*$, we obtain the result immediately by a lemma due to Chichra [2, p.38, Lemma 1]. In fact, we have only to take $N(z) = z(D^n f(z))'$ and $D(z) = D^n g(z)$ in the lemma. This completes the proof.

THEOREM 2.2

For $0 < \beta < \alpha$, $K^*(\alpha,n) \subset K^*(\beta,n)$.

PROOF

For $\beta = 0$, the proof is immediate from theorem 2.1. Therefore we let $\beta > 0$ and $f \in K^*(\alpha,n)$. Then, by theorem 2.1, there exist two functions p_1 and $p_2 \in P$ such that

$$(1-\alpha)\,\frac{z(D^n f(z))'}{D^n g(z)} + \alpha\,\frac{(z(D^n f(z))')'}{(D^n g(z))'} = p_1(z),$$

and

$$\frac{z(D^n f(z))'}{D^n g(z)} = p_2(z), \quad \text{where } g \in R_n.$$

Hence

$$(1-\beta)\,\frac{z(D^n f(z))'}{D^n g(z)} + \beta\,\frac{(z(D^n f(z))')'}{(D^n g(z))'} = \frac{\beta}{\alpha}\,p_1(z) + (1 - \frac{\beta}{\alpha})p_z(z). \quad (2.1)$$

From the convexity of the class P, it follows that the right hand side of (2.1) belongs to P and this gives us the required result.

THEOREM 2.3

Let $f: f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in K^*(\alpha,n)$.

Then

$$|a_2| < \frac{2+\alpha}{(1+\alpha)(1+n)} .$$

PROOF

Since $f \in K^*(\alpha,n)$, we can write

$$(1-\alpha)\ z(D^n f(z))'(D^n g(z))' + \alpha[z(D^n f(z))']'(D^n g(z)) = p(z)(D^n g(z))(D^n g(z))',$$

where $p \in P$ and $g \in R_n$.

Let $D^n g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ and $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$.

So

$$(1-\alpha)\ [z + \sum_{k=2}^{\infty} \frac{k(k+n-1)!}{n!(k-1)!} z^k][1 + \sum_{k=2}^{\infty} k\ b_k z^{k-1}]$$

$$+ \alpha[z + \sum_{k=2}^{\infty} \frac{k^2(k+n-1)!}{n!(k-1)!} a_k z^k][z + \sum_{k=2}^{\infty} b_k z^k]$$

$$= [1 + \sum_{k=1}^{\infty} c_k z^k][z + \sum_{k=2}^{\infty} b_k z^k][1 + \sum_{k=2}^{\infty} kb_k z^k].$$

Thus, equating the coefficient of z^2 on both sides, we have

$$(1-\alpha)[2b_2 + 2(n+1)a_2] + \alpha(b_2 + 4(n+1)a_2) = c_1 + 3b_2$$

or $\quad 2(n+1)(1+\alpha)a_2 = (1+\alpha)b_2 + c_1$

Now, since $D^n g \in S^*$, $|b_2| < 2$ and also $|c_1| < 2$, see [2]. Hence

$$|a_2| < \frac{(2 + \alpha)}{(n+1)(1+\alpha)} .$$

Using theorem 2.3, we have the following covering theorem.

THEOREM 2.4

Let $f \in K^*(\alpha,n)$. If B is the boundary of the image of E under f, then every point of B has a distance of at least $\frac{(1+\alpha)(1+n)}{4+3\alpha+2n(1-\alpha)}$ from the origin.

PROOF

Let $f(z) \neq c$, $c \neq 0$. Then $f_1(z) = \frac{cf(z)}{c-f(z)}$ is univalent in E.

Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\frac{cf(z)}{c-f(z)} = z + (a_2 + \frac{1}{c})z^2 + \ldots\ldots\ldots,$$

and since $f_1 \in S$, it follows that

$$|a_2 + \frac{1}{c}| < 2$$

or

$$|c| > \frac{(1+\alpha)(1+n)}{4+3\alpha+2n(1+\alpha)},$$

and this proves our result.

THEOREM 2.5

Let $f \in K^*(\alpha,n)$. Then there exist two functions F_1 and $F_2 \in K^*(1,n)$ such that

$$(1-\alpha)\, F_1(z) + \alpha f(z) = F_2(z).$$

PROOF

Since $f \in K^*(\alpha,n)$, there exists a $g \in R_n$ such that

$$(1-\alpha)\, \frac{z(D^n f(z))'}{D^n g(z)} + \alpha\, \frac{(z(D^n f(z))')'}{(D^n g(z))'} = p(z),\ p \in P.$$

From theorem 2.1, we know that

$$\frac{z(D^n f(z))'}{D^n g(z)} = p_1(z), \quad p_1 \varepsilon P.$$

Hence

$$(1-\alpha)\ p_1(z)\ (D^n g(z))' + \alpha(z(D^n f(z))')' = p(z)(D^n g(z))'. \qquad (2.2)$$

Since $g \varepsilon R_n$, there exist two functions F_1, $F_2 \varepsilon K^*(1,n)$ such that $p_1(z)(D^n g(z))' = (z(D^n F_1(z))')'$ and $p(z)(D^n g(z))' = (z(D^n F_2(z))')'$

Thus, from (2.2), we obtain

$$(1-\alpha)(z(D^n F_1(z))')' + \alpha(z(D^n f(z))')' = (z(D^n F_2(z))')'$$

or equivalently

$$(1-\alpha)\ D^n F_1(z) + \alpha\ D^n f(z) = D^n F_2(z)$$

and this gives us the required result.

THEOREM 2.6

Let $f \varepsilon K^*(0,n)$. Then $f \varepsilon K^*(1,n)$ for $|z| < r_0 = 2 - \sqrt{3}$.

PROOF

Consider the function ϕ defined as

$$\phi(z) = \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} k\ z^k .$$

$\phi(z)$ is convex for $|z| < r_0 = 2 - \sqrt{3}$. Let $f \varepsilon K^*(0,n)$ with respect to $g \varepsilon R_n$. Then $\frac{z(D^n f(z))'}{D^n g(z)} = p \varepsilon P$.

Now

$$\frac{(z(D^n f(z))')'}{(D^n g(z))'} = \frac{z[(\phi * D^n f)']}{(\phi * D^n g)}$$

$$= \frac{\phi * z(D^n f)'}{\phi * D^n g}$$

$$= \frac{\phi * p\, D^n g}{\phi * D^n g} .$$

Since $g \in R_n$, so $D^n g \in S^*$. Also ϕ is convex for $|z| < r_0 = 2 - \sqrt{3}$ and $p \in P$. Hence, by using a result due to Ruscheweyh and Shiel-Small [7], we conclude that $f \in F^*(1,n)$ for $|z| < r_0 = 2 - \sqrt{3}$.

ACKNOWLEDGEMENT. The author would like to thank the referee for his useful comments and suggestions.

REFERENCES

1. O. Ahuja, Integral operators of certain univalent functions, Int. J. Math. and Math. Sci. 4(1985), 653-662.

2. P.N. Chichra, New subclasses of the class of close-to-convex functions, Proc. Amer. Math. Soc., 62(1977), 37-43.

3. A.W. Goodman, Univalent Functions, Vol. I, II. Mariner Publishing Company, Florida, 1983.

4. K.I. Noor and D.K. Thomas, On quasi-convex univalent functions, Int. J. Math. and Math. Sci., 3(1980), 255-266.

5. K.I. Noor and N. Al-Dihan, A subclass of close-to-convex functions, Punj. Univ. J. Math. 14-15(1981/82), 183-192.

6. K.I. Noor, On certain classes of close-to-convex functions, Int. J. Math. and Math. Sci. 15(1992), to appear.

7. St. Ruscheweyh and T. Shiel-Small, Hadamard product of schlicht functions and the Polya-Schoenberg conjecture, Comm. Math. Helv. 48(1973), 119-135.

Received September 2, 1991

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