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AN EXTENSION OF KANTOROVICH INEQUALITY

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Dedicated to the memory of Professor Shizuo Kakutani

ABSTRACT. A simple proof of the Kantorovich inequality is presented, and consequently an extension of the inequality is proposed which seems neat.

1. In this note an *operator* means a bounded linear operator acting on a Hilbert space. For a positive invertible operator A, the interval $I = [m, M]$ is the convex hull of the spectrum of A. Let f be a (real-valued) continuous function defined on I and μ a probability measure on I, then the expectation value is defined by $\mathrm{E}[f] = \int_I f(t)d\mu(t)$. For the convenience, by the spectral theorem, an operator A is identified with the function t, $f(A)$ with $f(t)$, and the scalars are identified with the scalar multiples of the identity operator.

In these circumstances, the celebrated Kantorovich inequality is written as follows:

$$(Ax, x)(A^{-1}x, x) \leq \frac{(M+m)^2}{4Mm}, \quad \text{for a unit vector } x \in H. \tag{1}$$

There are a lot of proofs of the inequality [10], [14], [16] - [18], etc. Among them, the proof in [14] presents the following equivalent inequality:

$$\mathrm{E}[t]\mathrm{E}[1/t] \leq \frac{(M+m)^2}{4Mm}. \tag{2}$$

Let us cite the proof of (2) in [14]. Put

$$l(t) = \frac{M+m-t}{Mm},$$

then $1/t \leq l$, so that $\mathrm{E}[1/t] \leq \mathrm{E}[l]$, and

$$\mathrm{E}[t]\mathrm{E}[1/t] \leq \mathrm{E}[t]\mathrm{E}[l] = \mathrm{E}[t] \cdot l(\mathrm{E}[t]) = \frac{1}{Mm}\left((M+m)\mathrm{E}[t] - \mathrm{E}[t]^2\right).$$

Since the last term is a quadratic polynomial in $\mathrm{E}[t]$ and approaches its maximum at $\mathrm{E}[t] = (M+m)/2$, the desired (2) is proved.

Observing the above proof, we see that the essential tools are linearity and monotonicity of the expectation.

There are a large number of authors who have presented extensions of the Kantorovich inequality [2] - [6], [8] - [12], [14] - [18], etc..

In this note we shall modify the above proof in [14] to show an extension of Kantorovich inequality.

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2. For a continuous function f on $I = [m, M]$, we define a linear function

$$l_f(t) = a_f(t-m) + f(m), \quad a_f = \frac{f(M) - f(m)}{M - m}, \tag{3}$$

which corresponds to the line tying two points $(m, f(m))$ and $(M, f(M))$ on the curve $y = f(t)$ in a coordinate plane. For an extension of Kantorovich inequality we take general positive functions $f(t)$ and $h(t) = 1/g(t)$ (not $g(t)$ for simplicity of the later computation) in place of $1/t$ and t, respectively. Then we have a lemma, partially extended fact of [11, Theorem 6].

Lemma 1. *Let f and g be positive continuous functions on I, and assume that $f \leq l$ for a linear function l. Then for a positive operator A with its spectrum in I and for a unit vector $x \in H$,*

$$\frac{(f(A)x, x)}{g((Ax, x))} \leq \max_{t \in I} \frac{l(t)}{g(t)}. \tag{4}$$

In particular, if f is convex then $f \leq l_f$ and

$$\frac{(f(A)x, x)}{g((Ax, x))} \leq K(f, g) := \max_{t \in I} \frac{l_f(t)}{g(t)}. \tag{5}$$

Proof. Convexity of f implies $f \leq l_f$. Hence it suffices to prove the general case, that is, the inequality

$$\frac{\mathrm{E}[f]}{g(\mathrm{E}[t])} \leq \max_{t \in I} \frac{l(t)}{g(t)}. \tag{6}$$

From $f \leq l$ we see $\mathrm{E}[f] \leq \mathrm{E}[l]$, so that

$$\frac{\mathrm{E}[f]}{g(\mathrm{E}[t])} \leq \frac{\mathrm{E}[l]}{g(\mathrm{E}[t])} = \frac{l(\mathrm{E}[t])}{g(\mathrm{E}[t])}.$$

Since $m \leq \mathrm{E}[t] \leq M$, the desired inequality (6) is obtained. □

Now if we put $g = f$ in the particular case of Lemma 1, then we have an inequality which is equivalent to Mond-Pečarić convex inequality [12]:

Theorem 2. (cf. [12, Corollary 1], [11, Corollary 4].) *Let f be a positive continuous convex function, and assume A and x as in Lemma 1. Then*

$$\frac{(f(A)x, x)}{f((Ax, x))} \leq K(f) = \max_{t \in I} \frac{l_f(t)}{f(t)} \quad (K(f) = K(f, f)). \tag{7}$$

If f is continuously differentiable, $a_f \neq 0$ and

$$f'(m) < a_f < f'(M), \tag{8}$$

or f is strictly convex, then there is a point $t (= t^*) \in (m, M)$, at which $\frac{l_f(t)}{f(t)}$ attains its maximum, i.e.,

$$K(f) = \frac{l_f(t^*)}{f(t^*)} = \frac{a_f}{f'(t^*)}. \tag{9}$$

Proof. For (9), put $h(t) = \dfrac{l_f(t)}{f(t)}$. Then

$$h'(t) = \frac{1}{f(t)^2}\{a_f f(t) - l_f(t) f'(t)\} = \frac{1}{f(t)}\left(a_f - \frac{l_f(t)f'(t)}{f(t)}\right). \tag{10}$$

Note that $a_f = \dfrac{f(M)-f(m)}{M-m} = f'(\tau)$ for some $\tau \in (m, M)$ by the mean-value theorem, so that from (8) or strict convexity of f, we have

$$h'(m) = \frac{a_f - f'(m)}{f(m)} > 0 \quad \text{and} \quad h'(M) = \frac{a_f - f'(M)}{f(M)} < 0.$$

Hence $h'(t) = 0$ for a point (denoted by t^*) in (m, M), at which $h(t)$ attains its maximum. Since $a_f f(t) - l_f(t) f'(t) = 0$ for $t = t^*$ (from the first identity of (10)), we have

$$\max_{m \le t \le M} h(t) = h(t^*) = \frac{l_f(t^*)}{f(t^*)} = \frac{a_f}{f'(t^*)}.$$

□

As an application of the above theorem we have:

Corollary 3. (cf. [1, Theorem (Furuta)].)

$$\frac{(e^A x, x)}{e^{(Ax,x)}} \le \frac{k-1}{e \log k} k^{\frac{1}{k-1}} \quad (k = e^{M-m}).$$

Proof. Since $f(t) = e^t$ is strictly convex, the condition (8) is satisfied and for the corresponding function $h(t) = \dfrac{l_{e^t}(t)}{e^t}$ in the proof of the theorem, we see that $h'(t) = 0$ if

$$a - \{a(t-m) + e^m\} = 0 \quad (a = a_{e^t}).$$

The solution is then $t = t^* = \dfrac{a + am - e^m}{a} \in (m, M)$, so that by (9),

$$K(e^t) = \frac{a}{e^{t^*}} = a e^{-\frac{a+am-e^m}{a}} = \frac{k-1}{e \log k} e^{\frac{\log k}{k-1}} = \frac{k-1}{e \log k} k^{\frac{1}{k-1}}.$$

□

The constant $K(e^t)$ is called Specht ratio and its property has been studied in [1], [2], [4] - [6], [8], etc.

Specializing as $g(t) = t^p$ in Lemma 1, we have the following theorem which is due to T. Furuta.

Theorem 4. (cf. [7, p.189].) *Let $0 < m < M$ and $p \notin [0,1]$. Then with the same assumptions for f, A and x as before*

$$(Ax, x)^{-p}(f(A)x, x) \le K(f, t^p). \tag{11}$$

If

$$\frac{f(m)}{m} p < a_f < \frac{f(M)}{M} p \tag{12}$$

holds, then

$$K(f,t^p)=\frac{f(M)-f(m)}{p(M-m)}\left\{\frac{(p-1)(f(M)-f(m))}{p(mf(M)-Mf(m))}\right\}^{p-1}. \tag{13}$$

Proof. It suffices to show (13) with the assumption (12). Let $g(t)=t^p$ and $h(t)=t^{-p}l_f(t)$ $(t>0)$. Then since

$$h'(t)=t^{-p-1}(-pl_f(t)+a_ft)=t^{-p}\left(-p\frac{l_f(t)}{t}+a_f\right),$$

we see that the equation $h'(t)=0$ has a unique solution $t=t^*=\dfrac{p(a_fm-f(m))}{(p-1)a_f}$ in $(0,\infty)$, and that $h'(m)>0,\ h'(M)<0$ if (12) is satisfied. Hence the solution t^* is a point in (m,M), at which $h(t)$ attains its maximum. We then obtain

$$\begin{aligned} K(f,t^p)=h(t^*)&=\frac{a_f}{p}\left\{\frac{(p-1)a_f}{p(a_fm-f(m))}\right\}^{p-1}\\ &=\frac{f(M)-f(m)}{p(M-m)}\left\{\frac{(p-1)(f(M)-f(m))}{p(mf(M)-Mf(m))}\right\}^{p-1}, \end{aligned}$$

as desired.

□

The following result is an application of the above theorem.

Corollary 5. (cf. [7, p.191], [9, Theorem 3].) *If* $p\notin[0,1]$, *then*

$$(A^px,x)\le K(t^p)(Ax,x)^p, \tag{14}$$

where

$$K(t^p)=\begin{cases}\dfrac{(p-1)^{p-1}}{p^p}\cdot\dfrac{(M^p-m^p)^p}{(M-m)(M^pm-Mm^p)^{p-1}} & (p>1),\\ \dfrac{(-p)^{-p}}{(1-p)^{1-p}}\cdot\dfrac{(M^p-m^p)^{1-p}}{(M-m)(M^pm-Mm^p)^{-p}} & (p<0).\end{cases} \tag{15}$$

Proof. Let $f(t)=t^p$. Then since f is strictly convex, the inequality (12) in Theorem 4 holds. Hence from (13) we can obtain the desired $K(t^p)$.

□

The constant $K(p)=K(t^p)$ is called (generalized) Kantorovich constant. Its interesting properties and relations with Specht ratio have been presented in [2] - [6], [8], [9], etc..

By a similar argument as in Theorem 4 we can show the following:

Theorem 6. *Let* $0<m<M$, $p\notin[0,1]$, *and let* g *be a positive, continuously differentiable function on* I. *Then with the same assumptions for* A *and* x *as before,*

$$\frac{(A^px,x)}{g((Ax,x))}\le K(t^p,g). \tag{16}$$

If

$$m^p \frac{g'(m)}{g(m)} < a_{t^p} < M^p \frac{g'(M)}{g(M)} \tag{17}$$

holds, then the equation

$$a_{t^p} g(t) - l_{t^p}(t) g'(t) = 0 \tag{18}$$

has a solution (denoted by t^*) in (m, M), at which $\dfrac{l_{t^p}(t)}{g(t)}$ *attains its maximum, so that*

$$K(t^p, g) = \frac{l_{t^p}(t^*)}{g(t^*)} = \frac{a_{t^p}}{g'(t^*)} = \frac{1}{g'(t^*)} \cdot \frac{M^p - m^p}{M - m}. \tag{19}$$

An application of the above theorem is the following fact which is considered as a special case of a general result in [11].

Corollary 7. (cf. [11, Corollary 9].) *If* $1 \le m \le p \le M$, *then*

$$(A^p x, x) \le K(t^p, e^t) e^{(Ax,x)}, \tag{20}$$

where $K(t^p, e^t) = \dfrac{M^p - m^p}{M - m} e^{-\frac{(m+1)M^p - (M+1)m^p}{M^p - m^p}}$.

Proof. Put $g(t) = e^t$. Then (17) in the above theorem is satisfied, and (18) has a unique solution $t^* = \dfrac{a + am - m^p}{a}$ $\left(a = a_{t^p} = \dfrac{M^p - m^p}{M - m}\right)$ in (m, M). Hence from (19) we obtain

$$K(t^p, e^t) = \frac{a}{e^{t^*}} = a e^{-\frac{a + am + m^p}{a}} = \frac{M^p - m^p}{M - m} e^{-\frac{(m+1)M^p - (M+1)m^p}{M^p - m^p}}.$$

□

3. An extension of Kantorovich inequality due to Schopf [18] is:

$$(A^{n+1}x, x)(A^{n-1}x, x) \le \frac{(M+m)^2}{4Mm} (A^n x, x)^2 \quad \text{for all integers } n. \tag{21}$$

Here A is a positive operator with $(0 <)\ m \le A \le M$ and $x \in H$ is a unit vector.

A state ϕ is a positive linear functional on a C*-algebra $\mathbb{A}$ of operators acting on H such that $||\phi|| = \phi(1) = 1$. Now we show a generalization related to a state of the above inequality (21) by using an idea due to [17]:

Theorem 8. *Let ϕ be a state on a C*-algebra $\mathbb{A}$. Then for all positive operators A in $\mathbb{A}$ with $0 < m \le A \le M$ and for all real numbers r*

$$\phi(A^{r+1}) \phi(A^{r-1}) \le \frac{(M+m)^2}{4Mm} \phi(A^r)^2. \tag{22}$$

Proof. Since $m \le A \le M$, we see

$$A^{r-1}(A - M)(A - m) \le 0$$

or

$$A^{r+1} + Mm A^{r-1} \le (M + m) A^r,$$

so that

$$\phi(A^{r+1}) + Mm\phi(A^{r-1}) \leq (M+m)\phi(A^r).$$

Then by the arithmetic-geometric mean inequality, we have

$$2\,(Mm\phi(A^{r+1})\phi(A^{r-1}))^{1/2} \leq \phi(A^{r+1}) + Mm\phi(A^{r-1}),$$

from which we obtain the desired (22). □

The inequality (22) can be rewritten as follows:

$$1 \leq \frac{\phi(A^{r+1})\phi(A^{r-1})}{\phi(A^r)^2} \leq \frac{M+m)^2}{4Mm}, \tag{23}$$

where the left-hand side inequality is the well-known inequality of Liapounoff. Hence we can deduce by (23) the following:

Corollary 9. *The Kantorovich inequality is a reverse of Liapounoff's inequality.*

For two nonnegative operators A and B, the geometric mean $A\sharp B$ is defined [13] by

$$A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

(If A is not invertible, then $A\sharp B$ is defined as the limit of $(A+\epsilon)\sharp B$ for $\epsilon(>0)\downarrow 0$.) It is well known that the arithmetic-geometric mean inequality holds;

$$A\sharp B \leq \frac{1}{2}(A+B).$$

A unital positive map Φ between two C*-algebras is defined as a linear map such that $\Phi(1) = 1$ and $\Phi(A) \geq 0$ for $A \geq 0$. Then a Kantorovich type inequality with respect to a unital positive map, slight extension of [15, Theorem 1] is given similarly as before, in the following:

Theorem 10. *Let Φ be a unital positive map between two C*-algebras. Then for all positive operators A with $0 < m \leq A \leq M$ and for all real numbers r*

$$\Phi(A^{r+1})\sharp\Phi(A^{r-1}) \leq \frac{(M+m)}{2\sqrt{Mm}}\Phi(A^r).$$

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