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OPERATOR INEQUALITIES RELATED TO THE HEINZ-KATO-FURUTA INEQUALITY

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ABSTRACT. The Heinz-Kato-Furuta inequality; if A and B are positive operators on H satisfying $T^*T \le A^2$ and $TT^* \le B^2$ for a given operator T on H, then

$$|\langle T|T|^{\alpha+\beta-1}x, y\rangle| \le \| A^\alpha x \| \| B^\beta y \|$$

for all $\alpha, \beta \in [0,1]$ with $\alpha+\beta \ge 1$ and $x, y \in H$, has several extensions and improvements. On the other hand, the Bernstein inequality for selfadjoint operators is generalized for arbitrary operators. Recently we gave it Bessel type extensions. So we try to give a simultaneous Bessel type extension of the Heinz-Kato-Furuta inequality and the Bernstein inequality. As an application of the Furuta inequality, we obtain Furuta type extension of them. Moreover it is considered under the chaotic order, i.e., $\log A \ge \log B$ for positive invertible operators A and B. Finally we discuss a simultaneous extension of the Heinz-Kato-Furuta inequality and the Selberg inequality.

1. INTRODUCTION.

In what follows, an operator means a bounded linear one acting on a complex Hilbert space H. An operator T is positive, denoted by $T \ge 0$, if $\langle Tx, x\rangle \ge 0$ for all $x \in H$. The order $S \le T$ means that S and T are selfadjoint operators and $S - T$ is positive. Let $T = U|T|$ be the polar decomposition of T on H in the below.

We first cite the Heinz-Kato-Furuta inequality [20], [21] which is shown by a generalized Schwarz inequality via the Löwner-Heinz inequality:

The Heinz-Kato-Furuta inequality. *Let T be an operator on H. If A and B are positive operators on H such that $T^*T \le A^2$ and $TT^* \le B^2$, then*

$$|\langle T|T|^{\alpha+\beta-1}x, y\rangle| \le \| A^\alpha x \| \| B^\beta y \| \tag{1.1}$$

holds for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \ge 1$ and $x, y \in H$.

In the below, we call it the HKF inequality. We here remark that the Heinz-Kato inequality is just the case $\alpha + \beta = 1$ in above, cf. [4]. In [12, Theorem 2], we proposed the following improvement of the HKF inequality and gave conditions under which the equality holds:

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Theorem A. *Let T be an operator on H. If A and B are positive operators on H such that $T^*T \leq A^2$ and $TT^* \leq B^2$, then for each $x \in H$*

$$|\langle T|T|^{\alpha+\beta-1}x, y\rangle|^2 + \frac{|\langle |T|^{2\alpha}x, z\rangle|^2 \parallel |T^*|^{\beta}y \parallel^2}{\parallel |T|^{\alpha}z \parallel^2} \leq \parallel A^{\alpha}x \parallel^2 \parallel B^{\beta}y \parallel^2 \tag{1.2}$$

for all $\alpha, \beta \in [0,1]$ with $\alpha+\beta \geq 1$ and $y, z \in H$ such that $y \neq 0$, $T|T|^{\alpha+\beta-1}z \neq 0$ and $\langle T|T|^{\alpha+\beta-1}z, y\rangle = 0$. In the case $\alpha, \beta > 0$, the equality in (1.2) holds if and only if $A^{2\alpha}x = |T|^{2\alpha}x$, $B^{2\beta}y = |T^*|^{2\beta}y$ and $|T|^{\alpha+\beta-1}T^*y$ and $|T|^{2\alpha}(x - \frac{\langle |T|^{2\alpha}x, z\rangle}{\||T|^{\alpha}z\|^2}z)$ are proportional; the third condition is equivalent to the condition that $|T^*|^{2\beta}y$ and $T|T|^{\alpha+\beta-1}(x - \frac{\langle |T|^{2\alpha}x, z\rangle}{\||T|^{\alpha}z\|^2}z)$ are proportional.

It is easily seen that (1.2) is just Lin's result [24] in the case of $\alpha + \beta = 1$.

We recall the Bernstein inequality [1, p.319] which is used in testing convergence of eigenvector calculations:

The Bernstein inequality. *Let S be a selfadjoint operator on H. If e is a unit eigenvector corresponding to an eigenvalue λ of S, then*

$$|\langle x, e\rangle|^2 \leq \frac{\parallel x \parallel^2 \parallel Sx \parallel^2 - \langle x, Sx\rangle^2}{\parallel (S-\lambda)x \parallel^2} \tag{1.3}$$

for all $x \in H$ for which $Sx \neq \lambda x$.

It was extended to nonnormal operators, precisely dominant operators by Furuta [17] and moreover operators with normal eigenvalues [6]. Afterwards we pointed out that eigenvalues and its corresponding eigenvectors of adjoint operators are essential in this discussion [14], and that it can be regarded as an extension of Bessel type inequality [10, Theorem 1] as follows:

Theorem B. *Let S be an operator on H and e_i be a unit eigenvector corresponding to an eigenvalue $\bar{\lambda}_i$ of S^* for $i = 1, 2, \ldots, n$. Then for each $x \in H$ with $\prod_{i=1}^n (S-\lambda_i)x \neq 0$*

$$\sum_{i=1}^n |\langle u_{i-1}, e_i\rangle|^2 \leq \frac{\parallel x \parallel^2 \parallel \prod_{i=1}^n (S-\lambda_i)x \parallel^2 - |\langle x, \prod_{i=1}^n (S-\lambda_i)x\rangle|^2}{\parallel \prod_{i=1}^n (S-\lambda_i)x \parallel^2} \tag{1.4}$$

for $u_i = u_{i-1} - \langle u_{i-1}, e_i\rangle e_i$ with $u_0 = x$ for $i = 1, \ldots, n$.

In particular, if $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal set, then

$$\sum_{i=1}^n |\langle x, e_i\rangle|^2 \leq \frac{\parallel x \parallel^2 \parallel \prod_{i=1}^n (S-\lambda_i)x \parallel^2 - |\langle x, \prod_{i=1}^n (S-\lambda_i)x\rangle|^2}{\parallel \prod_{i=1}^n (S-\lambda_i)x \parallel^2}. \tag{1.5}$$

In [20], further extensions of the HKF inequality was obtained by the Furuta inequality [16].

Finally we cite the Selberg inequality which is a generalization of the Bessel inequality, [23], cf. [19], [15]:

The Selberg inequality. *If $z_1, z_2, \dots, z_n$ are nonzero vectors in H, then for each $x \in H$*

$$\sum_{i=1}^{n} \frac{|\langle x, z_i\rangle|^2}{\sum_{j=1}^{n} |\langle z_i, z_j\rangle|} \leq \| x \|^2 . \tag{1.6}$$

In [11, Theorem 4], we proposed a simultaneous extension of the HKF inequality and the Selberg inequality. Moreover we showed the further extension by the Furuta inequality.

2. Extensions of the HKF and Bernstein inequalities.

In this section, we give a simultaneous extension of Theorems A and B which will be mentioned in Theorem 2.3. For this, we present the following inequality along with [24, Theorem 4]. We denote by $\ker T$ the kernel of T.

Theorem 2.1. *Let T be an operator on H. Then for each $x, y \in H$*

$$|\langle T|T|^{\alpha+\beta-1}x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle |T|^{2\alpha}u_{i-1}, z_i\rangle|^2 \, \| \, |T^*|^{\beta} y \, \|^2}{\| \, |T|^{\alpha} z_i \, \|^2} \leq \langle |T|^{2\alpha}x, x\rangle \langle |T^*|^{2\beta}y, y\rangle \tag{2.1}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and $z_1, \dots, z_n \notin \ker T$ such that $\langle T|T|^{\alpha+\beta-1}z_i, y\rangle = 0$, where $u_i = u_{i-1} - \frac{\langle |T|^{2\alpha}u_{i-1}, z_i\rangle}{\| |T|^{\alpha} z_i \|^2} z_i$ with $u_0 = x$ for $i = 1, 2, \dots, n$. In the case $\alpha, \beta > 0$, the equality in (2.1) holds if and only if $|T^*|^{\beta}y$ and $U|T|^{\alpha}u_n$ are proportional.

Proof. Noting that $\langle T|T|^{\alpha+\beta-1}x, y\rangle = \langle U|T|^{\alpha}x, |T^*|^{\beta}y\rangle$ for all $x, y \in H$. Hence we have

$$\begin{aligned}
\langle U|T|^{\alpha}u_n, |T^*|^{\beta}y\rangle &= \langle U|T|^{\alpha}u_{n-1}, |T^*|^{\beta}y\rangle - \frac{\langle |T|^{2\alpha}u_{n-1}, z_n\rangle}{\| \, |T|^{\alpha}z_n \, \|^2} \langle U|T|^{\alpha}z_n, |T^*|^{\beta}y\rangle \\
&= \langle U|T|^{\alpha}u_{n-1}, |T^*|^{\beta}y\rangle = \cdots = \langle U|T|^{\alpha}x, |T^*|^{\beta}y\rangle \\
&= \langle T|T|^{\alpha+\beta-1}x, y\rangle,
\end{aligned}$$

and

$$\| \, |T|^{\alpha}u_n \, \|^2 = \| \, |T|^{\alpha}u_{n-1} \, \|^2 - \frac{|\langle |T|^{2\alpha}u_{n-1}, z_n\rangle|^2}{\| \, |T|^{\alpha}z_n \, \|^2} = \cdots = \| \, |T|^{\alpha}x \, \|^2 - \sum_{i=1}^{n} \frac{|\langle |T|^{2\alpha}u_{i-1}, z_i\rangle|^2}{\| \, |T|^{\alpha}z_i \, \|^2}$$

by the definition of u_i and $\langle T|T|^{\alpha+\beta-1}y, z_i\rangle = 0$. Hence we have by the Schwarz inequality

$$\begin{aligned}
|\langle T|T|^{\alpha+\beta-1}x, y\rangle|^2 &= |\langle U|T|^{\alpha}u_n, |T^*|^{\beta}y\rangle|^2 \\
&\leq \| \, U|T|^{\alpha}u_n \, \|^2 \| \, |T^*|^{\beta}y \, \|^2 = \| \, |T|^{\alpha}u_n \, \|^2 \| \, |T^*|^{\beta}y \, \|^2 \\
&= \| \, |T|^{\alpha}x \, \|^2 \| \, |T^*|^{\beta}y \, \|^2 - \sum_{i=1}^{n} \frac{|\langle |T|^{2\alpha}u_{i-1}, z_i\rangle|^2}{\| \, |T|^{\alpha}z_i \, \|^2} \| \, |T^*|^{\beta}y \, \|^2,
\end{aligned}$$

so we obtain the desired inequality (2.1). The equality condition is obvious by seeing the only inequality in the above. □

Theorem 2.1 is an extension of [12, Theorem 1] and [24, Theorem 4] for the cases $n = 1$ and $\alpha+\beta = 1$ respectively. Here we remark that $T|T|^{\alpha+\beta-1}z \neq 0$ are equivalent to $Tz \neq 0$ for all $z \in H$.

Remark 2.2. *Suppose that* $\dim \operatorname{ran}(T) = 1$, $|T^*|y \neq 0$ *and* $\alpha, \beta > 0$ *in Theorem 2.1. Then there doesn't exist* z_i *such that* $Tz_i \neq 0$ *and* $\langle T|T|^{\alpha+\beta-1}z_i, y\rangle = 0$. *Furthermore it implies that* $|T^*|^\beta y$ *and* $U|T|^\alpha x$ *are proportional, and so*

$$|\langle T|T|^{\alpha+\beta-1}x, y\rangle|^2 = \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2\beta}y, y\rangle.$$

Now we have the following theorem as a simultaneous extension of both Theorems A and B via the Löwner-Heinz inequality [25], i.e., $A \geq B \geq 0$ implies $A^p \geq B^p$ for all $p \in [0,1]$:

Theorem 2.3. *Let* T *be an operator on* H. *If* A *and* B *are positive operators on* H *such that* $T^*T \leq A^2$ *and* $TT^* \leq B^2$, *then for each* $x, y \in H$

$$|\langle T|T|^{\alpha+\beta-1}x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle |T|^{2\alpha}u_{i-1}, z_i\rangle|^2 \, \| \, |T^*|^\beta y \, \|^2}{\| \, |T|^\alpha z_i \, \|^2} \leq \| A^\alpha x \|^2 \| B^\beta y \|^2 \tag{2.2}$$

for all $\alpha, \beta \in [0,1]$ *with* $\alpha + \beta \geq 1$ *and* $z_1, \dots, z_n \notin \ker T$ *such that* $\langle T|T|^{\alpha+\beta-1}z_i, y\rangle = 0$, *where* $u_i = u_{i-1} - \frac{\langle |T|^{2\alpha}u_{i-1}, z_i\rangle}{\| |T|^\alpha z_i \|^2} z_i$ *with* $u_0 = x$ *for* $i = 1, 2, \dots, n$. *In the case* $\alpha, \beta > 0$, *the equality in* (2.2) *holds if and only if* $A^{2\alpha}x = |T|^{2\alpha}x$, $B^{2\beta}y = |T^*|^{2\beta}y$ *and* $|T^*|^\beta y$ *and* $U|T|^\alpha u_n$ *are proportional.*

Remark 2.4. *If we put* $n = 1$ *in Theorem 2.3, then we obtain Thorem A. Moreover if* $n = 1$ *the equality condition of* (2.2) *ensures one of* (1.2) *by* [12, Lemma]. *On the other hand, let* S, λ_i *and* e_i *for* $i = 1, 2, \dots, n$ *be as in Theorem B. Moreover if we put* $T = I$ *(the identity operator) and replace* $\frac{z_i}{\|z_i\|}$ *and* y *to* e_i *and* $\prod_{j=1}^n (S - \lambda_j)x$ *respectively in Theorem 2.1, then we obtain Thorem B by the following inequality*

$$|\langle x, \prod_{i=1}^n (S - \lambda_i)x\rangle|^2 + \sum_{i=1}^n |\langle u_{i-1}, e_i\rangle|^2 \, \| \prod_{i=1}^n (S - \lambda_i)x \|^2 \leq \| x \|^2 \| \prod_{i=1}^n (S - \lambda_i)x \|^2,$$

so Thorem 2.3 is a extension of Theorems A and B. Furthermore we can see that the equality condition of Theorem B (1.4) *holds if and only if* $\prod_{i=1}^n (S - \lambda_i)x$ *and* u_n *are proportional.*

The following corollary obtained by Theorem 2.1 is a generalization of [12, Theorem 4] and so [24, Theorem 3], too. For this we recall normal eigenvalues λ for T, i.e., there exists a nonzero vector $e \in H$ such that $Te = \lambda e$ and $T^*e = \overline{\lambda} e$.

Corollary 2.5. *Let* T *be an operator on* H. *Let* e_i *be an eigenvector corresponding to a nonzero normal eigenvalue* λ_i *of* T *for* $i = 1, 2, \dots, n$. *If* $y \in H$ *satisfies* $T^*y \neq 0$ *and* $\langle e_i, y\rangle = 0$ *for* $i = 1, 2, \dots, n$, *then for each* $x \in H$

$$\sum_{i=1}^n |\lambda_i|^2 |\langle u_{i-1}, e_i\rangle|^2 \leq \frac{\| Tx \|^2 \| \, |T^*|^\beta T^*y \, \|^2 - |\langle T|T|^\beta x, T^*y\rangle|^2}{\| \, |T^*|^\beta T^*y \, \|^2} \tag{2.3}$$

for all $\beta \geq 0$, *where* $u_i = u_{i-1} - \langle u_{i-1}, e_i\rangle e_i$ *with* $u_0 = x$ *for* $i = 1, 2, \dots, n$. *In the case* $\beta > 0$, *the equality in* (2.3) *holds if and only if* $|T^*|^\beta T^*y$ *and* Tu_n *are proportional.*

In particular, if $\{e_1, e_2, \ldots, e_n\}$ *is an orthonormal set, then*

$$\sum_{i=1}^{n} |\lambda_i|^2 |\langle x, e_i\rangle|^2 \le \frac{\| Tx \|^2 \| |T^*|^\beta T^* y \|^2 - |\langle T|T|^\beta x, T^*y\rangle|^2}{\| |T^*|^\beta T^* y \|^2}. \tag{2.4}$$

The equality in (2.4) *holds if and only if* $|T^*|^\beta T^*y$ *and* Tx *are proportional.*

Proof. We put $\alpha = 1$, $z_i = e_i$ and replace y to T^*y in Theorem 2.1. Since $\langle T|T|^\beta e_i, T^*y\rangle = 0$ by $\langle e_i, y\rangle = 0$ for $i = 1, 2, \ldots, n$, the assumption of Theorem 2.1 is satisfied and so it follows that

$$|\langle T|T|^\beta x, T^*y\rangle|^2 + \sum_{i=1}^{n} |\lambda_i|^2 |\langle u_{i-1}, e_i\rangle|^2 \, \| |T^*|^\beta T^*y \|^2 \le \| Tx \|^2 \| |T^*|^\beta T^*y \|^2 .$$

Hence we have the desired inequality (2.3).

If $\{e_i, e_2, \ldots, e_n\}$ is an orthonormal set, then the definition of u_i gives $\langle u_{i-1}, e_i\rangle = \langle u_{i-2}, e_i\rangle = \cdots = \langle u_0, e_i\rangle = \langle x, e_i\rangle$ for each $i = 1, 2, \ldots, n$, so the inequality (2.4) holds.

The equality condition is obvious by Theorem 2.1. □

We give a further extension of Corollary 2.5 by using normal approximate eigenvalues. Recall that λ is a normal approximate eigenvalue of T if there exists a sequence $\{x^{(n)}\}$ of unit vectors such that $\| (T - \lambda)x^{(n)} \| \to 0$ and $\| (T - \lambda)^* x^{(n)} \| \to 0$. For convenience, we cite the Berberian representation $T \to T^\circ$. We take a generalized limit Lim such that

$$\mathrm{Lim}_k |\langle x^{(k)}, y^{(k)}\rangle|^2 = \overline{\lim}_k |\langle x^{(k)}, y^{(k)}\rangle|^2.$$

The vector space V of all bounded sequences in H has a semi-inner product $< x^\circ, y^\circ >= \mathrm{Lim}\langle x^{(k)}, y^{(k)}\rangle$, so that a Hilbert space H° is given by the completion of V/N, where $N = \{x \in V; < x, z >= 0 \quad \text{for all } z \in V\}$. For an operator T on H, T° is defined by

$$T^\circ(\{x^{(k)}\} + N) = \{Tx^{(k)}\} + N.$$

Then it is an isometric $*$-isomorophism and the approximate eigenvalues of T converts to the eigenvalues of T°.

By this representation we show the following theorem which extends to results in [2], [6], [10] and [14] as in the case $n = 1$:

Theorem 2.6. *Let* T *be an operator on* H. *Let* $\{e_i^{(k)}\}$ *be a sequence of unit vectors corresponding to a nonzero normal approximate eigenvalue* λ_i *of* T *for* $i = 1, 2, \ldots, n$. *If* $y \in H$ *satisfies* $T^*y \ne 0$ *and* $\langle e_i^{(k)}, y\rangle = 0$ *for* $i = 1, 2, \ldots, n$, *then for each* $x \in H$

$$\sum_{i=1}^{n} \overline{\lim}_k |\lambda_i|^2 |\langle u_{i-1}^{(k)}, e_i^{(k)}\rangle|^2 \le \frac{\| Tx \|^2 \| |T^*|^\beta T^*y \|^2 - |\langle T|T|^\beta x, T^*y\rangle|^2}{\| |T^*|^\beta T^*y \|^2} \tag{2.5}$$

for all $\beta \in [0, 1]$, *where* $u_i^{(k)} = u_{i-1}^{(k)} - \langle u_{i-1}^{(k)}, e_i^{(k)}\rangle e_i^{(k)}$ *with* $u_0^{(k)} = x$ *for* $i = 1, 2, \ldots, n$.

In particular, if $\{e_1^{(k)}, e_2^{(k)}, \ldots, e_n^{(k)}\}$ *is an orthonormal set, then*

$$\sum_{i=1}^{n} |\lambda_i|^2 |\langle x, e_i^{(k)}\rangle|^2 \le \frac{\| Tx \|^2 \| |T^*|^\beta T^*y \|^2 - |\langle T|T|^\beta x, T^*y\rangle|^2}{\| |T^*|^\beta T^*y \|^2}. \tag{2.6}$$

Proof. We have

$$\begin{aligned}\langle T|T|^{\beta}x, T^*y\rangle &= \langle U|T|^{1+\beta}x, T^*y\rangle = \langle U|T|^{\beta}U^*U|T|x, T^*y\rangle \\ &= \langle |T^*|^{\beta}Tx, T^*y\rangle = \langle Tx, |T^*|^{\beta}T^*y\rangle.\end{aligned}$$

Hence the right hand of (2.3) is as follows by putting $f = \frac{|T^*|^{\beta}T^*y}{\| |T^*|^{\beta}T^*y\|}$:

$$\begin{aligned}\frac{\| Tx \|^2 \| |T^*|^{\beta}T^*y \|^2 - |\langle T|T|^{\beta}x, T^*y\rangle|^2}{\| |T^*|^{\beta}T^*y \|^2} &= \frac{\| Tx \|^2 \| |T^*|^{\beta}T^*y \|^2 - |\langle Tx, |T^*|^{\beta}T^*y\rangle|^2}{\| |T^*|^{\beta}T^*y \|^2} \\ &= \| Tx \|^2 - |\langle Tx, f\rangle|^2.\end{aligned}$$

By the Berberian representation, we obtain that

$$(T^{\circ} - \lambda_i)e_i^{\circ} = 0, \quad (T^{\circ} - \lambda_i)^*e_i^{\circ} = 0 \quad \text{and} \quad | < u_{i-1}^{\circ}, e_i^{\circ} > | = \overline{\lim_k}|\langle u_{i-1}^{(k)}, e_i^{(k)}\rangle|,$$

where $u_i^{\circ} = \{u_i^{(k)}\} + N$ and $e_i^{\circ} = \{e_i^{(k)}\} + N$. Hence it follows from Theorem 2.5 that

$$\begin{aligned}\sum_{i=1}^{n} \overline{\lim}_k |\lambda_i|^2 |\langle u_{i-1}^{(k)}, e_i^{(k)}\rangle|^2 &= \sum_{i=1}^{n} |\lambda_i|^2 | < u_{i-1}^{\circ}, e_i^{\circ} > |^2 \\ &\leq \| T^{\circ}x^{\circ} \| - | < T^{\circ}x^{\circ}, f^{\circ} > |^2 = \| Tx \| - |\langle Tx, f\rangle|^2,\end{aligned}$$

where x° and f° are the canonical embedding of x and f into H° respectively. We have desired inequality (2.5). By the same method the inequality (2.6) is obvious by (2.4). □

3. Extensions of Furuta's type inequality

The main tool in this section is the Furuta inequality [16]. We now cite it for convenience:

The Furuta inequality.
If $A \geq B \geq 0$, then for each $r \geq 0$,

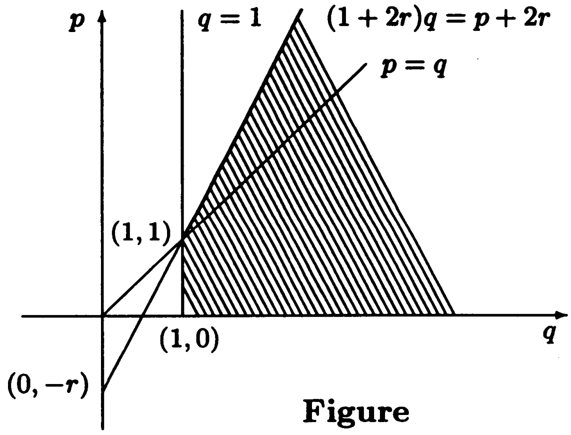
(i) $\quad (B^rA^pB^r)^{\frac{1}{q}} \geq (B^rB^pB^r)^{\frac{1}{q}}$

and

(ii) $\quad (A^rA^pA^r)^{\frac{1}{q}} \geq (A^rB^pA^r)^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with

$$(1 + 2r)q \geq p + 2r.$$



Figure

We refer [22] and [3] for mean theoretic proofs of it, and [18] for a one-page proof. The best possibility of the domain drawn in the Figure is proved by Tanahashi [26].

The HKF inequality is extended by the use of the Furuta inequality in [20], and so is Theorem A in [12, Theorem 3]. Theorem 2.3 also gives us improvement of the HKF inequality via the Furuta inequality.

Theorem 3.1. *Let T be an operator on H. If A and B are positive operators on H such that $T^*T \le A^2$ and $TT^* \le B^2$, then for each $x, y \in H$*

$$|\langle T|T|^{(1+r)\alpha+(1+s)\beta-1}x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle |T|^{2(1+r)\alpha}u_{i-1}, z_i\rangle|^2 \, \| \, |T^*|^{(1+s)\beta}y \, \|^2}{\| \, |T|^{(1+r)\alpha}z_i \, \|^2} \tag{3.1}$$
$$\le \langle (|T|^r A^{2p}|T|^r)^{\frac{(1+r)\alpha}{p+r}}x, x\rangle\langle (|T^*|^s B^{2q}|T^*|^s)^{\frac{(1+s)\beta}{p+s}}y, y\rangle$$

for all $p, q \ge 1$, $r, s > 0$, $\alpha, \beta \in [0,1]$ with $(1+r)\alpha + (1+s)\beta \ge 1$ and $z_1, \cdots, z_n \notin \ker T$ such that $\langle T|T|^{(1+r)\alpha+(1+s)\beta-1}z_i, y\rangle = 0$, where $u_i = u_{i-1} - \frac{\langle |T|^{2(1+r)\alpha}u_{i-1}, z_i\rangle}{\||T|^{(1+r)\alpha}z_i\|^2} z_i$ with $u_0 = x$ for $i = 1, 2, \cdots, n$. In the case $\alpha, \beta > 0$, the equality in (3.1) holds if and only if $|T|^{2(1+r)\alpha}x = (|T|^r A^{2p}|T|^r)^{\frac{(1+r)\alpha}{p+r}}x$, $|T^*|^{2(1+s)\beta}y = (|T^*|^s B^{2q}|T^*|^s)^{\frac{(1+s)\beta}{q+s}}y$ and $|T^*|^{(1+s)\beta}y$ and $U|T|^{(1+r)\alpha}u_n$ are proportional.

Proof. We replace α and β to $\alpha_1 = (1+r)\alpha$ and $\beta_1 = (1+s)\beta$ respectively in Theorem 2.1. Then

$$|\langle T|T|^{\alpha_1+\beta_1-1}x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle |T|^{2\alpha_1}u_{i-1}, z_i\rangle|^2 \, \| \, |T^*|^{\beta_1}y \, \|^2}{\| \, |T|^{\alpha_1}z_i \, \|^2} \le \langle |T|^{2\alpha_1}x, x\rangle\langle |T^*|^{2\beta_1}y, y\rangle.$$

Next we replace A, B, r and q to A^2, $|T|^2$, $\frac{r}{2}$ and $\frac{p+r}{(1+r)\alpha}$ respectively in the Furuta inequality. Then we have

$$|T|^{2\alpha_1} = |T|^{2(1+r)\alpha} = |T|^{2(p+r)\cdot\frac{(1+r)\alpha}{p+r}} \le (|T|^r A^{2p}|T|^r)^{\frac{(1+r)\alpha}{p+r}}$$

and

$$|T^*|^{2\beta_1} = |T^*|^{2(1+s)\beta} = |T^*|^{2(q+s)\cdot\frac{(1+s)\beta}{q+s}} \le (|T^*|^s B^{2q}|T^*|^s)^{\frac{(1+s)\beta}{q+s}}.$$

Combining three inequalities above, we obtain the desired inequality (3.1). As in the proof of [12, Theorem 3], the equality condition is easily checked. □

We see that if $n = 1$ in Theorem 3.1, [12, Theorem 3] is hold. We remark that the condition $(1+r)\alpha+(1+s)\beta \ge 1$ in above is unnecessary if T is either positive or invertible.

From the operator monotonicity of the logarithmic function, we intoroduced the chaotic order among positive invertible operators by $A \gg B$ if $\log A \ge \log B$ in [5], and obtained a characterization of the chaotic order in terms of Furuta's type operator inequality [7], [8] and [9]. Furthermore based on this, in [13, Theorem 4] we gave a chaotic order version of Theorem A by the Furuta inequality. We show a variant of Theorem 2.3 by chaotic order. For this, we use the following characterization of the chaotic order which is an extension of Ando's theorem [5], [7], [8], [9] and [27] for a polished proof.

Theorem C. *For positive invertible operators A and B, $A \gg B$ if and only if*

$$(B^r A^p B^r)^{\frac{1}{q}} \ge (B^r B^p B^r)^{\frac{1}{q}}$$

holds for $q \ge 1$, $p, r \ge 0$ with $2rq \ge p + 2r$.

The following chaotic version of Theorem 3.1 is an extension of [13, Theorem 4]:

Theorem 3.2. *Let T be an invertible operator on H. If A and B are positive invertible operators on H such that $A^2 \gg T^*T$ and $B^2 \gg TT^*$, then for each $x, y \in H$*

$$|\langle T|T|^{r\alpha+s\beta-1}x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle |T|^{2r\alpha}u_{i-1}, z_i\rangle|^2 \parallel |T^*|^{s\beta}y \parallel^2}{\parallel |T|^{r\alpha}z_i \parallel^2} \le \langle (|T|^r A^{2p}|T|^r)^{\frac{r\alpha}{p+r}}x, x\rangle\langle (|T^*|^s B^{2q}|T^*|^s)^{\frac{s\beta}{q+s}}y, y\rangle \tag{3.2}$$

for all $p, q \ge 0$, $r, s \ge 0$, $\alpha, \beta \in [0,1]$ and $z_1, \cdots, z_n \notin \ker T$ such that $\langle T|T|^{r\alpha+s\beta-1}z_i, y\rangle = 0$, where $u_i = u_{i-1} - \frac{\langle |T|^{2r\alpha}u_{i-1}, z_i\rangle}{\||T|^{r\alpha}z_i\|^2}z_i$ with $u_0 = x$ for $i = 1, 2, \dots, n$. In the case $\alpha, \beta > 0$, the equality in (3.2) holds if and only if $|T|^{2r\alpha}x = (|T|^r A^{2p}|T|^r)^{\frac{r\alpha}{p+r}}x$, $|T^*|^{2s\beta}y = (|T^*|^s B^{2q}|T^*|^s)^{\frac{s\beta}{q+s}}y$ and $|T^*|^{s\beta}y$ and $U|T|^{r\alpha}u_n$ are proportional.

Proof. The proof is similar to that of [13, Theorem 4]. By Theorem 2.1, we have

$$|\langle T|T|^{r\alpha+s\beta-1}x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle |T|^{2r\alpha}u_{i-1}, z_i\rangle|^2 \parallel |T^*|^{s\beta}y \parallel^2}{\parallel |T|^{r\alpha}z_i \parallel^2} \le \langle |T|^{2r\alpha}x, x\rangle\langle |T^*|^{2s\beta}y, y\rangle.$$

Moreover Theorem C says

$$|T|^{2r\alpha} \le (|T|^r A^{2p}|T|^r)^{\frac{r\alpha}{p+r}} \quad \text{and} \quad |T^*|^{2s\beta} \le (|T^*|^s B^{2q}|T^*|^s)^{\frac{s\beta}{q+s}}.$$

Combining three inequalities above, we obtain the desired inequality (3.2). As in the proof of Theorem 2.1 and [13, Theorem 4], the equality condition is easily checked. □

Now in [11, Theorem 4] we showed the following theorem as a simultaneous extension of the HKF inequality and the Selberg inequality:

Theorem D. *Let T be an operator on H. Then for each $x, y \in H$*

$$|\langle T|T|^{\alpha+\beta-1}x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle |T|^{2\alpha}x, z_i\rangle|^2 \parallel |T^*|^{\beta}y \parallel^2}{\sum_{j=1}^{n} |\langle |T|^{2\alpha}z_i, z_j\rangle|} \le \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2\beta}y, y\rangle \tag{3.3}$$

for all $\alpha, \beta \ge 0$ with $\alpha + \beta \ge 1$ and $z_i \notin \ker T$ such that $\langle T|T|^{\alpha+\beta-1}z_i, y\rangle = 0$ for $i = 1, 2, \dots, n$.

In [11, Theorem 8] Theorem D was extended by applying the Furuta inequality. We now show the chaotic version of it by applying Theorem C:

Theorem 3.3. *Let T be an invertible operator on H. If A and B are positive invertible operators on H such that $A^2 \gg T^*T$ and $B^2 \gg TT^*$, then for each $x, y \in H$*

$$|\langle T|T|^{r\alpha+s\beta-1}x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle |T|^{2r\alpha}x, z_i\rangle|^2 \parallel |T^*|^{s\beta}y \parallel^2}{\sum_{j=1}^{n} |\langle |T|^{2r\alpha}z_i, z_j\rangle|} \le \langle (|T|^r A^{2p}|T|^r)^{\frac{r\alpha}{p+r}}x, x\rangle\langle (|T^*|^s B^{2q}|T^*|^s)^{\frac{s\beta}{q+s}}y, y\rangle \tag{3.4}$$

for all $p, q \ge 0$, $r, s \ge 0$, $\alpha, \beta \in [0,1]$ and $z_i \notin \ker T$ such that $\langle T|T|^{r\alpha+s\beta-1}z_i, y\rangle = 0$ for $i = 1, 2, \dots, n$.

Proof. By replacing α and β to $r\alpha$ and $s\beta$ respectively in Theorem D, we have

$$|\langle T|T|^{r\alpha+s\beta-1}x,y\rangle|^2+\sum_{i=1}^{n}\frac{|\langle |T|^{2r\alpha}x,z_i\rangle|^2 \parallel |T^*|^{s\beta}y \parallel^2}{\sum_{j=1}^{n}|\langle |T|^{2r\alpha}z_i,z_j\rangle|}\leq\langle |T|^{2r\alpha}x,x\rangle\langle |T^*|^{2s\beta}y,y\rangle.$$

Hence we have the desired inequality from Theorem C. □

Furthermore we gave an alternative simultaneous extension of the HKF inequality and the Selberg inequality in [11, Theorem 3] and its extension in [11, Theorem 7] by applying the Furuta inequality. By the same method with Theorem 3.3, we shall show the following theorem as an extension of [11, Theorem 3] under the chaotic order:

Theorem 3.4. *Let T be an invertible operator on H. If A and B are positive invertible operators on H such that $A^2 \gg T^*T$ and $B^2 \gg TT^*$, then for each $x, y \in H$*

$$\text{(3.5)}\qquad |\langle T|T|^{r\alpha+s\beta-1}x,y\rangle|^2+\sum_{i=1}^{n}\frac{|\langle Tx,z_i\rangle|^2 \parallel |T^*|^{s\beta}y \parallel^2}{\sum_{j=1}^{n}|\langle |T^*|^{2(1-r\alpha)}z_i,z_j\rangle|}$$
$$\leq \langle (|T|^rA^{2p}|T|^r)^{\frac{r\alpha}{p+r}}x,x\rangle\langle (|T^*|^sB^{2q}|T^*|^s)^{\frac{s\beta}{q+s}}y,y\rangle$$

for all $p,q\geq 0$, $r,s\geq 0$, $\alpha,\beta\in[0,1]$ and $z_i\notin \ker T^*$ such that $\langle T|T|^{s\beta+1-r\alpha}z_i,y\rangle=0$ for $i=1,2,\dots,n$.

Proof. By replacing α and β to $r\alpha$ and $s\beta$ respectively in [11, Theorem 3], we have

$$|\langle T|T|^{r\alpha+s\beta-1}x,y\rangle|^2+\sum_{i=1}^{n}\frac{|\langle Tx,z_i\rangle|^2 \parallel |T^*|^{s\beta}y \parallel^2}{\sum_{j=1}^{n}|\langle |T^*|^{2(1-r\alpha)}z_i,z_j\rangle|}\leq\langle |T|^{2r\alpha}x,x\rangle\langle |T^*|^{2s\beta}y,y\rangle.$$

Hence we have the desired inequality from Theorem C. □

Next we interpolate between Theorems 3.1 and 3.2 by the use of Furuta's type operator inequality which interpolates the Furuta inequality and Theorem C.

Theorem 3.5. *Let T be an operator on H. If A and B are positive operators on H such that $|T|^\delta\leq A^\delta$ and $|T^*|^\delta\leq B^\delta$ for some $\delta\in[0,1]$, then for each $x,y\in H$*

$$\text{(3.6)}\qquad |\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x,y\rangle|^2+\sum_{i=1}^{n}\frac{|\langle |T|^{2(\delta+r)\alpha}u_{i-1},z_i\rangle|^2 \parallel |T^*|^{(\delta+s)\beta}y \parallel^2}{\parallel |T|^{(\delta+r)\alpha}z_i \parallel^2}$$
$$\leq \langle (|T|^rA^{2p}|T|^r)^{\frac{(\delta+r)\alpha}{p+r}}x,x\rangle\langle (|T^*|^sB^{2q}|T^*|^s)^{\frac{(\delta+s)\beta}{q+s}}y,y\rangle$$

for all $p\geq\delta$, $q\geq 1$, $r,s\geq 0$, $\alpha,\beta\in[0,1]$ with $(\delta+r)\alpha+(\delta+s)\beta\geq 1$ and $z_1,\cdots,z_n\notin \ker T$ such that $\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}z_i,y\rangle=0$, where $u_i=u_{i-1}-\frac{\langle |T|^{2(\delta+r)\alpha}u_{i-1},z_i\rangle}{\| |T|^{(\delta+r)\alpha}z_i\|^2}z_i$ with $u_0=x$ for $i=1,2,\dots,n$. In the case $\alpha,\beta>0$, the equality in (3.6) holds if and only if $|T|^{2(\delta+r)\alpha}x=(|T|^rA^{2p}|T|^r)^{\frac{(\delta+r)\alpha}{p+r}}x$, $|T^*|^{2(\delta+s)\beta}y=(|T^*|^sB^{2q}|T^*|^s)^{\frac{(\delta+s)\beta}{q+s}}y$ and $|T^*|^{(\delta+s)\beta}y$ and $U|T|^{(\delta+r)\alpha}u_n$ are proportional.

Proof. By Theorem 2.1, we have

$$|\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle |T|^{2(\delta+r)\alpha}u_{i-1}, z_i\rangle|^2 \, \| \, |T^*|^{(\delta+s)\beta}y \, \|^2}{\| \, |T|^{(\delta+r)\alpha}z_i \, \|^2}$$
$$\leq \langle |T|^{2(\delta+r)\alpha}x, x\rangle \langle |T^*|^{2(\delta+s)\beta}y, y\rangle.$$

Moreover the following inequality is known in [8]

$$|T|^{2(\delta+r)\alpha} \leq (|T|^r A^{2p} |T|^r)^{\frac{(\delta+r)\alpha}{p+r}} \quad \text{and} \quad |T^*|^{2(\delta+s)\beta} \leq (|T^*|^s B^{2q} |T^*|^s)^{\frac{(\delta+s)\beta}{q+s}}.$$

Combining three inequalities above, we obtain the desired inequality (3.6). As in the proof of [13, Theorem 5] the equality condition is easily checked. □

Now we have the following theorem interpolating between Theorem 3.3 and [11, Theorem 8].

Theorem 3.6. *Let T be an operator on H. If A and B are positive operators on H such that $|T|^\delta \leq A^\delta$ and $|T^*|^\delta \leq B^\delta$ for some $\delta \in [0,1]$, then for each $x, y \in H$*

$$(3.7) \qquad |\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle |T|^{2(\delta+r)\alpha}x, z_i\rangle|^2 \, \| \, |T^*|^{(\delta+s)\beta}y \, \|^2}{\sum_{j=1}^{n} |\langle |T|^{2(\delta+r)\alpha}z_i, z_j\rangle|}$$
$$\leq \langle (|T|^r A^{2p} |T|^r)^{\frac{(\delta+r)\alpha}{p+r}} x, x\rangle \langle (|T^*|^s B^{2q} |T^*|^s)^{\frac{(\delta+s)\beta}{q+s}} y, y\rangle$$

for all $p \geq \delta$, $q \geq 1$, $r, s > 0$, $\alpha, \beta \in [0,1]$ with $(\delta+r)\alpha + (\delta+s)\beta \geq 1$ and $z_i \notin \ker T$ such that $\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}z_i, y\rangle = 0$ for $i = 1, 2, \ldots, n$.

In addition, we show the following theorem interpolating between Theorem 3.4 and [11, Theorem 7].

Theorem 3.7. *Let T be an operator on H. If A and B are positive operators on H such that $|T|^\delta \leq A^\delta$ and $|T^*|^\delta \leq B^\delta$ for some $\delta \in [0,1]$, then for each $x, y \in H$*

$$(3.8) \qquad |\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x, y\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle Tx, z_i\rangle|^2 \, \| \, |T^*|^{(\delta+s)\beta}y \, \|^2}{\sum_{j=1}^{n} |\langle |T^*|^{2(1-(\delta+r)\alpha)}z_i, z_j\rangle|}$$
$$\leq \langle (|T|^r A^{2p} |T|^r)^{\frac{(\delta+r)\alpha}{p+r}} x, x\rangle \langle (|T^*|^s B^{2q} |T^*|^s)^{\frac{(\delta+s)\beta}{q+s}} y, y\rangle$$

for all $p \geq \delta$, $q \geq 1$, $r, s \geq 0$, $\alpha, \beta \in [0,1]$ with $(\delta+r)\alpha + (\delta+s)\beta \geq 1$ and $z_i \notin \ker T^*$ such that $\langle T|T|^{(\delta+s)\beta+1-(\delta+r)\alpha}z_i, y\rangle = 0$ for $i = 1, 2, \ldots, n$.

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