Nihonkai Math. J.
Vol.13(2002), 159-166

SOME EXTENSIONS OF GRÜSS' INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. (Discrete) Grüss' inequality, a complement of Čebyšev's inequality, is one which gives an upper bound of the absolute difference

$$\left|\frac{1}{n}\sum_{k=1}^{n} a_k b_k - \frac{1}{n^2}\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k\right|$$

for n-tuples $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of real numbers with certain conditions.

We give some extensions of Grüss' inequality by using certain convex functions. As an application, we show another weighted Ozeki's inequality which is a complement of the Cauchy-Schwartz inequality.

1. INTRODUCTION

Čebyšev's inequality

$$\frac{1}{n}\sum_{k=1}^{n} a_k b_k \geq \frac{1}{n^2}\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \tag{1.1}$$

for n-tuples $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of positive numbers with nonincreasing (or nondecreasing) order is well known. As a complement of this inequality, under the condition

$$\begin{aligned} 0 < m_1 < M_1,\ 0 < m_2 < M_2,\ m_1 \leq a_k \leq M_1 \\ \text{and} \quad m_2 \leq b_k \leq M_2 \quad (k = 1, \ldots, n), \end{aligned} \tag{1.2}$$

the following (discrete) Grüss' inequality [4] (cf. [7, p. 296]) holds:

$$\left|\frac{1}{n}\sum_{k=1}^{n} a_k b_k - \frac{1}{n^2}\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k\right| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2). \tag{1.3}$$

2000 *Mathematics Subject Classification.* 47A63.
Key words and phrases. Čebyšev's inequality, Grüss' inequality, Cauchy-Schwartz inequality, Ozeki's inequality.

A refinement of the above inequality due to M. Biernacki, H. Pidek and C. Ryll-Nardzewski [2] (cf. [7, p. 299]) is

$$\left|\frac{1}{n}\sum_{k=1}^{n} a_k b_k - \frac{1}{n^2}\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k\right| \le \frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)(M_1 - m_1)(M_2 - m_2). \tag{1.4}$$

There are a number of further (discrete or integral type) refinements and generalizations of Grüss' inequality; D. Andrica and C. Badea [1], G. T. Cargo and O. Shisha [3], J. E. Pečarić [9], [10] and etc.

In somewhat similar fashion as Grüss' inequality, the following Ozeki's inequality [8], [5] holds, as a complement of the Cauchy-Schwartz inequality: Under the condition (1.2)

$$\frac{1}{n}\sum_{k=1}^{n} a_k^2 \, \frac{1}{n}\sum_{k=1}^{n} b_k^2 - \left(\frac{1}{n}\sum_{k=1}^{n} a_k b_k\right)^2 \le \frac{1}{3}\,(M_1 M_2 - m_1 m_2)^2. \tag{1.5}$$

Recently some extensions of this inequality were given in [5], [6].

In this paper we give some extensions of Grüss' inequality by using convex functions, and show some refinements of Ozeki's inequality as applications.

2. Grüss' inequalities for convex functions

An n-tuple $a = (a_1, \ldots, a_n)$ with $m \le a_k \le M$ $(k = 1, \ldots, n)$ for $m < M$ is considered as a point in the n-dimensional cube $[m, M]^n$. Related to extreme points of the sets of monotonically ordered points in $[m, M]^n$, it is not difficult to see the following:

Lemma 2.1. *Let*

$$K = \{(a_1, \ldots, a_n) \in [m, M]^n; a_1 \le \cdots \le a_n\}$$

and

$$L = \{(a_1, \ldots, a_n) \in [m, M]^n; a_1 \ge \cdots \ge a_n\}.$$

Then both K and L are convex subsets, and their extreme points are vertices of the cube $[m, M]^n$.

An n-tuple $p = (p_1, \ldots, p_n)$ is called an n-weight if it satisfies

$$p_1, \ldots, p_n \ge 0 \quad \text{and} \quad \sum_{k=1}^{n} p_k = 1. \tag{2.1}$$

Put

$$P_l = \sum_{k=1}^{l} p_k \quad (l = 1, \ldots, n) \tag{2.2}$$

for an n-weight $p = (p_1, \ldots, p_n)$. For convenience, we write $I_n = \{1, ..., n\}$ and

$$\Delta = \{(i,j) \in I_n \times I_n; i < j\}. \tag{2.3}$$

The following lemma is a key point in this paper:

Lemma 2.2. *Let $a = (a_1, \ldots, a_n)$ be an n-tuple of real numbers satisfying*

$$M \geq a_1 \geq \cdots \geq a_n \geq m.$$

Then for any n-weight $p = (p_1, \ldots, p_n)$

$$\begin{aligned} D(a;p) := &\sum_{(i,j)\in\Delta} p_i p_j (a_i - a_j) \\ &\leq (M-m) \max_{1\leq k\leq n-1} P_k(1-P_k) \ \left(\leq \frac{1}{4}(M-m)\right). \end{aligned} \tag{2.4}$$

Proof. Since $D_p(a) = D(a;p)$ is a linear (hence convex) function on $L \subset [m, M]^n$, it follows from Lemma 2.1 that its maximum is attained at an extreme point of L, i.e., a vertex of $[m, M]^n$. Hence we may consider the values of $D_p(a)$ only for $a = a^{(l)}$, $l = 1, \ldots, n-1$, where

$$a^{(l)} = (\overbrace{M, \ldots, M}^{l}, \overbrace{m, \ldots, m}^{n-l}).$$

Then we have

$$\begin{aligned} D[l] := D_p(a^{(l)}) = &\sum_{(i,j)\in(I_l\times I_l^c)} p_i p_j (M-m) = (M-m) \sum_{i=1}^{l} \sum_{j=l+1}^{n} p_i p_j \\ &= (M-m) P_l (1-P_l) \leq (M-m) \max_{1\leq k\leq n-1} P_k(1-P_k). \end{aligned}$$

Since $P_k(1-P_k) \leq \frac{1}{4}$, we see that $D[l] \leq \frac{1}{4}(M-m)$. □

Now extending the notion of the cumulative sum (2.2) for an n-weight $p = (p_1, \ldots, p_n)$, we put

$$P(J) = \sum_{k\in J} p_k \quad \text{for } J \subset I_n.$$

We then have the following theorem which is regarded as an extension of Grüss' inequality.

Theorem 2.3. *Let $f(x)$ be a convex even function defined on $[m-M, M-m]$ $(0 < m < M)$ with $f(0) = 0$. Then for each n-tuple $a = (a_1, \ldots, a_n)$ satisfying $m \le a_k \le M$ $(k = 1, \ldots, n)$ and for each n-weight $p = (p_1, \ldots, p_n)$*

$$\begin{aligned} D_f(a;p) :&= \sum_{(i,j)\in\Delta} p_i p_j f(a_i - a_j) \\ &\le f(M-m) \max_{J\subset I_n} P(J)(1-P(J)). \end{aligned} \tag{2.5}$$

Proof. First note that by the assumptions on $f(x)$

$$D_f(a;p) = \frac{1}{2} \sum_{i,j\in I_n} p_i p_j f(a_i - a_j).$$

Furthermore, $D_{f,p}(a) = D_f(a;p)$ is a convex function on $[m, M]^n$. Hence it attains its maximum at a vertex of $[m, M]^n$. Let a be a vertex and put

$$J_a = \{k \in I_n; a_k = M\}.$$

Then since $f(m-M) = f(M-m)$, $f(0) = 0$ and

$$\sum_{(i,j)\in J_a\times J_a^c} p_i p_j = \sum_{(i,j)\in J_a^c\times J_a} p_i p_j = P(J_a)(1-P(J_a)),$$

we have

$$\begin{aligned} D_{f,p}(a) &= \frac{1}{2}\left\{ \sum_{(i,j)\in J_a\times J_a^c} p_i p_j f(M-m) + \sum_{(i,j)\in J_a^c\times J_a} p_i p_j f(m-M) \right\} \\ &= P(J_a)(1-P(J_a)) f(M-m) \\ &\le f(M-m) \max_{J\subset I_n} P(J)(1-P(J)) \ \left(\le \frac{1}{4} f(M-m)\right). \end{aligned}$$

□

Applying the above theorem to the functions $f(x) = |x|$ and x^2, we obtain the following two facts.

Corollary 2.4. ([3, Lemma 4.1]) *For any n-tuple a with the same assumptions as in Theorem 2.3 and for any n-weight p*

$$\sum_{(i,j)\in\Delta} p_i p_j |a_i - a_j| \le (M-m) \max_{J\subset I_n} P(J)(1-P(J)) \ \left(\le \frac{1}{4}(M-m)\right).$$

Corollary 2.5. ([1, Lemma]) *For any n-tuple a with the same assumptions as in Theorem* 2.3 *and for any n-weight p*

$$\sum_{(i,j)\in\Delta} p_i p_j (a_i - a_j)^2 \le (M-m)^2 \max_{J\subset I_n} P(J)(1-P(J)) \ \left(\le \frac{1}{4}(M-m)^2\right).$$

Now for two n-tuples $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, put

$$D(a,b;p) = \sum_{k=1}^{n} p_k a_k b_k - \sum_{k=1}^{n} p_k a_k \sum_{k=1}^{n} p_k b_k, \tag{2.6}$$

which is the difference derived from weighted Čebyšev's inequality. Then note that

$$D(a,b;p) = \sum_{(i,j)\in\Delta} p_i p_j (a_i - a_j)(b_i - b_j) \tag{2.7}$$

holds, as a weighted version of Korkine's identity [7, p. 242]. Applying Corollary 2.4, we obtain, by a short proof, the following generalization of (1.4) due to Andrica and Badea [1]:

Corollary 2.6. ([1, Theorem 2]) *Let a and b be n-tuples satisfying* (1.2). *Then for any n-weight p*

$$|D(a,b;p)| \le (M_1 - m_1)(M_2 - m_2) \max_{J\subset I_n} P(J)(1-P(J)). \tag{2.8}$$

Proof. It follows from (2.7) and Corollary 2.4 that

$$\begin{aligned}
|D(a,b;p)| &= \Big|\sum_{(i,j)\in\Delta} p_i p_j (a_i - a_j)(b_i - b_j)\Big| \\
&\le \sum_{(i,j)\in\Delta} p_i p_j |a_i - a_j||b_i - b_j| \le (M_2 - m_2) \sum_{(i,j)\in\Delta} p_i p_j |a_i - a_j| \\
&\le (M_1 - m_1)(M_2 - m_2) \max_{J\subset I_n} P(J)(1-P(J)).
\end{aligned}$$

□

Applying (2.7) and Lemma 2.2, we obtain the following corollary which is an improvement of [9, Theorem 8]:

Corollary 2.7. *Let a and b be n-tuples satisfying* (1.2), *and furthermore assume that a is monotonically decreasing* (*or increasing*). *Then for any n-weight p*

$$|D(a,b;p)| \le (M_1 - m_1)(M_2 - m_2) \max_{1\le k\le n-1} P_k(1-P_k). \tag{2.9}$$

3. Applications to Ozeki's inequality

Recently, as a refinement of Ozeki's inequality (1.5), we gave the following result [6, Theorem 3.2]: For each n-weight p

$$\begin{aligned} D_2(a,b;p) &:= \sum_{k=1}^{n} p_k a_k^2 \sum_{k=1}^{n} p_k b_k^2 - \left(\sum_{k=1}^{n} p_k a_k b_k\right)^2 \\ &\le M_1^2 M_2^2 \max_{J\subset I_n} \left\{ \frac{(1-\alpha\beta)^2}{4}(1-P(J))^2 + (1-\beta)^2 P(J)(1-P(J)) \right\}, \end{aligned} \tag{3.1}$$

where $\alpha = m_1/M_1$, $\beta = m_2/M_2$ and $\alpha \ge \beta$ is assumed.

In this section we discuss some applications of Corollaries 2.5 and 2.7, by which we simplify weighted Ozeki's inequalities.

Theorem 3.1. *Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be n-tuples satisfying the condition* (1.2). *Assume that $\alpha = m_1/M_1 \ge m_2/M_2 = \beta$. Then for any n-weight $p = (p_1, \dots, p_n)$*

$$D_2(a,b;p) \le \frac{M_1^2 M_2^2 (1-\alpha\beta)^2}{\alpha^2} \max_{J\subset I_n} P(J)(1-P(J)). \tag{3.2}$$

Proof. First note that

$$D_2(a,b;p) = \sum_{(i,j)\in\Delta} p_i p_j (a_i b_j - a_j b_i)^2$$

holds as a weighted version of Lagrange's formula (cf. [7, p. 84]). Put $c_k = b_k/a_k$ $(k = 1, \dots, n)$. Then $m_2/M_1 \le c_k \le M_2/m_1$, so that Corollary 2.5 implies

$$\begin{aligned} D_2(a,b;p) &= \sum_{(i,j)\in\Delta} p_i p_j a_i^2 a_j^2 (c_i - c_j)^2 \\ &\le M_1^4 \left(\frac{M_2}{m_1} - \frac{m_2}{M_1}\right)^2 \max_{J\subset I_n} P(J)(1-P(J)) \qquad (3.3) \\ &= \frac{M_1^2 M_2^2 (1-\alpha\beta)^2}{\alpha^2} \max_{J\subset I_n} P(J)(1-P(J)). \end{aligned}$$

□

Remark. Theorem 3.1 is another weighted version of Ozeki's inequality. In fact, put A, B and C the right-sides of Ozeki's inequality (1.5), of the above inequality (3.1) and of the one (3.2) in Theorem (3.1), respectively. For convenience, assume that $M_1 = M_2 = 1$. Then:

(i) Since $P(J)(1-P(J)) \leq 1/4$, we see that if $\alpha^2 \geq 3/4$ $(\alpha \geq \beta)$ then

$$A = \frac{(1-\alpha\beta)^2}{3} \geq \frac{(1-\alpha\beta)^2}{4\alpha^2} \geq \frac{(1-\alpha\beta)^2}{\alpha^2} P(J)(1-P(J)).$$

Hence $A \geq C$. This implies that in this case Theorem 3.1 is a refinement of Ozeki's inequality (1.5).

(ii) As for the relation between B and C, first it is easy to see that if α is sufficiently near 0 then $B \leq C$. Indeed, for $a = (1,1,1/4)$ and $b = (1/4,1,1)$, and $p_i = 1/3$ $(i = 1,2,3)$ we have $B = 0.2226... < 3.125 = C$. Next since

$$\begin{aligned}\lim_{\alpha \to 1} B &= \max_{J \subset I_n} \left\{ \frac{(1-\beta)^2}{4}(1-P(J))^2 + (1-\beta)^2 P(J)(1-P(J)) \right\} \\ &\geq (1-\beta)^2 \max_{J \subset I_n} P(J)(1-P(J)) = \lim_{\alpha \to 1} C,\end{aligned}$$

we see that $B \geq C$ for α sufficiently close to 1. Indeed, for $a = (1,1,0.9)$ and $b = (0.2,1,1)$, and $p_i = 1/3$, we have $B = 0.2169... > 0.1844... = C$.

We believe that Ozeki's inequality was originally represented in the following form:

Theorem 3.2. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be n-tuples satisfying*

$$m_1 \leq a_1 \leq \cdots \leq a_n \leq M_1 \quad \text{and} \quad m_2 \leq b_1 \leq \cdots \leq b_n \leq M_2.$$

Then for any n-weight $p = (p_1, \ldots, p_n)$

$$\begin{aligned}D_2(a,b;p) &\leq (M_1M_2 - m_1m_2)^2 \max_{1 \leq k \leq n-1} P_k(1-P_k) \\ &\left(\leq \frac{1}{4}(M_1M_2 - m_1m_2)^2 \right).\end{aligned} \tag{3.4}$$

Proof. First by weighted Čebyšev's inequality [7, p. 240] we have

$$\sum_{k=1}^n p_k a_k^2 \sum_{k=1}^n p_k b_k^2 - \sum_{k=1}^n p_k a_k^2 b_k^2 \leq 0. \tag{3.5}$$

Next put $c = (a_1b_1, \ldots, a_nb_n)$, then c is monotonically increasing and $m_1m_2 \leq c_k \leq M_1M_2$. Replacing both a and b by the same c in (2.9) of Corollary 2.7, we have

$$\begin{aligned}&\sum_{k=1}^n p_k a_k^2 b_k^2 - \left(\sum_{k=1}^n p_k a_k b_k \right)^2 \\ &\qquad (= D(c,c;p)) \leq (M_1M_2 - m_1m_2)^2 \max_{1 \leq k \leq n-1} P_k(1-P_k).\end{aligned} \tag{3.6}$$

Adding the inequalities (3.5) and (3.6), we obtain the desired inequality. □

In [6, Theorem 5.1], considering a close relation between $\alpha = m_1/M_1$ and $\beta = m_2/M_2$, we gave rather a complicated estimation of $D_2(a, b; p)$ ([6, Theorem 5.1]) with the same assumptions as in Theorem 3.2.

Acknowledgment. We would like to express our cordial thanks to the referee for nice advice to our first version.

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Received March 15, 2002 Revised August 28, 2002