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Nonlinear perturbations of a class of integrated semigroups on non-convex domains

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ABSTRACT. Nonlinear continuous perturbations of integrated semigroups are treated from the point of view of the theory of semigroups of nonlinear operators on nonconvex domains. Given an integrated semigroup $W(t)$ with generator A in a Banach space X, a general class of nonlinear perturbations on nonconvex domains is introduced by means of a lower semicontinuous functional φ. Generation and characterization of nonlinear semigroups are discussed in terms of semilinear stability condition and subtangential condition. The local Lipschitz continuity and growth condition for the nonlinear semigroups are restricted by φ on a Banach space X under consideration. In the case where both φ and the domains of perturbing operators are convex, a Hille-Yosida type theorem is obtained.

1. Introduction

Of concern in this paper are the semilinear problems in a Banach space $(X, |\cdot|)$ of the form

$$\text{(SP)} \qquad u'(t) = (A+B)u(t); \quad t > 0; \qquad u(0) = v.$$

Here A is assumed to be the generator of an integrated semigroup $\{W(t) : t \geq 0\}$ in X and B a possibly nonlinear operator from a subset D of $Y = \overline{D(A)}$ into X. It is assumed that B is continuous on bounded sets with respect to a lower semicontinuous functional φ on X such that $D \subset D(\varphi) = \{v \in X; \varphi(v) < \infty\}$. The functional φ is also employed to restrict the growth of mild solutions to (SP). The objective of this paper is to discuss generation and characterization of a nonlinear semigroup on D which provides mild solutions to (SP) in the case that the nonlinear operator B is not necessarily quasidissipative. The generation theorem is established

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under combination of a subtangential condition, a growth condition and a semilinear stability condition. One of the main points of our argument is to deal with the case in which D and φ are not necessarily convex and also $\overline{D(A)}$ is not necessarily dense in X.

Semilinear problems of the form (SP) arise in various fields of mathematical analysis and have been studied by many authors. Iwamiya [4] discussed time-dependent nonlinear perturbations of C_0-semigroups. He showed the generation theorem of nonlinear evolution operators under an explicit subtangential condition and the assumption that $B(t)$ is "quasidissipative" in a generalized sense. Thieme [13] treated the time-dependent Lipschitz perturbations of integrated semigroups under the assumption that the domain C of nonlinear operators $\{B(t)\}$ is convex. Instead of quasidissipativity condition for a nonlinear operator B, Georgescu and Oharu [3] employed a semilinear stability condition and explicit subtangential condition and they gave a generation theorem in the case that the linear operator A is densely defined.

On the other hand Oharu and Takahashi [9] discussed locally Lipschitz perturbations of C_0-semigroups under the convexity condition for the domain of the nonlinear operator B. They established the semilinear Hille-Yosida theorem in general Banach spaces. Their results were extended in Oharu and Takahashi [10], Matsumoto *et al.* [7] and Matsumoto [8] to the case in which the nonlinear operator B is locally quasidissipative and A is not necessarily densely defined.

In this paper we study the case where the linear operator A is not necessarily densely defined and establish a characterization theorem of nonlinear semigroups which provide mild solutions to (SP). This result is an extension of [3, Theorem 1].

This paper is organized as follows: In Section 2 a class of integrated semigroups is introduced and the basic results are outlined. In Section 3 our main results are stated along with some comments. Section 4 deals with the so-called local uniformity of the subtangential condition. In Section 5 the relationship between the semilinear stability condition and quasidissipativity condition is discussed . Moreover, a uniqueness theorem for the mild solution to (SP) is established. Section 6 is devoted to the construction of approximate solutions to (SP) through a precise discrete scheme consistent with (SP). Finally, the main results are verified in Section 7.

2. Integrated semigroups

In this section we introduce a class of integrated semigroups and state some basic facts on such integrated semigroups. In what follows, $(X^*, |\cdot|)$ denotes the dual space of X. For $v \in X$ and $f \in X^*$ the value of f at v is written as $\langle v, f\rangle$. The duality mapping of X is denoted by $\mathcal{F}$. For v, $w \in X$ the symbols $\langle v, w\rangle_i$ and $\langle v, w\rangle_s$ stand for the infimum and the supremum of the set $\{\langle v, f\rangle : f \in \mathcal{F}(w)\}$, respectively. In (SP) the operator A is assumed to be a closed linear operator in X whose domain is not necessarily dense in X. We write A^* for the dual operator of A. If A is densely defined in X, then A^* is defined as a closed linear operator in X^*. If the domain $D(A)$ is not dense in X, then A^* is multi-valued and the identity

$\langle Av, f\rangle = \langle v, g\rangle$ holds for $v \in D(A)$, $f \in D(A^*)$ and $g \in A^*f$; hence the value $\langle v, g\rangle$ does not depend upon the choice of $g \in A^*f$. It follows that $\langle v, g\rangle$ does not depend upon the choice of $g \in A^*f$ provided that $v \in Y$, where Y is the closure of $D(A)$:

$$Y \equiv \overline{D(A)}.$$

Hence for $w \in Y$ and $f \in D(A^*)$ it is justified and convenient to introduce the notation to represent the common value $\langle w, g\rangle$, $g \in A^*f$;

$$\langle w, A^*f\rangle \equiv \langle w, g\rangle, \qquad g \in A^*f.$$

We first give the definition of once integrated semigroup.

DEFINITION 2.1. A one-parameter family $\mathcal{W} \equiv \{W(t) : t \geq 0\}$ of bounded linear operators is said to be a *once integrated semigroup* on X if it has the following two properties (w.1) and (w.2):

(w.1) $$W(0) = 0 \quad \text{and} \quad W(\cdot)v \in C([0, \infty); X) \quad \text{for } v \in X.$$

(w.2) $$W(s)W(t)v = \int_0^s [W(r+t)v - W(r)v]dr \quad \text{for } s, t \geq 0 \text{ and } v \in X.$$

We say that $\mathcal{W}$ is *non-degenerate* if $W(t)v = 0$ for all $t > 0$ implies $v = 0$. If there exist constants $M > 1$ and $\omega \geq 0$ such that $|W(t)| \leq Me^{\omega t}$ for all $t \geq 0$, the once integrated semigroup $\mathcal{W}$ is said to be exponentially bounded.

In what follows, by an integrated semigroup is meant a non-degenerate exponentially bounded once integrated semigroup unless otherwise stated.

As shown in Thieme [12], given a once integrated semigroup $\mathcal{W} \equiv \{W(t)\}$ there exists a closed linear operator A in X such that $v \in D(A)$ and $w = Av$ are characterized by the property that the function $W(\cdot)v$ is continuously differentiable and satisfies

$$\frac{d}{dt}W(t)v = v + W(t)w \qquad \text{for } t > 0.$$

The operator A is called the *generator* of $\mathcal{W}$ and an integrated semigroup is uniquely determined by its generator. Two basic properties of the generator A are stated below Arendt [1, Proposition 3.3]:

PROPOSITION 2.1. *Let $\mathcal{W}$ be an integrated semigroup and A the generator of $\mathcal{W}$. Then the following are valid:*

(a) *$W(t)v \in D(A)$ and $AW(t)v = W(t)Av$ for $v \in D(A)$ and $t \geq 0$,*

(b) *$\int_0^t W(r)vdr \in D(A)$ and $W(t)v = A\int_0^t W(r)vdr + tv$ for $v \in X$ and $t \geq 0$.*

In the subsequent discussions, we are mainly concerned with closed linear operator A in X satisfying the following condition:

(H1) For each $\lambda > 0$ the resolvent $(I - \lambda A)^{-1}$ exists and satisfies

$$|(I - \lambda A)^{-1}| \leq 1.$$

The class of integrated semigroups treated in this paper is that of once integrated semigroups whose generators satisfy the above mentioned condition (H1). From condition (H1) we see that the part A_Y of A in the smaller Banach space $Y \equiv \overline{D(A)}$ has a dense domain in Y and generates a semigroup $\mathcal{T}_Y \equiv \{T_Y(t) : t \geq 0\}$ of class (C_0) on Y by the Hille-Yosida Theorem. Also, it has been shown in [11] that a closed linear operator A satisfying (H1) is the generator of an integrated semigroup $\mathcal{W} \equiv \{W(t)v : t \geq 0\}$ on X. The following structure theorem holds for any integrated semigroup $\mathcal{W}$.

THEOREM 2.1. (Structure Theorem) *Let A be a closed linear operator in X satisfying* (H1) *and $\mathcal{T}_Y$ the semigroup of class (C_0) on $Y \equiv \overline{D(A)}$ generated by the part A_Y of A in Y. Then the integrated semigroup $\mathcal{W}$ generated by A is represented as*

$$W(t)v = \lim_{\lambda \downarrow 0} \int_0^t T_Y(s)(I - \lambda A)^{-1} v ds \quad \text{for } t \geq 0 \text{ and } v \in X. \tag{2.1}$$

The above structure theorem is contained in [13] as a characteristic property of locally Lipschitz once integrated semigroups. We also refer to [7] and [8]. By Definition 2.1 we have the following relation.

$$T_Y(s)W(t)v = W(s+t)v - W(s)v \quad \text{for} \quad v \in X. \tag{2.2}$$

We often use the relation (2.2) in the subsequent sections. For further properties of integrated semigroups we refer to [1], [2], [6], [11], [12], [13].

3. Basic assumptions and a main result

In this section we introduce a class of semilinear operators in X and formulate the associated semilinear problems of the form (SP). In what follows we assume condition (H1) stated in the preceding section.

Let B be a possibly nonlinear operator in X which is defined on a subset D of the closed linear subspace $Y \equiv \overline{D(A)}$. If $D(A) \cap D \neq \emptyset$, then the sum $A + B$ defines a semilinear operator in $D(A + B) = D(A) \cap D$. Throughout this paper we call it a semilinear operator in X determined by A and B.

In order to impose the continuity and quasidissipativity conditions in a local sense for B, we employ a lower semicontinuous functional $\varphi : X \to [0, \infty]$ such that $D \subset D(\varphi) \equiv \{v \in X : \varphi(v) < \infty\}$. We make the following assumption on B:

(H2) For each $\alpha > 0$ the level set $D_\alpha \equiv \{v \in D : \varphi(v) \leq \alpha\}$ is closed in X and the operator B is continuous on D_α.

The continuity condition (H2) on the level sets $\{D_\alpha : \alpha > 0\}$ is much weaker than the continuity on the whole domain D in general and considerably useful for concrete applications to partial differential equations.

DEFINITION 3.1. Let $v \in D$. A continuous function $u(\cdot) : [0,\infty) \to X$ is said to be a *mild solution* to (SP), if $u(t) \in D$ for $t \geq 0$, $Bu(\cdot) \in C([0,\infty); X)$ and $u(\cdot)$ satisfies the equation

$$u(t) = T_Y(t)v + \lim_{\lambda \downarrow 0} \int_0^t T_Y(t-s)(I-\lambda A)^{-1} Bu(s)\, ds \qquad \text{for } t \geq 0. \tag{3.1}$$

Since (H2) is a local condition, (SP) may admit only local mild solutions. In order to obtain global existence of mild solutions to (SP), we employ a growth condition in terms of the real-valued function $\varphi(u(\cdot))$, namely,

$$\varphi(u(t)) \leq e^{at}[\varphi(v) + bt], \qquad t \geq 0, \tag{EG}$$

where a and b are nonnegative constants. This type of growth condition may be called the *exponential growth condition.*

A one-parameter family $\mathcal{S} \equiv \{S(t) : t \geq 0\}$ of possibly nonlinear operators from D into itself is called a *semigroup* on D, if it has two properties below:

(S1) $S(0)v = v$ and $S(t)S(s)v = S(t+s)v$ for s, $t \geq 0$ and $v \in D$.

(S2) For each $v \in D$, $S(\cdot)v \in C([0,\infty); X)$.

If in particular a nonlinear semigroup $\mathcal{S}$ on D provides mild solutions of (SP) in the sense that for each $v \in D$ the function $u(\cdot; v)$ defined by

$$u(t; v) = S(t)v \qquad \text{for } t \geq 0 \tag{3.2}$$

is a mild solution to (SP) on $[0,\infty)$, then we say that $\mathcal{S}$ is associated with the semilinear evolution problem (SP).

We say that a semigroup $\mathcal{S}$ on D is *locally equi-Lipschitz continuous* with respect to φ, if for each $\alpha > 0$ and $\tau > 0$ there is a number $\omega(\alpha, \tau)$ such that

$$|S(t)v - S(t)w| \leq e^{\omega(\alpha,\tau)t}|v - w| \qquad \text{for } t \in [0,\tau] \text{ and } v, w \in D_\alpha.$$

We are now in a position to state our main theorem.

THEOREM 3.1. *Let a, $b \geq 0$. Assume that A and B satisfy conditions* (H1) *and* (H2). *Then the following* (I) *and* (II) *are equivalent:*

(I) *There exists a nonlinear semigroup $\mathcal{S} \equiv \{S(t) : t \geq 0\}$ on D such that*

$$S(t)v = T_Y(t)v + \lim_{\lambda \downarrow 0} \int_0^t T_Y(t-s)(I-\lambda A)^{-1} BS(s)v\,ds, \tag{I.a}$$

$$\varphi(S(t)v) \leq e^{at}[\varphi(v) + bt], \text{ for each } t \geq 0 \text{ and } v \in D. \tag{I.b}$$

(I.c) *For each* $\alpha > 0$ *and* $\tau > 0$, *there exists* $\omega \equiv \omega(\alpha, \tau) \in \mathbb{R}$ *such that*

$$|S(t)v - S(t)w| \le e^{\omega t}|v - w|$$

for v, $w \in D_\alpha$ *and* $t \in [0, \tau]$.

(II) *For each* $v \in D$ *there exists a null sequence* $\{h_n\}$ *of positive numbers and a sequence* $\{v_n\}$ *in* D *such that*

(II.a) $$\lim_{n\to\infty} h_n^{-1}|T_Y(h_n)v + W(h_n)Bv - v_n| = 0,$$

(II.b) $$\overline{\lim_{n\to 0}} h_n^{-1}[\varphi(v_n) - \varphi(v)] \le a\varphi(v) + b.$$

(II.c) *For* $\alpha > 0$ *there is* $\omega_\alpha \in \mathbb{R}$ *such that*

$$\underline{\lim}_{h\downarrow 0} h^{-1}[|T_Y(h)(v - w) + W(h)(Bv - Bw)| - |v - w|] \le \omega_\alpha |v - w|$$

for v, $w \in D_\alpha$.

In addition to this assumptions, if the subset D and the functional φ are both convex and if B is locally Lipschitz continuous in the sense that
(H3) for each $\alpha > 0$ there exists $\omega_\alpha > 0$ such that

$$|Bv - Bw| \le \omega_\alpha |v - w| \quad \text{for } v, w \in D_\alpha,$$

then the following result is established in [7].

THEOREM 3.2. *Let* a, $b \ge 0$ *and* D, φ *are both convex. Assume that* A *and* B *satisfy conditions* (H1) *through* (H3). *then the statements* (I) *and* (II) *are equivalent to any one of* (III) *and* (IV) *below:*

(III) $D(A) \cap D$ *is dense in* D; *for* $\alpha > 0$ *there exists* $\lambda_0 \equiv \lambda_0(\alpha) > 0$ *such that to each* $v \in D_\alpha$ *and each* $\lambda \in (0, \lambda_0)$ *there corresponds an element* $v_\lambda \in D(A) \cap D$ *satisfying*

(III.a) $$v_\lambda - \lambda(A + B)v_\lambda = v,$$

(III.b) $$\varphi(v_\lambda) \le (1 - a\lambda)^{-1}[\varphi(v) + b\lambda].$$

(IV) *For each* $v \in D$ *there exists a null sequence* $\{h_n\}$ *of positive numbers and a sequence* $\{v_n\}$ *in* $D(A) \cap D$ *such that*

(IV.a) $$\lim_{n\to\infty} h_n^{-1}|v_n - h_n(A + B)v_n - v| = 0,$$

(IV.b) $$\limsup_{n\to\infty} h_n^{-1}[\varphi(v_n) - \varphi(v)] \le a\varphi(v) + b,$$

(IV.c) $$\lim_{n\to\infty} |v_n - v| = 0.$$

Clearly, condition (H3) implies semilinear stability condition (II.c). The equivalence between (I) through (IV) under (H3) and the convexity condition for D and φ follows from [7, Theorem 3.1]. It is straightforward to prove the implication (I) $\Longrightarrow$ (II). In fact, by Theorem 2.4 in [7], $\lim_{h\downarrow 0}(h^{-1})|S(h)v - T_Y(h)v - W(h)Bv| = 0$ for $v \in D$. This together with (I.b) implies that (II.a) and (II.b) hold for $v_h = S(h)v$. On the other hand,

$$h^{-1}[|T_Y(h)(v-w) + W(h)(Bv - Bw)| - |v-w|]$$

$$\leq h^{-1}[|T_Y(h)v + W(h)Bv - S(h)v| + |T_Y(h)w + W(h)Bw - S(h)w|]$$

$$+ h^{-1}[|S(h)v - S(h)w| - |v-w|].$$

Letting $h \downarrow 0$, we obtain

$$\underline{\lim}_{h\downarrow 0} h^{-1}[|T_Y(h)(v-w) + W(h)(Bv - Bw)| - |v-w|]$$

$$\leq \underline{\lim}_{h\downarrow 0} h^{-1}[|S(h)v - S(h)w| - |v-w|]$$

$$\leq \omega(\alpha, 1)|v-w|$$

for v, $w \in D_\alpha$. This shows that (II.c) holds. Therefore it remains to show the implication (II) $\Longrightarrow$ (I), which may be called a generation theorem.

4. Local uniformity of the subtangential condition

In this section we discuss the local uniformity of condition (II.a). The following two lemmas are straightforward generalizations of Lemmas 5.1 and 5.2 in [4].

LEMMA 4.1. *Let $\{s_n\}_{n\geq 0}$ be a nondecreasing sequence and $\{w_n\}_{n\geq 0}$ a sequence in D. Then the following identity holds:*

$$(4.1)\quad w_n - T_Y(s_n - s_0)w_0 - \sum_{k=0}^{n-1} T_Y(s_n - s_{k+1})W(s_{k+1} - s_k)Bw_k$$

$$= \sum_{k=0}^{n-1} T_Y(s_n - s_{k+1})[w_{k+1} - T_Y(s_{k+1} - s_k)w_k - W(s_{k+1} - s_k)Bw_k]$$

$$= \sum_{k=0}^{n-1} T_Y(s_n - s_{k+1})[w_{k+1} - T_Y(s_{k+1} - s_k)w_k] - \sum_{k=0}^{n-1}[W(s_n - s_k) - W(s_n - s_{k+1})]Bw_k.$$

The identity stated in the above Lemma is obtained by applying (2.2).

LEMMA 4.2. *Let $\varepsilon > 0$ and $M > 0$. Let $\{s_n\}_{n\geq 0}$ be a nondecreasing sequence and $\{v_n\}_{n\geq 0}$ a sequence in D satisfying $|Bv_n| \leq M$ and*

$$(4.2) \qquad |v_{n+1} - T_Y(s_{n+1} - s_n)v_n - W(s_{n+1} - s_n)Bv_n| \leq (s_{n+1} - s_n)\varepsilon$$

for $n \geq 0$. If $s_n \uparrow s$ as $n \to \infty$, then the sequence $\{v_n\}_{n\geq 0}$ is Cauchy in X.

PROOF. Let $0 \leq k < n$. It follows from Lemma 4.1 that

$$v_n - T_Y(s_n - s_k)v_k - \sum_{j=k}^{n-1} T_Y(s_n - s_{j+1})W(s_{j+1} - s_j)Bv_j$$

$$= \sum_{j=k}^{n-1} T_Y(s_n - s_{j+1})[v_{j+1} - T_Y(s_{j+1} - s_j)v_j - W(s_{j+1} - s_j)Bv_j].$$

This together with (4.2) implies that

$$\begin{aligned}
&|v_n - T_Y(s_n - s_k)v_k| \\
&\qquad \leq \sum_{j=k}^{n-1} |T_Y(s_n - s_{j+1})[v_{j+1} - T_Y(s_{j+1} - s_j)v_j - W(s_{j+1} - s_j)Bv_j]| \\
&\qquad\quad + \sum_{j=k}^{n-1} |T_Y(s_n - s_{j+1})W(s_{j+1} - s_j)Bv_j| \\
&\qquad \leq (s_n - s_k)(M + \varepsilon).
\end{aligned}$$

Using this estimate for m, $n \geq k$, we have

$$\begin{aligned}
&|v_m - v_n| \\
&\leq |v_m - T_Y(s_m - s_k)v_k| + |v_n - T_Y(s_n - s_k)v_k| \\
&\quad + |T_Y(s_m - s_k)v_k - T_Y(s_n - s_k)v_k| \\
&\leq (s_m + s_n - 2s_k)(M + \varepsilon) + |T_Y(s_m - s_k)v_k - T_Y(s_n - s_k)v_k|.
\end{aligned}$$

Letting m, $n \to \infty$, we see that $\overline{\lim}_{m,n\to\infty}|v_m - v_n| \leq 2(s - s_k)(M + \varepsilon)$. Since $s_k \to s$ as $k \to \infty$, it is proved that $\{v_n\}$ is a Cauchy sequence in X. □

The next result shows that subtangential condition (II.a) holds uniformly in a neighborhood of each $v \in D$.

PROPOSITION 4.1. *Suppose that conditions* (H1), (H2), (II.a) *and* (II.b) *hold. Let* $v \in D, \varepsilon \in (0,1)$, $\beta > \varphi(v)$ *and let* $r = r(v, \beta, \varepsilon)$ *be chosen such that*

$$|Bv - Bw| \leq \varepsilon/4 \text{ for } w \in D_\beta \cap U(v,r) \tag{4.3}$$

where $U(v,r) = \{w; |w - v| \leq r\}$. *We then choose* $M \geq 0$ *satisfying*

$$|Bw| \leq M \text{ for each } w \in D_\beta \cap U(v,r), \tag{4.4}$$

and define a positive number $h(v, \beta, \varepsilon)$ *by the supremum of the numbers* $h \in (0, r]$ *satisfying*

$$h(M+1) + \sup_{s \in [0,h]} |T_Y(s)v - v| \leq r \quad \text{and} \quad e^{ah}(\varphi(v) + (b+\varepsilon)h) \leq \beta. \tag{4.5}$$

Let $h \in [0, h(v,\beta,\varepsilon))$ *and* $w \in D$, *and assume that*

$$|w - T_Y(h)v| \leq h(M+1) \text{ and } \varphi(w) \leq e^{ah}[\varphi(v) + (b+\varepsilon)h]. \tag{4.6}$$

Then for each $\eta > 0$ *with* $h + \eta \leq h(v, \beta, \varepsilon)$ *there exists* $z \in D_\beta \cap U(v,r)$ *such that*

$$\eta^{-1}|z - T_Y(\eta)w - W(\eta)Bw| \leq \varepsilon \text{ and } \varphi(z) \leq e^{a\eta}[\varphi(w) + (b+\varepsilon)\eta]. \tag{4.7}$$

PROOF. Let $h \in [0, h(v,\beta,\varepsilon))$ and $w \in D$ and assume that (4.6) holds. We then find the desired elements z by constructing a sequence $\{s_n\}_{n\geq 0}$ in $[0,\eta]$ and the associated sequence $\{v_n\}_{n \geq 0}$ in D such that

(i) $s_0 = 0$, $v_0 = w$, $0 \leq s_{n-1} < s_n < \eta$;

(ii) $\lim_{n\to\infty} s_n = \eta$, $\lim_{n\to\infty} v_n = z$;

(iii) $|v_n - T_Y(s_n - s_{n-1})v_{n-1} - W(s_n - s_{n-1})Bv_{n-1}| \leq (\varepsilon/4)(s_n - s_{n-1})$;

(iv) $|v_n - T_Y(s_n + h)v| \leq (s_n + h)(M+1)$

(v) $\varphi(v_n) \leq e^{a(s_n - s_{n-1})}[\varphi(v_{n-1}) + (b + \varepsilon/4)(s_n - s_{n-1})]$;

(vi) $\varphi(v_n) \leq e^{as_n}[\varphi(v_0) + (b+\varepsilon/4)s_n]$;

(vii) $v_n \in D_\beta \cap U(v,r)$

for $n \geq 0$. Here, (i) is considered for $n \geq 1$ and inequalities (iii) and (v) are not formulated for $n = 0$.

The proof is given by an induction argument. First, v_0 satisfies (iv) by the first inequality in (4.6). Estimate (vi) is trivial for $n = 0$. From (4.5) and (4.6) one obtains

$$|w - v| \leq h(M+1) + |T_Y(h)v - v| \leq r \text{ and } \varphi(w) \leq \beta. \tag{4.8}$$

Hence (iv), (vi) and (vii) are satisfied for $n = 0$. Finally, inequalities (iii) and (v) are not considered for $n = 0$. The first step of the induction argument is now completed.

Suppose now that $\{s_n\}_{n=0}^N$ and $\{v_n\}_{n=0}^N$ have been constructed in such a way that (i) and (iii) through (vii) hold for $0 \le n \le N$. We first note that, by (vii), $|Bv_N| \le M$. Then by (II.a) and (II.b) one finds $\xi \in (0, \eta)$ and $v_{N,\xi} \in D$ such that $s_N + \xi < \eta$ and

$$\xi^{-1}|v_{N,\xi} - T_Y(\xi)v_N - W(\xi)Bv_N| \le \varepsilon/4\,, \tag{4.9}$$

$$\varphi(v_{N,\xi}) \le e^{a\xi}[\varphi(v_N) + (b + \varepsilon/4)\xi].$$

Let $\overline{h}_N$ be the supremum of such numbers ξ; hence $\overline{h}_N > 0$. We then choose an appropriate number $h_N \in (\overline{h}_N/2, \overline{h}_N)$ and set $s_{N+1} = s_N + h_N$. Also, we define v_{N+1} to be an element $v_{N,\xi}$ which is obtained for $\xi = h_N$ by (4.9). It should be noted that s_{N+1} and v_{N+1} are constructed without properties (iv), (vi) and (vii) with $n = N$. It is seen from the construction that (iii) and (v) hold for $n = N+1$. Also, applying (iii), we have

$$\begin{aligned} |v_{N+1} - T_Y(s_{N+1} - s_N)v_N| &\le |v_{N+1} - T_Y(s_{N+1} - s_N)v_N - W(s_{N+1} - s_N)Bv_N| \\ &\quad + |W(s_{N+1} - s_N)Bv_N| \\ &\le (s_{N+1} - s_N)(\varepsilon/4) + (s_{N+1} - s_N)|Bv_N| \\ &\le (s_{N+1} - s_N)(M+1). \end{aligned}$$

Condition (iv) with $n = N$ then implies

$$\begin{aligned} |v_{N+1} - T_Y(s_{N+1} + h)v| &\le |v_{N+1} - T_Y(s_{N+1} - s_N)v_N| \\ &\quad + |T_Y(s_{N+1} - s_N)v_N - T_Y(s_{N+1} + h)v| \\ &\le (s_{N+1} - s_N)(M+1) + |v_N - T_Y(s_N + h)v| \\ &\le (s_{N+1} + h)(M+1), \end{aligned}$$

which shows that (iv) is valid for $n = N+1$. Moreover, the above estimate implies

$$\begin{aligned} |v_{N+1} - v| &\le |v_{N+1} - T_Y(s_{N+1} + h)v| + |T_Y(s_{N+1} + h)v - v| \\ &\le (s_{N+1} + h)(M+1) + |T_Y(s_{N+1} + h)v - v| \end{aligned}$$

and so

$$|v_{N+1} - v| \le (\eta + h)(M+1) + \sup_{s \in [0, \eta + h]} |T_Y(s)v - v| \le r. \tag{4.10}$$

Since (v) holds for $n = N+1$, we have

$$e^{-as_{n+1}}\varphi(v_{n+1}) \le e^{-as_n}\varphi(v_n) + e^{-as_n}(b + \varepsilon/4)(s_{n+1} - s_n) \text{ for } 0 \le n \le N. \tag{4.11}$$

Summing up the inequalities in (4.11) with respect to $n = 0, \cdots, N$ gives

$$\begin{aligned}(4.12)\qquad \varphi(v_{N+1}) &\le e^{as_{N+1}}\varphi(v_0) + \sum_{k=0}^{N} e^{a(s_{N+1}-s_k)}(b+\varepsilon/4)(s_{k+1}-s_k)\\ &\le e^{as_{N+1}}[\varphi(v_0) + (b+\varepsilon/4)s_{N+1}],\end{aligned}$$

which implies the desired estimate (vi) for $n = N+1$. Finally, combining (4.6) and (vi) for $n = N+1$ we see that

$$\varphi(v_{N+1}) \le e^{a(s_{N+1}+h)}[\varphi(v) + (b+\varepsilon)(s_{N+1}+h)].$$

This together with (4.5) implies that $\varphi(v_{N+1}) \le \beta$ and, since $|v_{N+1} - v| \le r$ and that v_{N+1} satisfies (vii) for $n = N+1$. Thus we may extend the sequences $\{s_n\}_{n=0}^{N}$ and $\{v_n\}_{n=0}^{N}$ up to $N+1$. By induction, it is concluded that we can construct a sequence $\{s_n\}_{n\ge0}$ in $[0,\eta]$ and $\{v_n\}_{n\ge0}$ in D with the properties (i) and (iii) through (vii).

It now remains to prove that (ii) holds for the sequences $\{s_n\}_{n\ge0}$ and $\{v_n\}_{n\ge0}$ constructed above. We verify (ii) by contradiction. In view of the construction of the sequence $\{s_n\}_{n\ge0}$, we see that s_n converges to some $s \le \eta$. Hence, by Lemma 4.2 the sequence $\{v_n\}_{n\ge0}$ in D_β is convergent in X to some z and $z \in D_\beta$ by the closedness of D_β.

Suppose then that $s < \eta$. Then we may apply (II.a) and (II.b) to find a number $\delta \in (0, \eta - s)$ and an element $z_\delta \in D$ such that

$$(4.13)\qquad \delta^{-1}|T_Y(\delta)z + W(\delta)Bz - z_\delta| \le \varepsilon/5$$

and

$$(4.14)\qquad \varphi(z_\delta) \le e^{a\delta}[\varphi(z) + (b+\varepsilon/5)\delta].$$

Let N be an integer such that $s - s_n \le \delta/2$ for $n \ge N$. Let $n \ge N$ and let $\overline{h}_n$ be the supremum of $h > 0$ such that $s_n + h < \eta$ and (4.9) holds for N replaced by n and ξ replaced by h, as it was considered earlier. In the construction of the sequences $\{s_n\}_{n\ge0}$ and $\{v_n\}_{n\ge0}$ we have chosen $h_n \in (\overline{h}_n/2, \overline{h}_n)$ and set $s_{n+1} = s_n + h_n$ and $v_{n+1} = v_{N,h}$ with $n = N$ and $h = h_n$. Hence $0 < \overline{h}_n < 2h_n < 2(s - s_n) \le \delta < \eta - s$, and so $s_n + \delta < s_n + \eta - s < \eta$. By the maximality of $\overline{h}_n$, this means that we must have $\delta^{-1}|z_\delta - T_Y(\delta)v_n - W(\delta)Bv_n| > \varepsilon/4$, or $\varphi(z_\delta) > e^{a\delta}[\varphi(v_n) + (b+\varepsilon/4)\delta]$ for infinitely many $n \ge N$. Passing here to the limit as $n \to \infty$, we get $\delta^{-1}|z_\delta - T_Y(\delta)z - W(\delta)Bz| \ge \varepsilon/4$ or $\varphi(z_\delta) \ge e^{a\delta}[\varphi(z) + (b+\varepsilon/4)\delta]$, which contradicts (4.14) or (4.13).

Thus it is concluded that $\lim_{n\to\infty} s_n = \eta$. Finally, we demonstrate that the element z satisfies (4.7). Using Lemma 4.1, we obtain

$$\begin{aligned}&v_n - T_Y(s_n)w - W(s_n)Bw\\ &\quad= \sum_{k=0}^{n-1} T_Y(s_n - s_{k+1})[v_{k+1} - T_Y(s_{k+1}-s_k)v_k - W(s_{k+1}-s_k)Bv_k]\end{aligned}$$

$$+\sum_{k=0}^{n-1} T_Y(s_n - s_{k+1})W(s_{k+1} - s_k)Bv_k - W(s_n)Bw.$$

In view of (2.2) the second term on the right-hand side is rewritten as

$$\sum_{k=0}^{n-1} T_Y(s_n - s_{k+1})W(s_{k+1} - s_k)[Bv_k - Bv] + \sum_{k=0}^{n-1}[W(s_n - s_k) - W(s_n - s_{k+1})]Bv$$

$$= \sum_{k=0}^{n-1} T_Y(s_n - s_{k+1})W(s_{k+1} - s_k)[Bv_k - Bv] + W(s_n)Bv - W(0)Bv.$$

Since $W(0) = 0$, we have

$$\begin{aligned} &v_n - T_Y(s_n)w - W(s_n)Bw \\ &= \sum_{k=0}^{n-1} T_Y(s_n - s_{k+1})[v_{k+1} - T_Y(s_{k+1} - s_k)v_k - W(s_{k+1} - s_k)Bv_k] \\ &\quad + \sum_{k=0}^{n-1} T_Y(s_n - s_{k+1})W(s_{k+1} - s_k)[Bv_k - Bv] + W(s_n)[Bv - Bw]. \end{aligned}$$

So, by (iii), (vii), (4.3) and (4.8), we have

$$\begin{aligned} (4.15) \qquad &|v_n - T_Y(s_n)w - W(s_n)Bw| \\ &\leq \sum_{k=0}^{n-1} |v_{k+1} - T_Y(s_{k+1} - s_k)v_k - W(s_{k+1} - s_k)Bv_k| \\ &\quad + \sum_{k=0}^{n-1}(s_{k+1} - s_k)|Bv_k - Bv| + s_n|Bv - Bw| \\ &\leq \sum_{k=0}^{n-1}(s_{k+1} - s_k)\varepsilon/4 + \sum_{k=0}^{n-1}(s_{k+1} - s_k)\varepsilon/4 + s_n\varepsilon/4 = s_n\varepsilon. \end{aligned}$$

In view of (4.5) and the fact that $s_n < \eta$ for all n, we see that $s_n - s_{k+1} < h(v, \beta, \varepsilon) \leq r(v, \beta, \varepsilon)$ for $0 \leq k \leq n-1$ and $n \geq 1$. Letting $n \to \infty$ in (4.15), we obtain

$$\eta^{-1}|z - T_Y(\eta)w - W(\eta)Bw| \leq \varepsilon.$$

Also, it is seen from (vi) that

$$\varphi(z) \leq e^{a\eta}[\varphi(w) + (b + \varepsilon/4)\eta].$$

Thus it is concluded that z is the desired element. The proof of Proposition 4.1 is now complete. □

REMARK 4.1. If $h = 0$ and $w = v$ in Proposition 4.1, then the above assertion states that for every $\eta \in (0, h(v, \beta, \varepsilon)]$ there exists $z \in D_\beta \cap U(v, r)$ such that

$$\eta^{-1}|z - T_Y(\eta)v - W(\eta)Bv| \le \varepsilon, \quad \varphi(z) \le e^{a\eta}[\varphi(v) + (b + \varepsilon)\eta].$$

5. The semilinear stability condition

In this section we present there results related to semilinear stability condition (II.c). First we give a sufficient condition for (II.c) which is weaker than (H3) stated in our main theorem.

THEOREM 5.1. *Suppose* (H1), (H2) *and*

(H4) *For each $\alpha > 0$ there exists $\omega_\alpha \in \mathbb{R}$ such that*

$$(1 - h\omega_\alpha)|v - w| \le |(v - W(h)Bv) - (w - W(h)Bw)|$$

for $h > 0$ and v, $w \in D_\alpha$.

Then semilinear explicit subtangential condition (II.a) *and growth condition* (II.b) *together imply semilinear stability condition* (II.c).

PROOF. Let $\alpha > 0$, v, $w \in D_\alpha$, $\beta > \alpha$ and $\varepsilon > 0$. Then, as mentioned in Remark 4.1, for each $h \in (0, \varepsilon]$ one finds v_h, $w_h \in D_\beta$ satisfying

$$h^{-1}|T_Y(h)v + W(h)Bv - v_h| \le \varepsilon\,,\ h^{-1}|T_Y(h)w + W(h)Bw - w_h| \le \varepsilon, \tag{5.1}$$

$$|Bv - Bv_h| \le \varepsilon\,,\ |Bw - Bw_h| \le \varepsilon, \tag{5.2}$$

$$\varphi(v_h) \le e^{ah}[\varphi(v) + (b + \varepsilon)h]\,,\ \varphi(w_h) \le e^{ah}[\varphi(w) + (b + \varepsilon)h]. \tag{5.3}$$

Hence we obtain

$$\begin{aligned} &h^{-1}(|T_Y(h)(v - w) + W(h)(Bv - Bw)| - |v - w|) \\ &\le h^{-1}(|T_Y(h)v + W(h)Bv - v_h| + |T_Y(h)w + W(h)Bw - w_h|) \\ &\quad + h^{-1}(|v_h - w_h| - |v - w|) \le 2\varepsilon + h^{-1}(|v_h - w_h| - |v - w|). \end{aligned} \tag{5.4}$$

By (H4), there exists $\omega_\beta \in \mathbb{R}$ such that

$$|v_h - w_h| \le (1 - h\omega_\beta)^{-1}|(v_h - w_h) - W(h)(Bv_h - Bw_h)|. \tag{5.5}$$

Writing $v_h - w_h - W(h)(Bv_h - Bw_h)$ as $(v_h - T_Y(h)v - W(h)Bv) - (w_h - T_Y(h)w - W(h)Bw) - W(h)(Bv_h - Bv) + W(h)(Bw_h - Bw) + T_Y(h)(v - w)$ and applying (5.1), (5.2), we obtain

$$|v_h - w_h - W(h)(Bv_h - Bw_h)| \le 2h\varepsilon + 2h\varepsilon + |v - w|.$$

We then apply (5.5) to get

$$|v_h - w_h| \leq (1 - h\omega_\beta)^{-1}(|v - w| + 4h\varepsilon). \tag{5.6}$$

Combining (5.4) and (5.6) gives

$$\begin{aligned} &h^{-1}(|T_Y(h)(v - w) + W(h)(Bv - Bw)| - |v - w|) \\ &\leq 2\varepsilon + \omega_\beta|v - w|/(1 - h\omega_\beta) + 4\varepsilon/(1 - h\omega_\beta). \end{aligned} \tag{5.7}$$

Taking the limit inferior on both sides of (5.7) as $h \downarrow 0$, we have

$$\varliminf_{h\downarrow 0} h^{-1}(|T_Y(h)(v - w) + W(h)(Bv - Bw)| - |v - w|) \leq 6\varepsilon + \omega_\beta|v - w|.$$

Thus we obtain the desired result. □

The next result states that semilinear stability condition (Ⅱ.c) implies the so-called strong quasidissipativity of the part $(A + B)_{|Y}$ in Y of $A + B$.

THEOREM 5.2. *Assume that* (H1) *and* (H2) *hold. If* v, $w \in D \cap D(A)$ *and* $(A + B)v$, $(A + B)w \in Y$, *then we have*

$$\begin{aligned} &\lim_{h\downarrow 0} h^{-1}[|T_Y(h)(v - w) + W(h)(Bv - Bw)| - |v - w|]|v - w| \\ &= \langle (A + B)v - (A + B)w, v - w\rangle_s. \end{aligned}$$

PROOF. Let v, $w \in D(A) \cap D$ and fix any $f \in \mathcal{F}(v - w)$. Since $T_Y(h)v - v = W(h)Av$, it holds that

$$\begin{aligned} &h^{-1}(|T_Y(h)(v - w) + W(h)(Bv - Bw)| - |v - w|)|v - w| \\ &\geq \langle h^{-1}(T_Y(h)v - v) + h^{-1}W(h)Bv - h^{-1}(T_Y(h)w - w) - h^{-1}W(h)Bw, f\rangle \\ &= \langle h^{-1}W(h)(A + B)v - h^{-1}W(h)(A + B)w, f\rangle \end{aligned}$$

for $h > 0$. By Theorem 2.1 and condition that $(A + B)v \in Y$, we have

$$h^{-1}W(h)(A + B)v = h^{-1}\int_0^h T_Y(s)(A + B)v\,ds \to (A + B)v \quad \text{as } h \downarrow 0. \tag{5.8}$$

Hence

$$\begin{aligned} &\varliminf_{h\downarrow 0} h^{-1}[|T_Y(h)(v - w) + W(h)(Bv - Bw)| - |v - w|]|v - w| \\ &\qquad\qquad \geq \langle (A + B)v - (A + B)w, f\rangle. \end{aligned}$$

Since $f \in \mathcal{F}(v-w)$ is arbitrary, we obtain

$$(5.9) \qquad \underline{\lim}_{h\downarrow 0} h^{-1}[|T_Y(h)(v-w)+W(h)(Bv-Bw)|-|v-w|]|v-w|$$
$$\geq [(A+B)v-(A+B)w, v-w]_s.$$

Next, we prove the converse inequality. Using Young's inequality, we have

$$\overline{\lim}_{h\downarrow 0} h^{-1}\left[|(v-w)+W(h)[(A+B)v-(A+B)w]|-|v-w|\right]|v-w|$$
$$\leq \overline{\lim}_{h\downarrow 0} h^{-1}\left[|(v-w)+h[(A+B)v-(A+B)w]|-|v-w|\right]|v-w|$$
$$+\overline{\lim}_{h\downarrow 0} h^{-1}\left[|W(h)[(A+B)v-(A+B)w]-h[(A+B)v-(A+B)w]|\right]|v-w|$$
$$\leq \overline{\lim}_{h\downarrow 0}(2h)^{-1}\left[|(v-w)+h[(A+B)v-(A+B)w]|^2-|v-w|^2\right]$$
$$+\overline{\lim}_{h\downarrow 0} h^{-1}\left[|W(h)[(A+B)v-(A+B)w]-h[(A+B)v-(A+B)w]|\right]|v-w|.$$

Since the right-hand side converges to $[(A+B)v-(A+B)w, v-w]_s$ by (5.8), the proof is now complete. □

We now demonstrate that semilinear stability condition (Ⅱ.c) guarantees the uniqueness of mild solutions to the semilinear problem (SP).

THEOREM 5.3. *Suppose that* (H1), (H2) *and* (Ⅱ.c) *hold. Let* $\alpha > 0$. *Let* $u(\cdot)$ *and* $v(\cdot)$ *be mild solutions with initial data* $u(0)$, $v(0) \in D_\alpha$, *respectively. Then*

$$|u(t)-v(t)| \leq e^{\omega_\beta t}|u(0)-v(0)|$$

for $\tau > 0$, $\beta > e^{a\tau}(\alpha+b\tau)$ *and* $t \in [0,\tau]$.

PROOF. Let $t \in [0,\tau]$. By the definition of mild solutions to (SP) and Theorem 2.1, we have

$$u(t+h) = T_Y(h)u(t) + \lim_{\lambda\downarrow 0}\int_t^{t+h} T_Y(t+h-s)(I-\lambda A)^{-1}Bu(s)ds$$
$$= T_Y(h)u(t) + W(h)Bu(t) + \lim_{\lambda\downarrow 0}\int_0^h T_Y(s)(I-\lambda A)^{-1}(Bu(t+h-s)-Bu(t))ds.$$

Since

$$\overline{\lim}_{h\to 0} h^{-1}\left|\lim_{\lambda\downarrow 0}\int_0^h T_Y(s)(I-\lambda A)^{-1}(Bu(t+h-s)-Bu(t))\,ds\right|$$
$$\leq \overline{\lim}_{h\to 0}\int_0^h |Bu(t+h-s)-Bu(t)|\,ds = 0,$$

we have

$$(5.10)\quad \underline{\lim}_{h\downarrow 0} h^{-1}(|u(t+h)-v(t+h)|-|u(t)-v(t)|)$$

$$\leq \underline{\lim}_{h\downarrow 0} h^{-1}(|T_Y(h)(u(t)-v(t))+W(h)(Bu(t)-Bv(t))|-|u(t)-v(t)|).$$

The estimate (5.10) and (II.c) imply that

$$\underline{D}_+|u(t)-v(t)| \leq \omega_\beta |u(t)-v(t)|,$$

where $\underline{D}_+$ stands for the Dini-derivative. This implies the desired result. □

6. Construction of the approximate solutions

In this section we discuss the construction of approximate solutions to semilinear problem (SP) in terms of method of discretization in time. First, we need the following result for constructing local approximate solutions.

THEOREM 6.1. *Suppose that conditions* (H1), (H2), (II.a) *and* (II.b) *are satisfied. Let $v \in D$, $R > 0$, $\beta > \varphi(v)$. Choose positive numbers M and τ so that $|Bw| \leq M$ for $w \in D_\beta \cap U(v, R)$. Let $\tau > 0$ be chosen so that*

$$\tau(M+1) + \sup_{t\in[0,\tau]} |T_Y(t)v - v| \leq R \text{ and } e^{a\tau}[\varphi(v) + (b+1)\tau] < \beta.$$

Then for each $\varepsilon \in (0,1]$ there exists a sequence $\{t_i\}_{i=0}^N$ and a sequence $\{v_i\}_{i=0}^N$ in $D_\beta \cap U(v,R)$ such that

(i) $t_0 = 0$, $v_0 = v$, $t_N = \tau$;

(ii) $0 < t_{i+1} - t_i \leq \varepsilon$ *for* $0 \leq i \leq N-1$;

(iii) $|v_i - T_Y(t_i)v| \leq t_i(M+1)$ *and* $\varphi(v_i) \leq e^{at_i}[\varphi(v) + (b+\varepsilon)t_i]$ *for* $0 \leq i \leq N$;

(iv) $|v_{i+1} - T_Y(t_{i+1}-t_i)v_i - W(t_{i+1}-t_i)Bv_i| \leq (t_{i+1}-t_i)\varepsilon$ *and*
$\varphi(v_{i+1}) \leq e^{a(t_{i+1}-t_i)}[\varphi(v_i) + (b+\varepsilon)(t_{i+1}-t_i)]$ *for* $0 \leq i \leq N-1$;

(v) $v_i \in D_\beta \cap U(v,R)$ *for* $0 \leq i \leq N$;

(vi) *For* $0 \leq i \leq N-1$ *there is* $r_i \in (0,\varepsilon]$ *such that* $|Bw - Bv_i| \leq \varepsilon/4$
for $w \in U(v_i, r_i) \cap D_\beta$ *and* $(t_{i+1}-t_i)(M+1) + \sup_{t\in[0,t_{i+1}-t_i]} |T_Y(t)v_i - v_i| \leq r_i$.

PROOF. Set $t_0 = 0$ and $v_0 = x$. Suppose that $\{t_i\}_{i=0}^n$ and $\{v_i\}_{i=0}^n$ have been constructed in such a way that conditions (i) through (vi) are fulfilled. We then define

$$r_n = \sup\{r \in (0,\varepsilon]; |Bw - Bv_n| \leq \varepsilon/4 \text{ for } w \in D_\beta \cap U(v_n, r)\}$$

and

$$\eta_n = \sup\left\{ t > 0; t(M+1) + \sup_{s\in[0,t]} |T_Y(s)v_n - v_n| \leq r_n \text{ and } e^{at}\left(\varphi(v_n) + (b+\varepsilon)t\right) \leq \beta \right\}. \tag{6.1}$$

Following the proof of Proposition 4.1, we define $h_n = \min\{\tau - t_n, \eta_n\}$ and $t_{n+1} = t_n + h_n$. Applying Proposition 4.1 with $h = 0$, $\eta = h_n$, $v = w = v_n$ and $r = r_n$, one finds $v_{n+1} \in D_\beta \cap U(v_n, r_n)$ satisfying

$$|v_{n+1} - T_Y(h_n)v_n - W(h_n)Bv_n| \leq \varepsilon h_n \text{ and } \varphi(v_{n+1}) \leq e^{ah_n}\left(\varphi(v_n) + (b+\varepsilon)h_n\right).$$

By the induction hypothesis (i) through (vi), it is easily seen that

$$\begin{aligned}|v_{n+1} - T_Y(t_{n+1})v| \leq & |v_{n+1} - T_Y(h_n)v_n - W(h_n)Bv_n| \\ & + |T_Y(h_n)[v_n - T_Y(t_n)v]| + |W(h_n)Bv_n| \leq t_{n+1}(M+1)\end{aligned}$$

thereby

$$\begin{aligned}|v_{n+1} - v| &\leq t_{n+1}(M+1) + |T_Y(t_{n+1})v - v| \\ &\leq \tau(M+1) + \sup_{s\in[0,\tau]} |T_Y(s)v - v| \leq R\end{aligned}$$

and

$$\begin{aligned}\varphi(v_{n+1}) &\leq e^{a(t_{n+1}-t_n)}\left(\varphi(v_n) + (b+\varepsilon)(t_{n+1}-t_n)\right) \\ &\leq e^{at_{n+1}}\left(\varphi(v) + (b+\varepsilon)t_{n+1}\right).\end{aligned}$$

Thus we have constructed sequences $\{t_i\}_{i=0}^{n+1}$ and $\{v_i\}_{i=0}^{n+1}$ satisfying (i) through (vi). It now remains to show that τ can be attained in some finite, say N, steps. Suppose to the contrary that this is not the case. Then we would obtain two infinite sequences $\{t_i\}_{i\geq 0}$ in $[0, \tau)$ and $\{v_i\}_{i\geq 0}$ in $D_\beta \cap U(v, R)$. By Lemma 4.2, v_i converges to some $z \in D_\beta \cap U(v, R)$. It should be noted here that $\varphi(z) < \beta$ by (iii).

Let $\overline{r} \in (0, \varepsilon/2)$ and $\overline{k}_\varepsilon \geq 1$ be such that

$$|Bw - Bz| \leq \varepsilon/6 \quad \text{for each } w \in D_\beta \cap U(z, \overline{r}) \tag{6.2}$$

and

$$|Bv_k - Bz| \leq \varepsilon/12 \quad \text{and} \quad |v_k - z| \leq \min\{\varepsilon/12, \overline{r}/16\} \text{ for each } k \geq \overline{k}_\varepsilon. \tag{6.3}$$

Then (6.3) implies that $U(v_k, \overline{r}/2) \cap D_\beta \subseteq U(z, \overline{r}) \cap D_\beta$ for $k \geq \overline{k}_\varepsilon$. Hence, for $k \geq \overline{k}_\varepsilon$ and $w \in D_\beta \cap U(v_k, \overline{r}/2)$, we have

$$|Bw - Bv_k| \leq |Bw - Bz| + |Bv_k - Bz| \leq \varepsilon/4$$

by (6.2) and (6.3). These estimates together show that $r_k \geq \overline{r}/2$ for $k \geq \overline{k}_\varepsilon$.

We next choose $\delta = \delta(z, \overline{r})$ so that $|T(s)z - z| \leq \overline{r}/8$ for $s \in [0, \delta]$. By the choice of τ, there is $\overline{\delta} > 0$ such that

$$e^{a(\tau+\overline{\delta})} \left(\varphi(v) + (b+\varepsilon)(\tau + \overline{\delta})\right) \leq \beta. \tag{6.4}$$

In view of the construction of the sequence $\{t_i\}_{i\geq 0}$, we see that t_i converges to some $t \leq \tau$. In order to derive the contradiction, we put $\delta_0 = \tau + \overline{\delta} - t > 0$ and define

$$\xi = \min\{\overline{r}/4(M+1), \delta, \delta_0\}. \tag{6.5}$$

First, it is seen that $t_k + \xi \leq \tau + \overline{\delta}$ and

$$e^{a\xi}(\varphi(v_n) + (b+\varepsilon)\xi) \leq e^{a(t_n+\xi)}(\varphi(v) + (b+\varepsilon)(t_n + \xi)) \leq \beta.$$

From (6.3), (6.4) and (6.5) we deduce

$$\begin{aligned} \sup_{s\in[0,\xi]} |T(s)v_k - v_k| &\leq \sup_{s\in[0,\xi]} (|T(s)v_k - T(s)z| + |T(s)z - z| + |z - v_k|) \\ &\leq \sup_{s\in[0,\xi]} (2|v_k - z| + |T(s)z - z|) \leq \overline{r}/4 \end{aligned}$$

for $k \geq \overline{k}_\varepsilon$. These estimates together imply

$$\xi(M+1) + \sup_{s\in[0,\xi]} |T(s)v_k - v_k| \leq \overline{r}/2 \leq r_k.$$

This means that $\eta_k \geq \xi$ for $k \geq \overline{k}_\varepsilon$. We now recall that $h_k = \min\{\tau - t_k, \eta_k\}$. If there is $k_0 \geq \overline{k}_\varepsilon$ such that $\eta_{k_0} \geq \tau - t_{k_0}$, then $h_{k_0} = \tau - t_{k_0}$. This implies that $\tau = t_{k_0} + h_{k_0} = t_{k_0+1}$. This is a contradiction. If $\eta_k < \tau - t_k$ for $k \geq \overline{k}_\varepsilon$, then $h_k = \eta_k \geq \xi$ for $k \geq \overline{k}_\varepsilon$. This is again a contradiction. It is then concluded that τ is attained by some t_N, and the proof of Theorem 6.1 is complete. □

Using the finite sequences $\{t_i\}_{i=0}^N$ in $[0, \tau]$ and $\{v_i\}_{i=0}^N$ in $D_\beta \cap U(v, R)$ obtained for $v \in D$ by Theorem 6.1, we may define an approximate solution $u_\varepsilon : [0, \tau] \to X$ to (SP) by

$$u_\varepsilon(t) \equiv \begin{cases} T_Y(t - t_i)v_i + W(t - t_i)Bv_i & \text{for } t \in [t_i, t_{i+1}),\ 0 \leq i \leq N-1 \\ T_Y(\tau - t_{N-1})v_{N-1} + W(\tau - t_{N-1})Bv_{N-1} & \text{for } t = \tau. \end{cases} \tag{6.6}$$

In the next section we demonstrate that for any null sequence $\{\varepsilon_n\}$ of positive numbers the sequence of the corresponding approximate solutions $\{u_{\varepsilon_n}\}$ on $[0, \tau]$ converges uniformly to a mild solution of (SP) satisfying the exponential growth condition with respect to φ. To this end, we need a method for estimating the difference between approximate solutions.

In order to formulate the statement, we introduce four kinds of quantities which depend upon the choice of base data v, $\hat{v} \in D$ and error bounds ε, $\hat{\varepsilon} \in (0, 1/3)$. Let

a and b be constants given in condition (II.b) and fix any pair v, $\hat{v} \in D$ and any pair ε, $\hat{\varepsilon} \in (0, 1/3)$. First, we choose β so that

$$\beta > \max\{\varphi(v), \varphi(\hat{v})\}. \tag{6.7}$$

Next, by the continuity of B on D_β, one finds positive numbers $r = r(v, \beta, \varepsilon)$, $\hat{r} = r(\hat{v}, \beta, \hat{\varepsilon})$, $M(v, \beta, \varepsilon)$ and $M(\hat{v}, \beta, \hat{\varepsilon})$ such that

$$|Bw - Bv| \le \varepsilon/4 \text{ and } |Bw| \le M(v, \beta, \varepsilon) \text{ for } w \in D_\beta \cap U(v, r), \tag{6.8}$$

$$|Bw - B\hat{v}| \le \hat{\varepsilon}/4 \text{ and } |Bw| \le M(\hat{v}, \beta, \hat{\varepsilon}) \text{ for } w \in D_\beta \cap U(\hat{v}, \hat{r}). \tag{6.8$'$}$$

Choose a positive number M so that

$$M \ge \max\{M(v, \beta, \varepsilon), M(\hat{v}, \beta, \hat{\varepsilon})\}.$$

Then we define $h(v, \beta, \varepsilon)$ by

$$h(v, \beta, \varepsilon) = \sup\left\{ h > 0; h(M+1) + \sup_{s \in [0,h]} |T_Y(s)v - v| \le r \text{ and } e^{ah}[\varphi(v) + (b+\varepsilon)h] \le \beta \right\}. \tag{6.9}$$

In view of (6.7), it is seen that $h(v, \beta, \varepsilon) > 0$.

We are now in a position to state the following comparison theorem.

THEOREM 6.2. *Suppose that conditions* (H1), (H2) *and* (II) *are satisfied. Let v, $\hat{v}$ be any pair of base data in D and ε, $\hat{\varepsilon}$ a pair of error bounds in $(0, 1/3)$. Let $h \in [0, h(v, \beta, \varepsilon))$, $\hat{h} \in [0, h(\hat{v}, \beta, \hat{\varepsilon}))$ and w, $\hat{w} \in D$ be chosen so that*

$$|w - T_Y(h)v| \le h(M+1), \quad \varphi(w) \le e^{ah}[\varphi(v) + (b+\varepsilon)h] \tag{6.10a}$$

and

$$|\hat{w} - T_Y(\hat{h})\hat{v}| \le \hat{h}(M+1), \quad \varphi(\hat{w}) \le e^{a\hat{h}}[\varphi(\hat{v}) + (b+\hat{\varepsilon})\hat{h}]. \tag{6.10b}$$

Then for each $\delta > 0$ and each $\eta > 0$ satisfying $h+\eta \le h(v, \beta, \varepsilon)$ and $\hat{h}+\eta \le h(\hat{v}, \beta, \hat{\varepsilon})$ there exist $z \in D_\beta \cap U(v, r)$ and $\hat{z} \in D_\beta \cap U(\hat{v}, \hat{r})$ such that

$$|z - T_Y(\eta)w - W(\eta)Bw| < 2\eta\varepsilon, \quad \varphi(z) \le e^{a\eta}[\varphi(w) + (b+\varepsilon)\eta], \tag{6.11a}$$

$$|\hat{z} - T_Y(\eta)\hat{w} - W(\eta)B\hat{w}| < 2\eta\hat{\varepsilon}, \quad \varphi(\hat{z}) \le e^{a\eta}[\varphi(\hat{w}) + (b+\hat{\varepsilon})\eta], \tag{6.11b}$$

and such that

$$|z - \hat{z}| \le e^{\omega_\beta \eta}|w - \hat{w}| + \eta e^{\overline{\omega}_\beta \eta}(\varepsilon + \hat{\varepsilon} + \delta) \tag{6.12}$$

where $\overline{\omega}_\beta = \max\{\omega_\beta, 0\}$.

PROOF. First, we note that $w \in D_\beta \cap U(v,r)$ and $\hat{w} \in D_\beta \cap U(\hat{v},\hat{r})$ since

$$|w-v| \le |w - T_Y(h)v| + |T_Y(h)v - v| \le h(M+1) + |T_Y(h)v - v| \le r$$

by the definition of $h(v,\beta,\varepsilon)$ and since the corresponding estimate holds for $\hat{v}$, $\hat{w}$, $\hat{h}$ and $\hat{r}$. Let η be a positive number satisfying $h+\eta \le h(v,\beta,\varepsilon)$ and $\hat{h}+\eta \le h(\hat{v},\beta,\hat{\varepsilon})$.

We then demonstrate that three sequences $\{s_n\}_{n\ge0}$, $\{v_n\}_{n\ge0}$ and $\{\hat{v}_n\}_{n\ge0}$ can be inductively constructed in such a way that

(i) $\quad s_0 = 0,\ v_0 = w,\ \hat{v}_0 = \hat{w}$

(ii) $\quad 0 < s_{n-1} < s_n$ and $\lim_{n\to\infty} s_n = \eta$

(iii-a) $\quad |v_n - T_Y(s_n - s_{n-1})v_{n-1} - W(s_n - s_{n-1})Bv_{n-1}| \le (s_n - s_{n-1})\varepsilon$

(iii-b) $\quad |\hat{v}_n - T_Y(s_n - s_{n-1})\hat{v}_{n-1} - W(s_n - s_{n-1})B\hat{v}_{n-1}| \le (s_n - s_{n-1})\hat{\varepsilon}$

(iv-a) $\quad \varphi(v_n) \le e^{a(s_n - s_{n-1})}[\varphi(v_{n-1}) + (b+\varepsilon)(s_n - s_{n-1})]$

(iv-b) $\quad \varphi(\hat{v}_n) \le e^{a(s_n - s_{n-1})}[\varphi(\hat{v}_{n-1}) + (b+\hat{\varepsilon})(s_n - s_{n-1})]$

(v) $\quad e^{-\omega_\beta(s_n - s_{n-1})}|T_Y(s_n - s_{n-1})(v_{n-1} - \hat{v}_{n-1}) + W(s_n - s_{n-1})(Bv_{n-1} - B\hat{v}_{n-1})|$
$\quad\quad \le |v_{n-1} - \hat{v}_{n-1}| + (s_n - s_{n-1})\delta$

(vi-a) $\quad |v_n - T_Y(s_n)v_0| \le s_n(M+1)$

(vi-b) $\quad |\hat{v}_n - T_Y(s_n)\hat{v}_0| \le s_n(M+1)$

(vii-a) $\quad \varphi(v_n) \le e^{a(s_n+h)}[\varphi(v) + (b+\varepsilon)(s_n+h)]$

(vii-b) $\quad \varphi(\hat{v}_n) \le e^{a(s_n+\hat{h})}[\varphi(\hat{v}) + (b+\hat{\varepsilon})(s_n+\hat{h})]$

(viii-a) $\quad v_n \in U(v,r) \cap D_\beta$

(viii-b) $\quad \hat{v}_n \in U(\hat{v},\hat{r}) \cap D_\beta$

hold for $n \ge 0$. The estimates (iii-a) and (iii-b) ensure the convergence of the sequences $\{v_n\}$ and $\{\hat{v}_n\}$ and the inequalities (iv-a) and (iv-b) lead us to the exponential growth condition for mild solutions. The estimates (v) will be used to obtain (6.12).

First, we infer from (6.10a) and (6.10b) that (vi-a) through (viii-b) are all valid for $n = 0$. Estimates (iii-a) through (v) are not formulated for $n = 0$. In the same way as in the proof of Proposition 4.1, the first step of the induction argument is completed in this sense.

We then suppose that three finite sequences $\{s_n\}_{n=0}^N$, $\{v_n\}_{n=0}^N$ and $\{\hat{v}_n\}_{n=0}^N$ have been constructed in such a way that (i), (iii-a) through (viii-b) are satisfied.

Let $\overline{h}_N$ be the supremum of the positive numbers ξ such that $s_N + \xi \le \eta$ and

$$e^{-\omega_\beta \xi}|T_Y(\xi)(v_N - \hat{v}_N) + W(\xi)(Bv_N - B\hat{v}_N)| \le |v_N - \hat{v}_N| + \xi\delta. \tag{6.13}$$

We then fix any $h_N \in (\overline{h}_N/2, \overline{h}_N)$ and put $s_{N+1} \equiv s_N + h_N$. We note that $\varphi(v_N) < \beta$ and $\varphi(\hat{v}_N) < \beta$ by (viii-a) and (viii-b). Hence we may apply Proposition 4.1 to find v_{N+1}, $\hat{v}_{N+1} \in D$ satisfying

$$|v_{N+1} - T_Y(s_{N+1} - s_N)v_N - W(s_{N+1} - s_N)Bv_N| \le (s_{N+1} - s_N)\varepsilon, \tag{6.14a}$$

(6.14b) $$|\hat{v}_{N+1} - T_Y(s_{N+1} - s_N)\hat{v}_N - W(s_{N+1} - s_N)B\hat{v}_N| \le (s_{N+1} - s_N)\hat{\varepsilon},$$

(6.15a) $$\varphi(v_{N+1}) \le e^{a(s_{N+1}-s_N)}[\varphi(v_N) + (b+\varepsilon)(s_{N+1} - s_N)],$$

(6.15b) $$\varphi(\hat{v}_{N+1}) \le e^{a(s_{N+1}-s_N)}[\varphi(\hat{v}_N) + (b+\hat{\varepsilon})(s_{N+1} - s_N)].$$

This shows that s_{N+1}, v_{N+1} and $\hat{v}_{N+1}$ are constructed without properties (v) through (vii-b) with $n = N+1$. Then, letting $\xi = h_N$ in (6.13) we see that (v) is satisfied for $n = N+1$. Our next aim is to show that the constructed s_{N+1}, v_{N+1} and $\hat{v}_{N+1}$ satisfy (vi-a), (vii-a), (vi-b) and (vii-b) for $n = N+1$. Applying (vi-a) with $n = N$ and (6.14a) we obtain

$$\begin{aligned} &|v_{N+1} - T_Y(s_{N+1})v_0| \\ &\quad \le |v_{N+1} - T_Y(s_{N+1} - s_N)v_N - W(s_{N+1} - s_N)Bv_N| \\ &\qquad + |T_Y(s_{N+1})v_0 - T_Y(s_{N+1} - s_N)v_N| + |W(s_{N+1} - s_N)Bv_N| \\ &\quad \le (s_{N+1} - s_N)\varepsilon + |v_N - T_Y(s_N)v_0| + (s_{N+1} - s_N)M \le s_{N+1}(M+1) \end{aligned}$$

and, $|\hat{v}_{N+1} - T_Y(s_{N+1})\hat{v}_0| < s_{N+1}(M+1)$ in the same way. This proves that v_{N+1}, $\hat{v}_{N+1}$ satisfy (vi-a) and (vi-b) for $n = N+1$, respectively. To show that v_{N+1} and $\hat{v}_{N+1}$ satisfy (vii-a) and (vii-b) for $n = N+1$, respectively, we apply (6.15a) and the induction hypothesis to obtain

$$\begin{aligned} \varphi(v_{N+1}) &< e^{a(s_{N+1}-s_N)}[e^{a(s_N+h)}(\varphi(v) + (b+\varepsilon)(s_N+h)) + (b+\varepsilon)(s_{N+1} - s_N)] \\ &< e^{a(s_{N+1}+h)}[\varphi(v) + (b+\varepsilon)(s_{N+1}+h)] \end{aligned}$$

and

$$\varphi(\hat{v}_{N+1}) \le e^{a(s_{N+1}+\hat{h})}[\varphi(\hat{v}) + (b+\hat{\varepsilon})(s_{N+1}+\hat{h})].$$

It now remains to show that $\lim_{n\to\infty} s_n = \eta$. Suppose to the contrary that $s_n \to s < \eta$. Then, by Lemma 4.2, there would exist some elements z, $\hat{z} \in D$ such that $v_n \to z$ and $\hat{v}_n \to \hat{z}$. It follows from the closedness of D_β that z, $\hat{z} \in D_\beta$. On the other hand, by semilinear stability condition (II.c), there must exist some $\xi \in (0, \eta - s)$ such that

(6.16) $$e^{-\omega_\beta \xi}|T_Y(\xi)(z - \hat{z}) + W(\xi)(Bz - B\hat{z})| \le |z - \hat{z}| + (1/2)\xi\delta,$$

where δ is the number employed in the estimate (v). Choose $N \ge 1$ so that $s - s_n \le \xi/2$ for each $n \ge N$. Set $\xi_n \equiv s - s_n + \xi$. Then $s_n + \xi_n = s + \xi < \eta$ and $\bar{h}_n < 2h_n < 2(s - s_n) < \xi_n$. Hence it would follow from (6.13) that

$$e^{-\omega_\beta \xi_n}|T_Y(\xi_n)(v_n - \hat{v}_n) + W(\xi_n)(Bv_n - B\hat{v}_n)| > |v_n - \hat{v}_n| + \xi_n\delta$$

for $n \geq N$. Now letting $n \to \infty$, we see that

$$e^{-\omega_\beta \xi}|T_Y(\xi)(z-\hat{z}) + W(\xi)(Bz - B\hat{z})| \geq |z - \hat{z}| + \xi\delta.$$

This contradicts (6.16) and hence it is proved that $\lim_{n\to\infty} s_n = \eta$. We now show that the limits z and $\hat{z}$ are the desired elements. First, using (iv-a) and recalling $v_0 = w$, we have

$$\varphi(z) \leq e^{a\eta}[\varphi(w) + (b+\varepsilon)\eta]$$

and, in the same way,

$$\varphi(\hat{z}) \leq e^{a\eta}[\varphi(\hat{w}) + (b+\hat{\varepsilon})\eta].$$

Next, by Lemma 4.1, we see that $v_n - T_Y(s_n)w - W(s_n)Bw$ can be rewritten as

$$\sum_{k=0}^{n-1} T_Y(s_n - s_{k+1})[v_{k+1} - T_Y(s_{k+1} - s_k)v_k - W(s_{k+1} - s_k)Bv_k]$$

$$+ \sum_{k=0}^{n-1} T_Y(s_n - s_{k+1})W(s_{k+1} - s_k)[Bv_k - Bv] + W(s_n)(Bv - Bw)$$

Hence, applying this equality, we have

$$|v_n - T_Y(s_n)w - W(s_n)Bw|$$

$$\leq \sum_{k=0}^{n-1}(s_{k+1} - s_k)\varepsilon + \sum_{k=0}^{n-1}(s_{k+1} - s_k)|Bv_k - Bv| + s_n|Bv - Bw| \leq (3\varepsilon/2)s_n.$$

In the same way as above, we obtain the estimate

$$|\hat{v}_n - T_Y(s_n)\hat{w} - B\hat{w}| \leq (3\hat{\varepsilon}/2)s_n.$$

Passing to the limit as $n \to \infty$ in the above estimates, we have

$$|z - T_Y(\eta)w - W(\eta)Bw| < 2\varepsilon\eta \quad \text{and} \quad |\hat{z} - T_Y(\eta)\hat{w} - W(\eta)B\hat{w}| < 2\hat{\varepsilon}\eta.$$

These inequalities shows that (6.11a) and (6.11b) hold.

Finally, we show that the elements z and $\hat{z}$ satisfy (6.12). From (iii-a) and (v) one can deduce

$$|v_{k+1} - \hat{v}_{k+1}|$$

$$\leq |T_Y(s_{k+1} - s_k)(v_k - \hat{v}_k) + W(s_{k+1} - s_k)(Bv_k - B\hat{v}_k)|$$

$$+ |v_{k+1} - T_Y(s_{k+1} - s_k)v_k - W(s_{k+1} - s_k)Bv_k|$$

$$+ |\hat{v}_{k+1} - T_Y(s_{k+1} - s_k)\hat{v}_k - W(s_{k+1} - s_k)B\hat{v}_k|$$

$$\leq e^{\omega_\beta(s_{k+1}-s_k)}[|v_k - \hat{v}_k| + (s_{k+1}-s_k)\delta] + (s_{k+1}-s_k)(\varepsilon + \hat{\varepsilon})$$

for $0 \leq k \leq n$. Putting $\overline{\omega}_\beta = \max\{\omega_\beta, 0\}$ and summing up these inequalities side by side, we obtain

$$|v_{n+1} - \hat{v}_{n+1}| \leq e^{\omega_\beta s_{n+1}}|w - \hat{w}| + s_{n+1}e^{\overline{\omega}_\beta \eta}(\varepsilon + \hat{\varepsilon} + \delta). \tag{6.17}$$

Passing to the limit as $n \to \infty$ in (6.17) we conclude that (6.12) holds. The proof of Theorem 6.2 is now complete. □

7. Global existence for mild solutions

In this section we construct a global mild solution to (SP). To this end, we mainly employ Theorems 5.3 and 6.2. Theorem 5.3 gives *a priori* estimate concerning the uniform continuity of mild solutions to (SP). Hence we may apply a standard continuation argument for local mild solutions. We first establish a local existence theorem.

THEOREM 7.1. *Suppose that conditions* (H1), (H2) *and* (Ⅱ) *are satisfied. Let* $v \in D$, $R > 0$, $\varepsilon_0 \in (0, 1/3)$ *and* $\beta > \varphi(v)$. *Let* $M > 0$ *and* $\tau > 0$ *be such that* $|Bw| \leq M$ *for* $w \in D_\beta \cap U(v, R)$, $\tau(M+1) + \sup_{t\in[0,\tau]}|T_Y(t)v - v| \leq R$ *and* $e^{a\tau}[\varphi(v) + (b + \varepsilon_0)\tau] < \beta$. *Then there exists a unique mild solution* $u(\cdot)$ *to* (SP) *on* $[0, \tau]$ *satisfying the initial condition* $u(0) = v$ *and the growth condition* $\varphi(u(t)) \leq e^{at}[\varphi(v) + bt]$.

REMARK 7.1. It is noted that conditions (6.7), (6.8) are satisfied for ε, r, $M(v, \beta, \varepsilon)$ replaced, respectively, by ε_0, R, M.

PROOF OF THEOREM 7.1. Let $\{\varepsilon_n\}_{n\geq1}$ be any null sequence in $(0, \varepsilon_0)$. For each ε_n we apply the argument employed in the proof of Theorem 6.1 to construct decreasing sequences $\{r_i^n\}_{i=0}^{N_n-1}$, $\{\eta_i^n\}_{i=0}^{N_n-1}$ in $(0, \varepsilon_n]$, a sequence $\{t_i^n\}_{i=0}^{N_n}$ in $[0, \tau]$ and a sequence $\{v_i^n\}_{i=0}^{N_n}$ in $D_\beta \cap U(v, R)$, such that (i) through (vi-a) listed in Theorem 6.1 are valid for $\varepsilon = \varepsilon_n$ and $N = N_n$, and such that the partition $P_n = \{t_i^n\}_{i=0}^{N_n}$ of $[0, \tau]$ is finer than the previous partition $P_{n-1} = \{t_k^{n-1}\}_{k=0}^{N_{n-1}}$ of $[0, \tau]$. First, one can construct $\{r_i^1\}_{i=0}^{N_1-1}$, $\{\eta_i^1\}_{i=0}^{N_1-1}$, $\{t_i^1\}_{i=0}^{N_1}$ in $[0, \tau]$ and $\{v_i^1\}_{i=0}^{N_1}$ for $\varepsilon = \varepsilon_1$, in the same way as in the proof of Theorem 6.1. Suppose that we have constructed sequences $\{t_i^n\}_{i=0}^{N_n}$ in $[0, \tau]$ and $\{v_i^n\}_{i=0}^{N_n}$. We then construct $\{t_k^{n+1}\}_{k=0}^{N_{n+1}}$ and $\{v_k^{n+1}\}_{k=0}^{N_{n+1}}$ by setting $h_k^{n+1} \equiv \min\{\eta_k^{n+1}, t_{i+1}^n - t_k^{n+1}\}$ and $t_{k+1}^{n+1} \equiv t_k^{n+1} + h_k^{n+1}$ provided that $t_i^n \leq t_k^{n+1} < t_{i+1}^n$. It should be noted here that h_k^{n+1} is defined by taking the minimum of η_k^{n+1} and $t_{i+1}^n - t_k^{n+1}$, instead of taking the minimum of η_k^{n+1} and $\tau - t_k^{n+1}$ (as in Theorem 6.1).

In accordance with this construction we define a sequence of approximate solutions $u_n(\cdot) : [0, \tau] \to X$ by

(7.1)

$$u_n(t) \equiv \begin{cases} T_Y(t - t_i^n)v_i^n + W(t - t_i^n)Bv_i^n & \text{for } t \in [t_i^n, t_{i+1}^n) \\ & \text{and } 0 \leq i \leq N_n - 1, \\ T_Y(\tau - t_{N_n-1}^n)v_{N_n-1}^n + W(\tau - t_{N_n-1}^n)Bv_{N_n-1}^n & \text{for } t = \tau. \end{cases}$$

Then, for $t \in [t_i^n, t_{i+1}^n)$, we obtain

$$\begin{aligned}
&|u_n(t) - v_{i+1}^n| \\
&\quad \le |T_Y(t - t_i^n)v_i^n - T_Y(t_{i+1}^n - t_i^n)v_i^n| + |W(t - t_i^n)Bv_i^n - W(t_{i+1}^n - t_i^n)Bv_i^n| \\
&\qquad + |v_{i+1}^n - T_Y(t_{i+1}^n - t_i^n)v_i^n - W(t_{i+1}^n - t_i^n)Bv_i^n| \\
&\quad \le |T_Y(t_{i+1}^n - t)v_i^n - v_i^n| + (t_{i+1}^n - t)|Bv_i^n| + (t_{i+1}^n - t_i^n)\varepsilon_n \\
&\quad \le (t_{i+1}^n - t_i^n)(M+1) + |T_Y(t_{i+1}^n - t)v_i^n - v_i^n| \le \varepsilon_n
\end{aligned}$$

by (iv-a) and (vi-a) in Theorem 6.1 and (2.2). In particular, $d(u_n(t), D_\beta) \le \varepsilon_n$ for $t \in [0, \tau]$.

We then demonstrate that the sequence $u_n(\cdot)$ is uniformly convergent on $[0, \tau]$. We apply Theorem 6.2 to estimate the difference between two approximate solutions $u_n(\cdot)$ and $u_m(\cdot)$.

Let $1 \le n < m$, $t \in [0, \tau)$ and choose $0 \le i \le N_n - 1$ and $0 \le j \le N_m - 1$ such that $t \in [t_i^n, t_{i+1}^n) \cap [t_j^m, t_{j+1}^m)$, or let $t = \tau$. First, we introduce a new subdivision $\{s_l\}_{l=0}^{j+1}$ of $[0, t]$ by $s_l = t_l^m$ for $0 \le l \le j$ and $s_{j+1} = t$. We then apply Theorem 6.2 with $\delta = \varepsilon_m$ to construct the sequences $\{z_l\}_{l=0}^{j+1}$ and $\{\hat{z}_l\}_{l=0}^{j+1}$ satisfying $z_0 = \hat{z}_0 = v$ and (7.2) through (7.7) below.

If $s_l = t_k^n$, we put $v = v_k^n$, $\hat{v} = v_l^m$, $w = v$, $\hat{w} = \hat{v}$, $h = \hat{h} = 0$, $\eta = s_{l+1} - s_l$ in Theorem 6.2 and construct $z_{l+1} \in D_\beta \cap U(v_k^n, r_k^n)$ and $\hat{z}_{l+1} \in D_\beta \cap U(v_l^m, r_l^m)$ satisfying

$$\begin{aligned}
(7.2) \qquad &|z_{l+1} - T_Y(s_{l+1} - s_l)v_k^n - W(s_{l+1} - s_l)Bv_k^n| < 2(s_{l+1} - s_l)\varepsilon_n, \\
&|\hat{z}_{l+1} - T_Y(s_{l+1} - s_l)v_l^m - W(s_{l+1} - s_l)Bv_l^m| < 2(s_{l+1} - s_l)\varepsilon_m,
\end{aligned}$$

$$\begin{aligned}
(7.3) \qquad &\varphi(z_{l+1}) \le e^{a(s_{l+1} - s_l)}[\varphi(v_k^n) + (b + \varepsilon_n)(s_{l+1} - s_l)], \\
&\varphi(\hat{z}_{l+1}) \le e^{a(s_{l+1} - s_l)}[\varphi(v_l^m) + (b + \varepsilon_m)(s_{l+1} - s_l)]
\end{aligned}$$

and

$$(7.4) \qquad |z_{l+1} - \hat{z}_{l+1}| \le e^{\omega_\beta(s_{l+1} - s_l)}[|v_k^n - v_l^m| + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)].$$

If $s_l \in (t_k^n, t_{k+1}^n)$, we put $v = v_k^n$, $\hat{v} = v_l^m$, $w = z_l$, $\hat{w} = \hat{v}$, $h = s_l - t_k^n$, $\hat{h} = 0$ and $\eta = s_{l+1} - s_l$ in Theorem 6.2 and infer that $z_{l+1} \in D_\beta \cap U(v_k^n, r_k^n)$ and $\hat{z}_{l+1} \in D_\beta \cap U(v_l^m, r_l^m)$ satisfy

$$\begin{aligned}
(7.5) \qquad &|z_{l+1} - T_Y(s_{l+1} - s_l)z_l - W(s_{l+1} - s_l)Bz_l| < 2(s_{l+1} - s_l)\varepsilon_n \\
&|\hat{z}_{l+1} - T_Y(s_{l+1} - s_l)v_l^m - W(s_{l+1} - s_l)Bv_l^m| < 2(s_{l+1} - s_l)\varepsilon_m,
\end{aligned}$$

$$(7.6) \qquad \varphi(z_{l+1}) \le e^{a(s_{l+1} - s_l)}[\varphi(z_l) + (b + \varepsilon_n)(s_{l+1} - s_l)]$$

$$\varphi(\hat{z}_{l+1}) \leq e^{a(s_{l+1}-s_l)}[\varphi(v_l^m) + (b+\varepsilon_m)(s_{l+1}-s_l)]$$

and

$$|z_{l+1} - \hat{z}_{l+1}| \leq e^{\omega_\beta(s_{l+1}-s_l)}[|z_l - v_l^m| + (s_{l+1}-s_l)(\varepsilon_n + 2\varepsilon_m)]. \tag{7.7}$$

It should be noted here that in (7.5), (7.6) and (7.7) the element z_l is employed instead of the element v_k^n, and the time interval (t_k^n, t_{k+1}^n) may contain several s_l's. In order to apply Theorem 6.2 in the latter case, we show the estimates

$$|z_l - T_Y(s_l - t_k^n)v_k^n| \leq (s_l - t_k^n)(M+1) \tag{7.8}$$

and

$$\varphi(z_l) \leq e^{a(s_l - t_k^n)}[\varphi(v_k^n) + (b+\varepsilon_n)(s_l - t_k^n)] \text{ for each } s_l \in (t_k^n, t_{k+1}^n), \tag{7.9}$$

which correspond to the estimates in (6.10a).

Suppose that $t_k^n = s_{l_0}$ and $s_{l_0+1} \in (t_k^n, t_{k+1}^n)$. Then (7.2) implies

$$|z_{l_0+1} - T_Y(s_{l_0+1} - t_k^n)v_k^n - W(s_{l_0+1} - t_k^n)Bv_k^n| \leq 2(s_{l_0+1} - t_k^n)\varepsilon_n ,$$

and so

$$|z_{l_0+1} - T_Y(s_{l_0+1} - t_k^n)v_k^n| \leq (s_{l_0+1} - t_k^n)(M + 2\varepsilon_n) \leq (s_{l_0+1} - t_k^n)(M+1), \tag{7.10}$$

which implies that $z_{l_0+1} \in D_\beta \cap U(v_k^n, r_k^n)$. Also, by (7.3), we have

$$\varphi(z_{l_0+1}) \leq e^{a(s_{l_0+1} - t_k^n)}[\varphi(v_k^n) + (b+\varepsilon_n)(s_{l_0+1} - t_k^n)]. \tag{7.11}$$

Next let $s_l \in (t_k^n, t_{k+1}^n)$ and $l = l_0 + 2$. Then by (7.5) with $l = l_0 + 1$, condition (6.8) and the fact that $z_{l-1} \in D_\beta(v_k^n, r_k^n)$, we have

$$\begin{aligned}
&|z_l - T_Y(s_l - t_k^n)v_k^n| \\
&\leq |z_l - T_Y(s_l - s_{l-1})z_{l-1} - W(s_l - s_{l-1})Bz_{l-1}| + |T_Y(s_l - s_{l-1})z_{l-1} - T_Y(s_l - t_k^n)v_k^n| \\
&\quad + (s_l - s_{l-1})(|Bz_{l-1} - Bv_k^n| + |Bv_k^n|) \\
&\quad \leq 2(s_l - s_{l-1})\varepsilon_n + |z_{l-1} - T_Y(s_{l-1} - t_k^n)v_k^n| + (s_l - s_{l-1})(M + \varepsilon_n/4).
\end{aligned}$$

Hence we infer that (7.8) holds for $l = l_0 + 2$. Also, by (7.6) with $l = l_0 + 1$, we have

$$\begin{aligned}
\varphi(z_l) &\leq e^{a(s_l - s_{l-1})}[\varphi(z_{l-1}) + (b+\varepsilon_n)(s_l - s_{l-1})] \\
&\leq e^{a(s_l - t_k^n)}[\varphi(v_k^n) + (b+\varepsilon_n)(s_l - t_k^n)].
\end{aligned}$$

This shows that (7.9) is satisfied for $l = l_0 + 2$. The case for the next point $s_{l+1} \in (t_k^n, t_{k+1}^n)$ can be treated in the same way. Thus, we conclude that (7.8) and (7.9) are valid for all s_l in (t_k^n, t_{k+1}^n).

We now estimate the difference $u_m(\cdot) - u_n(\cdot)$ on $[0, \tau]$. We write

$$|u_m(t) - u_n(t)| \le |u_m(t) - \hat{z}_{j+1}| + |\hat{z}_{j+1} - z_{j+1}| + |z_{j+1} - u_n(t)|, \tag{7.12}$$

and make an estimate of each term on the right-hand side of (7.12). We begin by estimating the first term. We infer from the definition of $u_m(t)$, (7.2) or (7.5) with $l = j$ that

$$|\hat{z}_{j+1} - T_Y(t - t_j^m)v_j^m - W(t - t_j^m)Bv_j^m| \le 2(t - t_j^m)\varepsilon_m. \tag{7.13}$$

We then make an estimate of the third term. If t_j^m is a common point of P_n and P_m, then (7.2) and the definition of $u_n(t)$ imply $|z_{j+1} - u_n(t)| < 2(t - t_i^n)\varepsilon_n$ for $t \in [t_i^n, t_{i+1}^n) \cap [t_j^m, t_{j+1}^m)$. Next, we suppose that t_j^m is not a common point for P_n and P_m and then estimate $|z_{j+1} - T_Y(s_{j+1} - t_i^n)v_i^n - W(s_{j+1} - t_i^n)Bv_i^n|$ under the assumption that $s_j \in (t_i^n, t_{i+1}^n)$. In Lemma 4.1, we put $s_{j_0} = t_i^n$, $w_{j_0} = v_i^n$, $w_l = z_l$ for $l = j_0 + 1, \cdots, j$. Then we have

$$\begin{aligned}
z_{j+1} - u_n(t) &= z_{j+1} - T_Y(s_{j+1} - t_i^n)v_i^n - W(s_{j+1} - t_i^n)Bv_i^n \\
&= \sum_{l=j_0+1}^{j} T_Y(s_{j+1} - s_{l+1})[z_{l+1} - T_Y(s_{l+1} - s_l)z_l - W(s_{l+1} - s_l)Bz_l] \\
&\quad + T_Y(s_{j+1} - s_{j_0+1})[z_{j_0+1} - T_Y(s_{j_0+1} - t_i^n)v_i^n - W(s_{j_0+1} - t_i^n)Bv_i^n] \\
&\quad + \sum_{l=j_0}^{j} T_Y(s_{j+1} - s_{l+1})W(s_{l+1} - s_l)[Bz_l - Bv_i^n].
\end{aligned}$$

Applying (7.2), (7.5) and (6.8) to the above equality, we obtain

$$|z_{j+1} - u_n(t)| \le 3(s_{j+1} - t_i^n)\varepsilon_n. \tag{7.14}$$

We next make an estimate of the second term on the right-hand side of (7.12). For this purpose we apply (7.2) or (7.5) and the property (iv-a) in Theorem 6.1 to obtain the estimate

$$\begin{aligned}
|\hat{z}_l - v_l^m| &\le |\hat{z}_l - T_Y(s_l - s_{l-1})v_{l-1}^m - W(s_l - s_{l-1})Bv_{l-1}^m| \\
&\quad + |v_l^m - T_Y(s_l - s_{l-1})v_{l-1}^m - W(s_l - s_{l-1})Bv_{l-1}^m| \\
&\le 3(s_l - s_{l-1})\varepsilon_m.
\end{aligned} \tag{7.15}$$

Suppose that $[t_k^n, t_{k+1}^n] = [s_{l_0}, s_{l_1}]$ for $k = 0, \cdots, j-1$. Then, we have

$$\begin{aligned}
|z_{l_1} - v_{k+1}^n| &\le |z_{l_1} - T_Y(s_{l_1} - s_{l_0})v_k^n - W(s_{l_1} - s_{l_0})Bv_k^n| \\
&\quad + |v_{k+1}^n - T_Y(s_{l_1} - s_{l_0})v_k^n - W(s_{l_1} - s_{l_0})Bv_k^n|.
\end{aligned} \tag{7.16}$$

The second term on the right-hand side of (7.16) is estimated by Theorem 6.1 as $|v_{k+1}^n - T_Y(s_{l_1} - s_{l_0})v_k^n - W(s_{l_1} - s_{l_0})Bv_k^n| \le (t_{k+1}^n - t_k^n)\varepsilon_n$. It is seen from Lemma 4.1 that the first term is written as

$$
\begin{aligned}
z_{l_1} &- T_Y(s_{l_1} - s_{l_0})v_k^n - W(s_{l_1} - s_{l_0})Bv_k^n \\
&= \sum_{l=l_0+1}^{l_1-1} T_Y(s_{l_1} - s_{l+1})[z_{l+1} - T_Y(s_{l+1} - s_l)z_l - W(s_{l+1} - s_l)Bz_l] \\
&\quad + T_Y(s_{l_1 - s_{l_0+1}})[z_{l_0+1} - T_Y(s_{l_0+1} - t_k^n)v_k^n - W(s_{l_0+1} - t_k^n)Bv_k^n] \\
&\quad + \sum_{l=l_0}^{l_1-1} T_Y(s_{l_1} - s_{l+1})W(s_{l+1} - s_l)[Bz_l - Bv_k^n].
\end{aligned}
$$

Using this identity, (7.2), (7.5) and (6.8), we have

$$
\begin{aligned}
|z_{l_1} &- T_Y(s_{l_1} - s_{l_0})v_k^n - W(s_{l_1} - s_{l_0})Bv_k^n| \\
&\le \sum_{l=l_0+1}^{l_1-1} |z_{l+1} - T_Y(s_{l+1} - s_l)z_l - W(s_{l+1} - s_l)Bz_l| \\
&\quad + |z_{l_0+1} - T_Y(s_{l_0+1} - t_k^n)v_k^n - W(s_{l_0+1} - t_k^n)Bv_k^n| \\
&\quad + \sum_{l=l_0}^{l_1-1}(s_{l+1} - s_l)|Bz_l - Bv_k^n| \le 3(t_{k+1}^n - t_k^n)\varepsilon_n.
\end{aligned}
$$

Hence, it holds that

$$
(7.17) \qquad |z_{l_1} - v_{k+1}^n| \le 4(t_{k+1}^n - t_k^n)\varepsilon_n \text{ for } k = 0, \cdots, j-1.
$$

Finally, we estimate the second term on the right-hand side of (7.12). If s_l is a common point t_k^n of P_n and P_m, then we apply (7.4) to have

$$
\begin{aligned}
(7.18) \quad |z_{l+1} - \hat{z}_{l+1}| &\le e^{\omega_\beta t}[|v_k^n - z_l| + |z_l - \hat{z}_l| + |\hat{z}_l - v_l^m| + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)] \\
&\le e^{\omega_\beta t}[4(t_k^n - t_{k-1}^n)\varepsilon_n + |z_l - \hat{z}_l| + 3(s_l - s_{l-1})\varepsilon_m + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)].
\end{aligned}
$$

If s_l is not a common point of P_n and P_m, then we use (7.7) to obtain

$$
\begin{aligned}
(7.19) \qquad |z_{l+1} - \hat{z}_{l+1}| &\le e^{\omega_\beta t}[|z_l - \hat{z}_l| + |\hat{z}_l - x_l^m| + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)] \\
&\le e^{\omega_\beta t}[|z_l - \hat{z}_l| + 3(s_l - s_{l-1})\varepsilon_m + (s_{l+1} - s_l)(\varepsilon_n + 2\varepsilon_m)].
\end{aligned}
$$

Summing up the inequalities (7.18) and (7.19) with respect to $l = 1, \ldots, j$ and using the inequality

$$
|z_1 - \hat{z}_1| \le e^{\omega_\beta t}(s_1 - s_0)(\varepsilon_n + 2\varepsilon_m),
$$

which follows from (7.4), we obtain

(7.20) $$|z_{j+1}-\hat{z}_{j+1}| \le \sum_{l=0}^{j} e^{\omega_\beta t}(s_{l+1}-s_l)(\varepsilon_n+2\varepsilon_m) + 3\sum_{l=1}^{j} e^{\omega_\beta t}(s_l-s_{l-1})\varepsilon_m$$

$$+4\sum_{k=1}^{i} e^{\omega_\beta t}(t_k^n-t_{k-1}^n)\varepsilon_n \le e^{\omega_\beta t}[t(\varepsilon_n+2\varepsilon_m)+3\varepsilon_m t_j^m+4\varepsilon_n t_i^n].$$

This gives the desired estimate for the second term on the right-hand side of (7.12).

Combining (7.13), (7.14) and (7.20), we deduce

(7.21) $$|u_m(t)-u_n(t)| \le 2\varepsilon_m(t-t_j^m)+3\varepsilon_n(t-t_i^n)$$

$$+e^{\omega_\beta t}[(\varepsilon_n+2\varepsilon_m)t+3\varepsilon_m t_j^m+4\varepsilon_n t_i^n] \le 5\tau e^{\omega_\beta \tau}(\varepsilon_n+\varepsilon_m).$$

This means that the sequence $\{u_n(\cdot)\}$ of the approximate solutions converges uniformly on $[0,\tau]$ to some X-valued function $u(\cdot)$ on $[0,\tau]$. Since $d(u_n(t),D_\beta)\le\varepsilon_n$ as mentioned after the definition of $u_n(\cdot)$, it follows that $u(t)\in D_\beta$ for each $t\in[0,\tau]$.

The limit function $u(\cdot)$ so obtained on $[0,\tau]$ gives the desired mild solution to (SP) on $[0,\tau]$. To verify this we define a step function $\gamma_n(\cdot)$ by

(7.22) $$\gamma_n(t) \equiv \begin{cases} t_i^n & \text{for } t\in[t_i^n,t_{i+1}^n),\ 0\le i\le N_n-1 \\ t_{N_n-1}^n & \text{for } t=\tau \end{cases},$$

and an X-valued function $w_n(t)$ by

(7.23) $$w_n(t) \equiv T_Y(t)v+\lim_{\lambda\downarrow 0}\int_0^t T_Y(t-s)(I-\lambda A)^{-1}Bu_n(\gamma_n(s))ds \quad \text{for } t\in[0,\tau].$$

In view of the definition of $u_n(\cdot)$ and $\gamma_n(\cdot)$, it is seen that the function $w_n(\cdot)$ is strongly continuous on $[0,\tau]$. Suppose $t\in[t_i^n,t_{i+1}^n)$ for $i=1,2,\dots,N_n-1$. Since

$$\lim_{\lambda\downarrow 0}\int_{t_k^n}^{t_{k+1}^n} T_Y(t-\eta)(I-\lambda A)^{-1}Bv_k^n\,d\eta = W(t-t_k^n)Bv_k^n - W(t-t_{k+1}^n)Bv_k^n,$$

we may write the difference $u_n(t)-w_n(t)$ as

(7.24)

$$u_n(t)-w_n(t) = T_Y(t-t_i^n)v_i^n - T_Y(t)v + W(t-t_i^n)Bv_i^n$$

$$-\sum_{k=0}^{i-1}\lim_{\lambda\downarrow 0}\int_{t_k^n}^{t_{k+1}^n} T_Y(t-\eta)(I-\lambda A)^{-1}Bv_k^n\,d\eta - \lim_{\lambda\downarrow 0}\int_{t_i^n}^{t} T_Y(t-\eta)(I-\lambda A)^{-1}Bv_i^n\,d\eta$$

$$=\sum_{k=0}^{i-1} T_Y(t-t_{k+1}^n)[v_{k+1}^n - T_Y(t_{k+1}^n-t_k^n)v_k^n - W(t_{k+1}^n-t_k^n)Bv_k^n].$$

Now the norm of the right-hand side is bounded above by $\sum_{k=0}^{i-1}(t_{k+1}^n - t_k^n)\varepsilon_n = t_i^n\varepsilon_n$. Thus $|u_n(t) - w_n(t)| \leq t\varepsilon_n$. It is also shown in the same way that the above estimate holds for $t = \tau$. Therefore the function $w_n(\cdot)$ converges uniformly on $[0, \tau]$ to $u(\cdot)$ and it follows that $u(\cdot)$ is strongly continuous on $[0, \tau]$. Since $\gamma_n(t)$ converges uniformly to t and $u_n(\gamma_n(t))$, $u(t) \in D_\beta$ for $t \in [0, \tau]$, the continuity of B on D_β asserts that

$$Bu_n(\gamma_n(t)) \to Bu(t) \text{ as } n \to \infty \text{ uniformly on} [0, \tau].$$

One can now pass to the limit as $n \to \infty$ in (7.23) to conclude that

$$u(t) = T_Y(t)v + \lim_{\lambda \downarrow 0} \int_0^t T_Y(t-s)(I - \lambda A)^{-1} Bu(s)ds$$

holds for $t \in [0, \tau]$. Also, we have

$$\varphi(u_n(\gamma_n(t))) = \varphi(v_i^n) \leq e^{at_i^n}[\varphi(v) + (b + \varepsilon_n)t_i^n],$$

for $t \in [t_i^n, t_{i+1}^n)$, $i = 0, \ldots, N_n - 1$ and for $t = \tau$. Letting $n \to \infty$ in the above estimate and applying the lower semicontinuity of φ, we have

$$\varphi(u(t)) \leq \varliminf_{n \to \infty} \varphi(u_n(\gamma_n(t))) \leq e^{at}[\varphi(v) + bt) \text{ for } t \in [0, \tau].$$

This concludes that the limit function $u(\cdot)$ gives a unique mild solution to (SP) on $[0, \tau]$. The proof of Theorem 7.1 is complete. □

Theorem 7.1 together with the standard continuation argument gives the following global existence result.

THEOREM 7.2. *Suppose that a semilinear operator $A + B$ satisfies conditions* (H1), (H2) *and* (II). *Then for each v in D there exists a unique global mild solution $u(\cdot) \equiv u(\cdot; v)$ to* (SP) *on* $[0, \infty)$.

In view of Theorem 7.2, we may demonstrate that condition (II) implies the assertion (I) in our main result, Theorem 3.1. Let $v \in D$ and let $u(\cdot; v)$ be the associated mild solution to (SP) on $[0, \infty)$ given by Theorem 7.2. For any $t \geq 0$, we define an operator $S(t)$ from D into itself by $S(t)v = u(t; v)$, $v \in D$. Then it is seen that the family $\mathcal{S} \equiv \{S(t); t \geq 0\}$ forms a semigroup on D satisfying conditions (II.a), (II.b) and (II.c). This shows that (II) implies (I). Consequently, the proof of Theorem 3.1 in the non-convex case is now complete.

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