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Group von Neumann algebras associated with locally compact groups

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Abstract

The relation between two semi-finite von Neumann algebras associated with a locally compact group is investigated.

1 Introduction

Let G be a locally compact group. Then it is known that there exists an action α of G on the additive group $\mathbf{R}$ of real numbers such that the semi-direct product $\mathbf{R} \times_\alpha G$ is unimodular. Hence the group von Neumann algebra $\lambda(\mathbf{R} \times_\alpha G)$ generated by the left regular representation of $\mathbf{R} \times_\alpha G$ is semi-finite. On the other hand, using the modular action σ on the group von Neumann algebra $\lambda(G)$ generated by the left regular representation of G, we also have a semi-finite von Neumann algebra $\lambda(G) \times_\sigma \mathbf{R}$, the crossed product of $\lambda(G)$ by σ.

In this note, we shall compare the structures of these two semi-finite von Neumann algebras.

2 Results

We first recall some basic facts on locally compact groups and von Neumann algebras (see standard text-books in the reference).

Let G be a locally compact group with a fixed left invariant Haar measure and let Δ be the modular function of G. If we define an action α on the additive group $\mathbf{R}$ of real numbers by

$$\alpha_g(t) = \Delta(g)\, t,\ t \in \mathbf{R},\ g \in G,$$

then the semi-direct product $\mathbf{R} \times_\alpha G$ is unimodular, so that the group von Neumann algebra $\lambda(\mathbf{R} \times_\alpha G)$ generated by the left regular representation of $\mathbf{R} \times_\alpha G$ is semi-finite.

On the other hand, let $\lambda(G)$ be the group von Neumann algebra generated by the left regular representation of G and take a natural weight φ on $\lambda(G)$ such that the modular operator Δ_φ arising from φ is given by

$$(\Delta_\varphi^{it}\xi)(g) = \Delta(g)^{it}\xi(g),\ \xi \in L^2(G),\ g \in G,\ t \in \mathbf{R}.$$

Then the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbf{R}}$ on $\lambda(G)$ is computed by

$$\sigma_t^\varphi(\lambda_g) = \Delta(g)^{it}\lambda_g,\ g \in G,\ t \in \mathbf{R},$$

where λ_g is the left translation operator by g on $L^2(G)$ defined by

$$(\lambda_g \xi)(h) = \xi(g^{-1}h),\ \xi \in L^2(G),\ g, h \in G,$$

and it is known that the crossed product $\lambda(G) \times_{\sigma^\varphi} \mathbf{R}$ of $\lambda(G)$ by σ^φ is semi-finite.

In order to fix notation used later, λ_t and χ_t $(t \in \mathbf{R})$ denote unitary operators on $L^2(\mathbf{R})$ defined by

$$\begin{array}{l} (\lambda_t f)(s) = f(s-t) \\ (\chi_t f)(s) = e^{-ist} f(s) \end{array},\ f \in L^2(\mathbf{R}),$$

respectively, and $\mathcal{F}$ means the Fourier transformation on $L^2(\mathbf{R})$ given by

$$(\mathcal{F}f)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f(t) dt,\ f \in L^2(\mathbf{R}).$$

We note the commutation relation

$$\chi_s \lambda_t = e^{-ist} \lambda_t \chi_s,\ s, t \in \mathbf{R}.$$

Proposition 1 *$\lambda(\mathbf{R} \times_\alpha G)$ is isomorphic to $L^\infty(\mathbf{R}) \times_{\tilde{\alpha}} G$, where the action $\tilde{\alpha}$ of G on $L^\infty(\mathbf{R})$ is given by*

$$(\tilde{\alpha}_g(f))(t) = f(\Delta(g)\, t),\ f \in L^\infty(\mathbf{R}),\ t \in \mathbf{R},\ g \in G.$$

Proof. Since

$$\mathcal{F} \lambda_t \mathcal{F}^* = \chi_t,\ t \in \mathbf{R},$$

we have

$$\begin{aligned} \lambda(\mathbf{R} \times_\alpha G) &\cong \lambda(\mathbf{R}) \times_{\bar{\alpha}} G \qquad (\bar{\alpha}_g(\lambda_t) = \lambda_{\Delta(g)t}) \\ &\cong L^\infty(\mathbf{R}) \times_{\tilde{\alpha}} G \qquad (\tilde{\alpha}_g(\chi_t) = \chi_{\Delta(g)t}). \end{aligned}$$

q.e.d.

Proposition 2 *$\lambda(G) \times_{\sigma^\varphi} \mathbf{R}$ is isomorphic to $L^\infty(\mathbf{R}) \times_\beta G$, where the action β of G on $L^\infty(\mathbf{R})$ is given by*

$$(\beta_g(f))(t) = f(t + \log \Delta(g)),\ f \in L^\infty(\mathbf{R}),\ t \in \mathbf{R},\ g \in G.$$

Proof. It is straightforward to see that

$$\lambda(G) \times_{\sigma^\varphi} \mathbf{R} \cong \{\lambda_g \otimes 1,\ \Delta_\varphi^{it} \otimes \lambda_t \mid g \in G,\ t \in \mathbf{R}\}''.$$

If we define a unitary operator W on $L^2(G) \otimes L^2(\mathbf{R})$ by

$$(W\xi)(g, t) = \Delta(g)^{-it} \xi(g, t),\ \xi \in L^2(G) \otimes L^2(\mathbf{R}),$$

then we get

$$\begin{aligned} W(\lambda_g \otimes 1)W^* &= \lambda_g \otimes \chi_{\log \Delta(g)},\ g \in G, \\ W(\Delta_\varphi^{it} \otimes \lambda_t)W^* &= 1 \otimes \lambda_t,\ t \in \mathbf{R}. \end{aligned}$$

Thus we obtain

$$\begin{aligned}
& \{\lambda_g \otimes 1,\ \Delta_\varphi^{it} \otimes \lambda_t \mid g \in G,\ t \in \mathbf{R}\}'' \\
\cong\ & \{\lambda_g \otimes \chi_{\log \Delta(g)},\ 1 \otimes \lambda_t \mid g \in G,\ t \in \mathbf{R}\}'' \\
\cong\ & \{\lambda_t \otimes 1,\ \chi_{\log \Delta(g)} \otimes \lambda_g \mid t \in \mathbf{R},\ g \in G\}'' \\
\cong\ & \{\chi_t \otimes 1,\ \lambda_{-\log \Delta(g)} \otimes \lambda_g \mid t \in \mathbf{R},\ g \in G\}''.
\end{aligned}$$

Here the second isomorphism follows from the flip and the third is given by the Fourier transformation. Since

$$\lambda_{-\log \Delta(g)} \chi_t \lambda^*_{-\log \Delta(g)} = e^{-it \log \Delta(g)} \chi_t,\ t \in \mathbf{R},\ g \in G,$$

we have the conclusion.

q.e.d.

Corollary 3

$$\lambda(\mathbf{R} \times_\alpha G) \cong (\lambda(G) \times_{\sigma^\varphi} \mathbf{R}) \oplus (\lambda(G) \times_{\sigma^\varphi} \mathbf{R}).$$

Proof.

$$\begin{aligned}
\lambda(\mathbf{R} \times_\alpha G) &\cong L^\infty(\mathbf{R}) \times_{\tilde{\alpha}} G \\
&\cong (L^\infty(-\infty, 0) \oplus L^\infty(0, \infty)) \times_{\tilde{\alpha}} G \\
&\cong (L^\infty(-\infty, 0) \times_{\tilde{\alpha}} G) \oplus (L^\infty(0, \infty) \times_{\tilde{\alpha}} G) \\
&\cong (L^\infty(\mathbf{R}) \times_\beta G) \oplus (L^\infty(\mathbf{R}) \times_\beta G) \\
&\cong (\lambda(G) \times_{\sigma^\varphi} \mathbf{R}) \oplus (\lambda(G) \times_{\sigma^\varphi} \mathbf{R}).
\end{aligned}$$

q.e.d.

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