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NONLINEAR NONLOCAL TRANSPORT-DIFFUSION EQUATIONS ARISING IN PHYSIOLOGY

NOBUYUKI KATO AND TOSHIYUKI YAMAGUCHI

ABSTRACT. We study a transport-diffusion initial value problem arising in mathematical models of muscle contraction. The equation has the transport term whose coefficient is a time function depending on the solution in a nonlinear and nonlocal way. In this paper, we investigate the unique existence of a strong solution in a function space BUC. Our results allow the inhomogeneous term to depend on the solution in a nonlinear way, such as $\gamma(t)f(x)(1-u^p)-g(x)u^q$ and $\gamma(t)f(x)(1-u)^p-g(x)u^q$.

Key words and phrases: Muscle contraction, nonlocal transport-diffusion equation, semilinear evolution equation

1. INTRODUCTION

In this paper we study the initial value problem for a nonlinear nonlocal transport-diffusion equation with a small diffusion coefficient $\varepsilon \in\,]0,1]$:

$$u_t - \varepsilon u_{xx} + z'(t)u_x = \varphi(x,t,z(t),u), \quad (x,t) \in \mathbb{R}\times[0,T], \tag{1.1}$$

$$z(t) = L\Big(\int_{\mathbb{R}} w(x)u(x,t)dx\Big), \quad t\in[0,T], \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{1.3}$$

Here $u : \mathbb{R}\times[0,T] \to \mathbb{R}$ and $z : [0,T] \to \mathbb{R}$ are unknown functions, z' stands for the time-derivative. The functions φ, L, w and u_0 are given functions specified later.

Study of the above equation is related to the nonlinear nonlocal first order hyperbolic problem: Find $u : \mathbb{R}\times[0,T] \to \mathbb{R}$ and $z : [0,T] \to \mathbb{R}$ for which

$$u_t + z'(t)u_x = \varphi(x,t,z(t),u), \quad (x,t) \in \mathbb{R}\times[0,T] \tag{1.4}$$

and (1.2)–(1.3) are satisfied. This hyperbolic problem is formulated as a rheological model describing the so-called cross-bridge dynamics observed in the muscle contraction phenomena in physiology. For the model problem, see [1, 4, 5, 7, 8] and the references therein. The constitutive unit of muscle structure is called a sarcomere which consists of particles of myosin (thick filament) and actin (thin filament). The cross-bridges are chemical links between myosin and actin filaments. According to the sliding filament theory of Huxley [8], the phenomenon of muscle contraction is a consequence of relative sliding motion between these two filaments and this sliding occurs when the cross-bridges attach the actin filaments and act as springs. The quantity $u(x,t)$ essentially represents a density of cross-bridges attached to an actin at distance x from the equilibrium position and time t. The function z stands for the contractile movement of filaments which is determined by the contractile force $\int_{\mathbb{R}} w(x)u(x,t)dx$.

In the model problem (1.1)–(1.3), the "slipping effect", which is represented as a viscosity term $-\varepsilon u_{xx}$, is taken into account. See [2, 3] for such models. In the case of bounded domain in $\mathbb{R}$, Colli and Grasselli [2] have shown the local existence of a strong solution of (1.1)–(1.3) under the Dirichlet boundary condition. In the case of the whole space $\mathbb{R}$, Colli and Grasselli [3] have shown the global existence of weak solutions for the special case $\varphi(x,t,z,u) = F(x,t,z) - G(x,t,z)u$.

In this paper we shall establish the global existence and uniqueness of strong solutions to (1.1)–(1.3) by using the approach employed in [3] together with the theory of abstract semilinear evolution equations. Although we require certain stronger regularity assumptions of φ and u_0 than those in [2, 3], our setting allows φ to have more general forms: for instance, it is possible to take not only the form $\varphi(x,t,z,u) \equiv \gamma(t)f(x)(1-u) - g(x)u$ (which was originally introduced by Huxley) but also its generalizations $\gamma(t)f(x,z)(1-|u|^{p-1}u) - g(x,z)|u|^{q-1}u$ and $\gamma(t)f(x,z)|1-u|^{p-1}(1-u) - g(x,z)|u|^{q-1}u$ which involve power nonlinearities. Further, it is expected that when ε tends to zero, the solution of (1.1)–(1.3) approaches to the solution of (1.2)–(1.4). Concerning such limiting behaviour, we shall discuss it in the forthcoming paper [9].

The paper is organized as follows. In Section 2 we formulate our problem and give our assumptions and main result. Section 3 is devoted to solving semilinear evolution problems obtained from (1.1) and (1.3) by regarding z as a fixed given function. Then we introduce equivalent formulations to the problem (1.1)–(1.3) in Section 4. In Section 5 we verify the uniqueness and global existence of strong solutions to (1.1)–(1.3) by applying the contraction mapping principle argument in a successive way. Section 6 is devoted to the proofs of some technical lemmas containing a priori estimates of z, which is important in the argument of previous section. In this paper we treat the above-mentioned problems in the function space BUC with the Hölder continuity. For the convenience of the readers, we contain a detailed explanations on such spaces in Appendix.

2. Transport-diffusion problem

In this section we make assumptions for the given functions and data, give a precise formulation of the transport-diffusion problem (1.1)–(1.3), and then state the main results concerning the existence and uniqueness of solutions. In what follows, BUC stands for the space of bounded and uniformly continuous functions and $BUC^{\eta,\frac{\eta}{2}}$ for the space of Hölder continuous functions in BUC of two variables. The space of Hölder continuous functions will be denoted by $C^{0,\eta}$ with $0 < \eta < 1$; by $C^{0,1}$ we mean the space of Lipschitz continuous functions. For the precise definitions and properties, see Appendix.

Let $T > 0$ be fixed. We assume the following hypotheses.

(H1) $L :]a,b[\to \mathbb{R}$ is a continuous, strictly decreasing function, where $-\infty \leqslant a < 0 < b \leqslant +\infty$, and its inverse function λ $(:= L^{-1}) : \mathbb{R} \to]a,b[$ satisfies $\lambda(0) = 0$, $\lambda(\xi) \to a$ (resp. $\to b$) as $\xi \to +\infty$ (resp. $\to -\infty$). Moreover, for any $R > 0$, there exists a constant $M_1(R) > 0$ such that

$$|\xi_1 - \xi_2| \leqslant M_1(R)|\lambda(\xi_1) - \lambda(\xi_2)| \quad \text{for all } \xi_1,\ \xi_2 \in\ [-R, R].$$

(H2) $w \in C^1(\mathbb{R})$ is an increasing function satisfying $w(0) = 0$ and $dw/dx \in W^{1,\infty}(\mathbb{R})$.

(H3) $\varphi \in C(\mathbb{R} \times [0,T] \times \mathbb{R} \times \mathbb{R})$ has the following properties:

(i) for some $\eta \in]0,1]$, $\varphi(\cdot,\cdot,z,u) \in BUC^{\eta,\frac{\eta}{2}}(\mathbb{R} \times [0,T])$ uniformly for (z,u) in bounded subsets of $\mathbb{R} \times \mathbb{R}$ (see Appendix for the definition);

(ii) for any $R > 0$, there exists a constant $M_2(R) > 0$ such that

$$|\varphi(x,t,z_1,u) - \varphi(x,t,z_2,u)| \leqslant M_2(R)|z_1 - z_2|$$

for all (x,t,z_1,u), $(x,t,z_2,u) \in \mathbb{R} \times [0,T] \times [-R,R] \times [-R,R]$;

(iii) φ has the partial derivative $\partial_u \varphi$ with respect to u which is uniformly continuous and bounded on $\mathbb{R} \times [0,T] \times [-R,R] \times [-R,R]$ for any $R > 0$;

(iv) φ is decreasing in u on $[0,+\infty[$;

(v) for any $(x,t,z) \in \mathbb{R} \times [0,T] \times \mathbb{R}$,

$$\varphi(x,t,z,u) \geqslant 0 \quad \text{if } u \leqslant 0 \quad \text{and} \quad \varphi(x,t,z,u) \leqslant 0 \quad \text{if } u \geqslant 1 \ ;$$

(vi) there exists a nonnegative function $\mathcal{F} \in L^\infty(\mathbb{R}\times]0,T[)$ which satisfies $x^2\mathcal{F} \in L^\infty(0,T;$ $L^1(\mathbb{R}))$ and

$$\varphi(x,t,z,u) \leqslant \mathcal{F}(x,t)$$

for *a.e.* $(x,t) \in \mathbb{R}\times]0,T[$, $z \in \mathbb{R}$ and $0 \leqslant u \leqslant 1$;

(vii) for any $R > 0$ there exists a constant $M_3(R) > 0$ satisfying

$$\varphi(x,t,z,u) \geqslant -M_3(R)(1+|x|)u \quad \text{if } 0 \leqslant u \leqslant 1$$

for $(x,t,z) \in \mathbb{R} \times [0,T] \times [-R,R]$;

(viii) for any $R > 0$, there exist $q > 1$ and $h \in L^q(0,T)$ such that for any function $u \in L^\infty(\mathbb{R})$ satisfying $x^2 u \in L^1(\mathbb{R})$ and $\|u\|_{L^\infty(\mathbb{R})} \leqslant R$,

$$\int_{\mathbb{R}} (1+|x|)|\varphi(x+z_1,t,z_1,u(x)) - \varphi(x+z_2,t,z_2,u(x))|dx$$
$$\leqslant h(t)|z_1 - z_2|\left\{1 + \int_{\mathbb{R}} (1+|x|)|u(x)|dx\right\}$$

for *a.e.* $t \in]0,T[$ and $z_i \in [-R,R]$, $i = 1,\ 2$.

(H4) The initial value u_0 belongs to $BUC(\mathbb{R})$ and satisfies $0 \leqslant u_0 \leqslant 1$ on $\mathbb{R}$, $x^2 u_0 \in L^1(\mathbb{R})$ and

$$a < \int_{\mathbb{R}} w(x)u_0(x)dx < b.$$

Remark 2.1. (1) By $x^2\mathcal{F} \in L^\infty(0,T;L^1(\mathbb{R}))$ and $x^2 u_0 \in L^1(\mathbb{R})$, we mean $\int_{\mathbb{R}} x^2\mathcal{F}(x,\cdot)dx \in L^\infty(0,T)$ and $\int_{\mathbb{R}} x^2 u_0(x)dx < +\infty$, respectively.

(2) From (H2), we see that w is Lipschitz continuous. We denote by C_w the Lipschitz constant of w.

(3) Condition (H3 iii) implies that for any $R > 0$ there exists a constant $M_4(R) > 0$ such that

$$|\varphi(x,t,z,u_1) - \varphi(x,t,z,u_2)| \leqslant M_4(R)|u_1 - u_2|$$

for (x,t,z,u_1), $(x,t,z,u_2) \in \mathbb{R} \times [0,T] \times [-R,R] \times [-R,R]$.

(4) It is easily seen from (H3 vi) and (H4) that $\mathcal{F} \in L^\infty(0,T;\ L^p(\mathbb{R}))$ and $u_0 \in L^p(\mathbb{R})$ for any $p \geqslant 1$. (See Lemma 3.3 in Section 3.) Besides, the integral in (H4) makes sense, since w is Lipschitz continuous, $w(0) = 0$, and $\int_{\mathbb{R}} |x| u_0(x) dx < +\infty$.

(5) Conditions (H1), (H2), (H3 v), and (H4) are similar to those in [2] or [3], although we have required stronger regularities for φ and u_0 than those in [2, 3] to treat the case in which φ is nonlinear in u.

Examples. As a function φ satisfying (H3), one may take functions of the form

$$\varphi(x,t,z,u) = \gamma(t) f(x,z)(1 - |u|^{p-1}u) - g(x,z)|u|^{q-1}u$$

$$\text{and} \quad \varphi(x,t,z,u) = \gamma(t) f(x,z)|1-u|^{p-1}(1-u) - g(x,z)|u|^{q-1}u.$$

Here it suffices to assume that p, $q \in [1,+\infty[$, and that the functions γ, f and g are nonnegative and satisfy the following conditions: $\gamma \in C^{0,\frac{\eta}{2}}[0,T]$ $(0 < \eta \leqslant 1)$, $f(x,\cdot)$, $g(x,\cdot) \in C^{0,1}_{loc}(\mathbb{R})$ uniformly in $x \in \mathbb{R}$, and $f(\cdot,z)$, $g(\cdot,z) \in BUC^\eta(\mathbb{R})$ uniformly in z on bounded subsets of $\mathbb{R}$. Further, $f \in L^\infty(\mathbb{R}^2)$, $x^2 \|f(x,\cdot)\|_{L^\infty(\mathbb{R}_z)} \in L^1(\mathbb{R}_x)$ and for any $R > 0$ there is a constant $C(R) > 0$ such that

$$\int_{\mathbb{R}} (1 + |y|) |f(y + z_1, z_1) - f(y + z_2, z_2)| dy \leqslant C(R)|z_1 - z_2|,$$

$$|g(x + z_1, z_1) - g(x + z_2, z_2)| \leqslant C(R)|z_1 - z_2|$$

for all $x \in \mathbb{R}$ and $z_i \in [-R,R]$ $(i = 1,2)$.

In these cases, one can take $\mathcal{F}(x,t) = \gamma(t)\|f(x,\cdot)\|_{L^\infty(\mathbb{R}_z)}$, $h(t) = C_R \max\{\gamma(t), 1\}$ with $C_R = C(R)\max\{R^p + 1, R^{q-1}\}$ and $C_R = C(R)\max\{(R+1)^p, R^{q-1}\}$, respectively.

We are now in a position to formulate the transport-diffusion problem.

Problem (P_1). Given initial function u_0 satisfying (H4).

Find $u \in BUC(\mathbb{R} \times [0,T]) \cap BUC^{2+\eta,\frac{\eta}{2}}(\mathbb{R} \times [\delta,T])$ for any $\delta \in]0,T]$ and $z \in C^{0,1}[0,T]$ such that $u(x,\cdot) \in C^{0,1}[\delta,T]$ uniformly for $x \in \mathbb{R}$ and any $\delta \in]0,T]$, $u_t(\cdot,t) \in BUC(\mathbb{R})$ for *a.e.* $t \in]0,T[$, $wu \in L^\infty(0,T;L^1(\mathbb{R}))$, and such that

$$a < \int_{\mathbb{R}} w(x)u(x,t)dx < b, \quad t \in [0,T], \tag{2.1}$$

$$z(t) = L\Big(\int_{\mathbb{R}} w(x)u(x,t)dx\Big), \quad t \in [0,T], \tag{2.2}$$

$$u_t(x,t) - \varepsilon u_{xx}(x,t) + z'(t)u_x(x,t) = \varphi(x,t,z(t),u(x,t)), \quad x \in \mathbb{R},\ a.e.\ t \in]0,T[, \tag{2.3}$$

where $z' = dz/dt$, and

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{2.4}$$

Our main result is then stated as follows:

Main Theorem. *Assume that conditions* (H1)–(H3) *are satisfied and let* u_0 *satisfy* (H4). *Then there exists a unique solution* (u,z) *of Problem* (P_1). *Moreover,* $0 \leqslant u \leqslant 1$ *on* $\mathbb{R}\times[0,T]$.

By change of variables $x \mapsto x + z(t)$, the equations (2.2)–(2.4) are transformed into

$$z(t) = L\Big(\int_{\mathbb{R}} w_z^*(x,t)v(x,t)dx\Big), \quad t \in [0,T]; \tag{2.5}$$

$$v_t(x,t) - \varepsilon v_{xx}(x,t) = \varphi_z^*(x,t,v(x,t)), \quad (x,t) \in \mathbb{R}\times]0,T], \tag{2.6}$$

$$v(x,0) = u_0(x+z(0)), \quad x \in \mathbb{R}. \tag{2.7}$$

Here for $\zeta \in C[0,T]$,

$$\begin{aligned} \varphi_\zeta^*(x,t,\xi) &:= \varphi(x+\zeta(t),t,\zeta(t),\xi), \quad (x,t,\xi) \in \mathbb{R}\times[0,T]\times\mathbb{R}, \\ w_\zeta^*(x,t) &:= w(x+\zeta(t)), \quad (x,t) \in \mathbb{R}\times[0,T]. \end{aligned} \tag{2.8}$$

This transformation has an advantage that the resultant equations (2.5)–(2.7) make sense if z is only continuous because the derivative z' disappears. Therefore, instead of equations (2.2)–(2.4), we deal with equations (2.5)–(2.7). Accordingly, our subsequent discussions may be organized in the following way:

I. Given $z \in C[0,T]$, we solve the semilinear problem (2.6)–(2.7). (Section 3)
II. We reformulate the original problem (P_1) as equivalent problems in conjunction with (2.5)–(2.7). (Section 4)
III. Finally, we seek $z \in C[0,T]$ which solves the equivalent problems. (Section 5)

3. Semilinear problems

We begin by considering the following semilinear problem :

$$\begin{cases} \dfrac{\partial \nu_z}{\partial t} - \varepsilon\dfrac{\partial^2 \nu_z}{\partial x^2} = \varphi_z^*(x,t,\nu_z), \quad (x,t) \in \mathbb{R}\times]r,T], \\ \nu_z(x,r) = \omega(x), \quad x \in \mathbb{R}, \end{cases} \tag{$P_z;r,\omega$}$$

where z is given in $C[0,T]$, $0 \leqslant r < T$ and $\omega \in BUC(\mathbb{R})$. To solve the problem $(P_z;r,\omega)$, we employ a theory of abstract semilinear evolution equations (cf. [6, 10]). For this purpose, we prepare some technical propositions.

Lemma 3.1. *Let* $z \in C[0,T]$. *Assume that* $z \in C^{0,\beta}[r,T]$ *for some* β *with* $\frac{1}{2} \leqslant \beta \leqslant 1$. *Then the function* $\varphi_z^*(x,t,u)$ *defined by* (2.8) *has the following properties*:
(i) $\varphi_z^* \in C(\mathbb{R}\times[0,T]\times\mathbb{R})$;
(ii) $\varphi_z^*(\cdot,\cdot,u) \in BUC^{\eta,\frac{\eta}{2}}(\mathbb{R}\times[r,T])$ *uniformly for* u *in bounded subsets of* $\mathbb{R}$;
(iii) *the partial derivative* $\partial_u\varphi_z^*$ *with respect to* u *exists and is uniformly continuous and bounded on* $\mathbb{R}\times[0,T]\times[-R,R]$ *for each* $R > 0$.

Proof. (i) By the continuity of φ and z, it is easily seen that φ_z^* is continuous.
(ii) By (H3 i), it is straightforward to check that $\varphi_z^*(\cdot,\cdot,u) \in BUC(\mathbb{R}\times[0,T])$ uniformly for u in bounded subsets of $\mathbb{R}$. Let $R > 0$, (x,t), $(y,s) \in \mathbb{R}\times[0,T]$, and $|u| \leqslant R$. By (H3

i–ii) we have

$$
\begin{aligned}
|\varphi_z^*(x,t,u)-\varphi_z^*(y,s,u)| &= |\varphi(x+z(t),t,z(t),u)-\varphi(y+z(s),s,z(s),u)|\\
&\leqslant |\varphi(x+z(t),t,z(t),u)-\varphi(y+z(t),s,z(t),u)|\\
&\quad + |\varphi(y+z(t),s,z(t),u)-\varphi(y+z(t),s,z(s),u)|\\
&\quad + |\varphi(y+z(t),s,z(s),u)-\varphi(y+z(s),s,z(s),u)|\\
&\leqslant \sup_{\substack{|\xi|\leqslant\|z\|_T\\ |v|\leqslant R}} [\varphi(\cdot,\cdot,\xi,v)]_{\mathbb{R},[0,T],\eta,\frac{\eta}{2}}(|x-y|^\eta+|t-s|^{\frac{\eta}{2}})+C_1|z(t)-z(s)|\\
&\quad + \sup_{\substack{|\xi|\leqslant\|z\|_T\\ |v|\leqslant R}} [\varphi(\cdot,\cdot,\xi,v)]_{\mathbb{R},[0,T],\eta,\frac{\eta}{2}}|z(t)-z(s)|^\eta
\end{aligned}
$$

where C_1 depends on $\|z\|_T$; hereafter $\|z\|_T$ stands for $\|z\|_{C[0,T]}$. For the quantity $[\varphi(\cdot,\cdot,\xi,v)]_{\mathbb{R},[0,T],\eta,\frac{\eta}{2}}$, see Appendix. Since z is Hölder continuous on $[r,T]$ with exponent β with $\frac{1}{2}\leqslant\beta\leqslant 1$, we obtain

$$
|\varphi_z^*(x,t,u)-\varphi_z^*(y,s,u)|\leqslant C_2(|x-y|^\eta+|t-s|^{\frac{\eta}{2}}),\quad (x,t),\ (y,s)\in\mathbb{R}\times[r,T]
$$

for $|u|\leqslant R$, where C_2 is independent of x, y, t, s, and u.

(iii) From (H3 iii) the assertion is obvious. ■

Set $X_0 := BUC(\mathbb{R})$, $X_1 := \{u\in X_0 \mid u_{xx} \text{ exists in } X_0\}$ and let $X_\alpha := (X_0,X_1)^0_{\alpha,\infty}$ be the interpolation space between X_0 and X_1 for $0\leqslant\alpha\leqslant 1$ (see [6, 4.11 Definition, p.50] for the definition). For $z\in C[0,T]$, define an operator $F_z : [0,T]\times X_0\to X_0$ by

$$
F_z(t,u)(x) := \varphi_z^*(x,t,u(x)),\quad x\in\mathbb{R}
$$

for $t\in[0,T]$ and $u\in X_0$.

Proposition 3.1. *The operator F_z is well-defined and has the following properties:*

(i) *There exists an increasing function $\rho_z : [0,+\infty[\ \to [0,+\infty[$ such that for any $R>0$,*

$$
\|F_z(t,u)-F_z(t,v)\|_{X_0}\leqslant\rho_z(R)\|u-v\|_{X_0}
$$

for all $t\in[0,T]$ and u, $v\in X_0$ with $\|u\|_{X_0}$, $\|v\|_{X_0}\leqslant R$.

(ii) *If in addition $z\in C^{0,\beta}[r,T]$ with $\frac{1}{2}\leqslant\beta\leqslant 1$, then there is an increasing function $\rho_{r,z} : [0,+\infty[\ \to [0,+\infty[$ such that for any $R>0$,*

$$
\|F_z(t,u)-F_z(s,v)\|_{X_0}\leqslant\rho_{r,z}(R)(|t-s|^{\frac{\eta}{2}}+\|u-v\|_{X_0})
$$

for all t, $s\in[r,T]$ and u, $v\in X_0$ with $\|u\|_{X_0}$, $\|v\|_{X_0}\leqslant R$.

Proof. (i) This is an easy consequence of (H3 iii). See Remark 2.1 (3).

(ii) By virtue of Lemma 3.1, the assertion follows from [6, Proposition 15.16 and Remark 15.18 (b)]. ■

Defining an operator A with domain $D(A)=X_1$ by

$$
Au := \varepsilon u_{xx}\quad\text{for } u\in D(A),
$$

the initial value problem $(P_z;r,\omega)$ can be rewritten as the following abstract initial value problem in X_0:

$$
\begin{cases}
\dfrac{d\nu_z}{dt}(t) = A\nu_z(t)+F_z(t,\nu_z(t)), & t\in\ ]r,T]\\
\nu_z(r)=\omega\in X_0.
\end{cases}
\qquad (AP_z;r,\omega)
$$

The next proposition is well known. See [6, Theorem 1.5].

Proposition 3.2. *The operator A generates an analytic semigroup $\{T(t)\}_{t\geqslant 0}$ of contractions on X_0. Moreover, $T(t)$ is given by*

$$(T(t)u)(x) = \int_{\mathbb{R}} K_\varepsilon(x-y,t)u(y)dy, \quad x \in \mathbb{R},\ t>0 \tag{3.1}$$

for $u \in X_0$, where

$$K_\varepsilon(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}}\exp\Big(-\frac{x^2}{4\varepsilon t}\Big), \quad x \in \mathbb{R},\ t>0.$$

Definition 3.1. (i) By a classical solution of $(P_z; r, \omega)$, we mean a function $\nu \in BUC(\mathbb{R}\times[r,T]) \cap BUC^{2,1}(\mathbb{R}\times[\delta,T])$ for all $\delta \in]r,T]$ satisfying $(P_z; r,\omega)$.

(ii) By a regular solution of $(P_z; r,\omega)$, we mean a function $\nu \in BUC(\mathbb{R}\times[r,T]) \cap BUC^{2+\eta,1+\frac{\eta}{2}}(\mathbb{R}\times[\delta,T])$ for all $\delta \in]r,T]$, satisfying $(P_z; r,\omega)$.

Definition 3.2. (i) If $\omega \in X_\alpha$, a function $\nu \in C([r,T];X_\alpha)\cap C^1(]r,T];X_0)$ $(0 \leqslant \alpha \leqslant 1)$ satisfying $(AP_z; r,\omega)$ for each $t \in]r,T]$ is called a classical solution of $(AP_z; r,\omega)$.

(ii) A function $\nu \in C([r,T];X_0)$ is called a strong solution of $(AP_z; r,\omega)$, if $\nu(t)$ is differentiable for *a.e.* $t \in]r,T[$ and $\nu(t) \in X_1$ for $t \in]r,T[$ and satisfies $(AP_z; r,\omega)$ for *a.e.* $t \in]r,T[$.

(iii) A function $\nu \in C([r,T];X_0)$ satisfying

$$\nu(t) = T(t-r)\omega + \int_r^t T(t-s)F_z(s,\nu(s))ds \quad r \leqslant t \leqslant T,$$

is called a mild solution of $(AP_z; r,\omega)$.

Proposition 3.3.

(i) *Each classical solution of $(AP_z; r,\omega)$ is a strong solution of $(AP_z; r,\omega)$.*

(ii) *Each strong solution ν of $(AP_z; r,\omega)$ is a mild solution of $(AP_z; r,\omega)$.*

(iii) *There exists at most one mild solution of $(AP_z; r,\omega)$.*

Proof. (i) is clear from Definition 3.2. The proof of (ii) is obtained in a standard way. (iii) is a consequence of Proposition 3.1 (i) and Gronwall's Lemma. ■

Proposition 3.4. *Suppose that $z \in C[0,T]\cap C^{0,\beta}[r,T]$, $\frac{1}{2} \leqslant \beta \leqslant 1$. Then*

(i) *each classical solution of $(P_z; r,\omega)$ is a classical solution of $(AP_z; r,\omega)$;*

(ii) *each classical solution of $(AP_z; r,\omega)$ with initial value $\omega \in X_\alpha$, $0 \leqslant \alpha \leqslant 1$, is a regular solution of $(P_z; r,\omega)$.*

Proof. Owing to Lemma 3.1 and Proposition 3.1 (i), the assertions are derived from [6, Theorem 25.2 and Remark 25.3 (a)]. ■

Proposition 3.5. *Let $z \in C[0,T]\cap C^{0,\beta}[r,T]$, $\frac{1}{2} \leqslant \beta \leqslant 1$ and $\omega \in X_\alpha$, $0 \leqslant \alpha \leqslant 1$. Then*

(i) *$(AP_z; r,\omega)$ has a unique classical solution ν_z;*

(ii) *the problem $(P_z; r,\omega)$ with $0 \leqslant \omega(x) \leqslant 1$ has a unique regular solution ν_z satisfying $0 \leqslant \nu_z \leqslant 1$ on $\mathbb{R}\times[r,T]$.*

Proof. Due to [6, Section 25], (i) follows from Lemma 3.1. Part (i) together with Proposition 3.4 (ii) yields the existence of a unique regular solution ν_z of $(P_z; r,\omega)$. By virtue of (H3 v), it is easily seen that $\nu \equiv 0$ (resp. $\nu \equiv 1$) is a super-(resp. sub-)solution of $(P_z; r,\omega)$. Since $0 \leqslant \omega(x) \leqslant 1$ for $x \in \mathbb{R}$, it follows from [6, Theorem 25.6] that $0 \leqslant \nu_z \leqslant 1$. ■

When z is given to be a continuous function, we have:

Theorem 3.1. *For $z \in C[0,T]$ and $\omega \in X_0$ with $0 \leqslant \omega \leqslant 1$, the problem $(AP_z; r, \omega)$ has a unique mild solution $\nu_z \in C([r,T]; X_0)$ satisfying*

$$\nu_z(x,t) = \int_{\mathbb{R}} K_\varepsilon(x-y, t-r)\omega(y)dy \tag{3.2}$$
$$+ \int_r^t \int_{\mathbb{R}} K_\varepsilon(x-y, t-\tau)\varphi_z^*(y,\tau,\nu_z(y,\tau))dyd\tau, \quad (x,t) \in \mathbb{R} \times\,]r,T],$$

and $0 \leqslant \nu_z \leqslant 1$ on $\mathbb{R} \times [r,T]$. Moreover, suppose that $z_n \in C^{0,\beta}[r,T]$, $\frac{1}{2} \leqslant \beta \leqslant 1$, and $z_n \to z$ in $C[r,T]$ and that $\omega_n \to \omega$ in X_0 and $0 \leqslant \omega_n(x) \leqslant 1$. Let ν_n be a classical solution of $(AP_{z_n}; r, \omega_n)$, which exists by Proposition 3.5 (i). Then $\nu_n \to \nu_z$ in $C([r,T]; X_0)$ as $n \to \infty$.

Proof. Let $z_n \in C^{0,\beta}[r,T]$, $\frac{1}{2} \leqslant \beta \leqslant 1$, and let $z_n \to z$ in $C[r,T]$, and $\omega_n \to \omega$ in X_0. Let ν_n be a unique classical solution of $(AP_{z_n}; r, \omega_n)$. Then, ν_n is also a mild solution, i.e.,

$$\nu_n(t) = T(t-r)\omega_n + \int_r^t T(t-s)F_{z_n}(s, \nu_n(s))ds, \quad r \leqslant t \leqslant T, \tag{3.3}$$

and from Propositions 3.4 and 3.5, ν_n satisfies $0 \leqslant \nu_n(x,t) \leqslant 1$ for $(x,t) \in \mathbb{R} \times [r,T]$. Since $T(t)$ is a contraction semigroup on X_0, it follows from (3.3) that

$$\|\nu_n(t) - \nu_m(t)\|_{X_0} \leqslant \|\omega_n - \omega_m\|_{X_0} + \int_r^t \|F_{z_n}(s,\nu_n(s)) - F_{z_m}(s,\nu_m(s))\|_{X_0} ds$$
$$\leqslant \|\omega_n - \omega_m\|_{X_0} + \int_r^t \|F_{z_n}(s,\nu_n(s)) - F_z(s,\nu_n(s))\|_{X_0} ds \tag{3.4}$$
$$+ \int_r^t \|F_z(s,\nu_n(s)) - F_z(s,\nu_m(s))\|_{X_0} ds + \int_r^t \|F_z(s,\nu_m(s)) - F_{z_m}(s,\nu_m(s))\|_{X_0} ds$$

for $t \in [r,T]$. Using the relations $0 \leqslant \nu_n \leqslant 1$, the uniform boundedness of $z_n(s)$ and (H3 i–ii), we have

$$\|F_{z_n}(s,\nu_n(s)) - F_z(s,\nu_n(s))\|_{X_0} \leqslant C_3 \|z_n - z\|^\eta_{C[r,T]}, \quad s \in [r,T] \tag{3.5}$$

for all $n \in \mathbb{N}$. On the other hand, it follows from Proposition 3.1 (i) that

$$\|F_z(s,\nu_n(s)) - F_z(s,\nu_m(s))\|_{X_0} \leqslant \rho_z(1)\|\nu_n(s) - \nu_m(s)\|_{X_0}, \quad s \in [r,T] \tag{3.6}$$

for all n, m. From (3.4)–(3.6) we obtain for any $t \in [r,T]$,

$$\|\nu_n(t) - \nu_m(t)\|_{X_0} \leqslant \|\omega_n - \omega_m\|_{X_0} + C_3 T(\|z_n - z\|^\eta_{C[r,T]} + \|z_m - z\|^\eta_{C[r,T]})$$
$$+ \rho_z(1) \int_r^t \|\nu_n(s) - \nu_m(s)\|_{X_0} ds.$$

By Gronwall's Lemma,

$$\|\nu_n(t) - \nu_m(t)\|_{X_0}$$
$$\leqslant \left\{\|\omega_n - \omega_m\|_{X_0} + C_3 T(\|z_n - z\|^\eta_{C[r,T]} + \|z_m - z\|^\eta_{C[r,T]})\right\} \exp(\rho_z(1)T), \quad t \in [r,T].$$

Taking the supremum over $t \in [r, T]$ and then passing to the limit as n, $m \to \infty$, we find that $\{\nu_n\}$ is a Cauchy sequence in $C([r, T]; X_0)$, and so there exists $\nu_z \in C([r, T]; X_0)$ such that $\nu_n \to \nu_z$ in $C([r, T]; X_0)$ as $n \to \infty$. Since $\nu_z(x, t)$ is the uniform limit of $\nu_n(x, t)$, ν_z satisfies $0 \leqslant \nu_z \leqslant 1$ on $\mathbb{R} \times [r, T]$.

Let us prove that ν_z is a mild solution of $(AP_z; r, \omega)$. It follows from Proposition 3.1 (i) and (3.5) that

$$\begin{aligned}
&\left\| \int_r^t T(t-s) F_{z_n}(s, \nu_n(s)) ds - \int_r^t T(t-s) F_z(s, \nu_z(s)) ds \right\|_{X_0} \\
&\leqslant \int_r^t \| F_{z_n}(s, \nu_n(s)) - F_z(s, \nu_z(s)) \|_{X_0} ds \\
&\leqslant \int_r^t \| F_{z_n}(s, \nu_n(s)) - F_z(s, \nu_n(s)) \|_{X_0} ds + \int_r^t \| F_z(s, \nu_n(s)) - F_z(s, \nu_z(s)) \|_{X_0} ds \\
&\leqslant T C_3 \| z_n - z \|_{C[r,T]}^{\eta} + T \rho_z(1) \| \nu_n - \nu_z \|_{C([r,T]; X_0)} \quad \longrightarrow 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence, letting $n \to \infty$ in (3.3), we have

$$\nu_z(t) = T(t-r)\omega + \int_r^t T(t-s) F_z(s, \nu_z(s)) ds, \quad t \in [r, T]. \tag{3.7}$$

Thus ν_z is a mild solution of $(AP_z; r, \omega)$. Since the mild solution of $(AP_z; r, \omega)$ is unique by Proposition 3.3 (iii), ν_z does not depend on the choice of $\{z_n\}$. Finally, (3.2) is a consequence of (3.1) and (3.7). ■

In the rest of this section, we give some properties of the solutions ν_z to $(AP_z; 0, \omega)$ or $(P_z; 0, \omega)$ with $\omega(\cdot) = u_0(\cdot + z(0))$ for later use.

Lemma 3.2. *Let ν_z be a mild solution of $(AP_z; 0, \omega)$ with $\omega = u_0(\cdot + z(0))$. Then there exists a constant $C_4 > 0$, independent of z and ε, such that for any $z \in C[0, T]$ and $\alpha \in [0, 2]$,*

$$\int_{\mathbb{R}} |x|^\alpha \nu_z(x, t) dx \leqslant C_4 (1 + \|z\|_T^2), \quad t \in [0, T].$$

Corollary 3.1. (i) *For any $R > 0$, there is a constant $M_5(R) > 0$ such that for any $z \in C[0, T]$ with $\|z\|_T \leqslant R$ and $\alpha \in [0, 2]$,*

$$\int_{\mathbb{R}} |x|^\alpha \nu_z(x, t) dx \leqslant M_5(R), \quad t \in [0, T].$$

(ii) *For each $z \in C[0, T]$, $w_z^* \nu_z \in L^\infty(0, T; L^1(\mathbb{R}))$.*

Proof of Corollary 3.1. (i) is obvious from Lemma 3.2. Let $z \in C[0, T]$. Recalling Remark 2.1 (2), we get

$$|w_z^*(x, t) \nu_z(x, t)| = |w(x + z(t))| \nu_z(x, t) \leqslant C_w (|x| + |z(t)|) \nu_z(x, t)$$

for $(x, t) \in \mathbb{R} \times [0, T]$. Thus it follows from Lemma 3.2 that $w_z^* \nu_z \in L^\infty(0, T; L^1(\mathbb{R}))$. ■

Before proving Lemma 3.2, we recall the following elementary fact:

Lemma 3.3. *If $f \in L^\infty(\mathbb{R})$ and $x^2 f \in L^1(\mathbb{R})$ then $f \in L^p(\mathbb{R})$ for any $1 \leqslant p < \infty$ and*

$$\|f\|_{L^p}^p \leqslant \|f\|_{L^\infty}^{p-1}(\|x^2 f\|_{L^1} + 2\|f\|_{L^\infty})$$

is satisfied; moreover $xf \in L^1(\mathbb{R})$.

Proof of Lemma 3.2. Since $|x|^\alpha \leqslant 1 + x^2$ for $\alpha \in [0, 2]$, it suffices to show that there exists a constant $C_5 > 0$, independent of z and ε, such that

$$\int_{\mathbb{R}} x^2 \nu_z(x,t)dx \leqslant C_5(1 + \|z\|_T^2), \quad t \in [0,T] \tag{3.8}$$

for $z \in C[0,T]$. Indeed, (3.8) implies $x^2\nu_z \in L^\infty(0,T; L^1(\mathbb{R}))$; then since $0 \leqslant \nu_z \leqslant 1$, it follows from Lemma 3.3 that

$$\int_{\mathbb{R}} |x|^\alpha \nu_z(x,t)dx \leqslant \int_{\mathbb{R}} (1 + x^2)\nu_z(x,t)dx \leqslant 2\Big(\int_{\mathbb{R}} x^2\nu_z(x,t)dx + 1\Big),$$

which implies the desired estimate.

Let us show that (3.8) holds. Using (3.2), (2.8) and (H3 vi), we have

$$\begin{aligned}\int_{\mathbb{R}} x^2\nu_z(x,t)dx &= \int_{\mathbb{R}} x^2 \int_{\mathbb{R}} K_\varepsilon(x-y,t)u_0(y+z(0))dydx \\ &\quad + \int_{\mathbb{R}} x^2 \int_0^t K_\varepsilon(x-y,t-\tau)\varphi(y+z(\tau),\tau,z(\tau),\nu_z(y,\tau))dyd\tau dx \\ &\leqslant \int_{\mathbb{R}} u_0(y+z(0)) \int_{\mathbb{R}} x^2 K_\varepsilon(x-y,t)dxdy \\ &\quad + \int_0^t \int_{\mathbb{R}} \mathcal{F}(y+z(\tau),\tau) \int_{\mathbb{R}} x^2 K_\varepsilon(x-y,t-\tau)dxdyd\tau, \quad t \in [0,T].\end{aligned}$$

By a change of variables,

$$\begin{aligned}\int_{\mathbb{R}} x^2 K_\varepsilon(x-y,t)dx &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} (\sqrt{4\varepsilon t}\xi + y)^2 e^{-\xi^2}d\xi \\ &\leqslant C_6 \int_{\mathbb{R}} (4\varepsilon t\xi^2 + y^2)e^{-\xi^2}d\xi \leqslant C_7(1+y^2)\end{aligned}$$

for all $y \in \mathbb{R}$ and $t \in [0,T]$. Hence,

$$\begin{aligned}&\int_{\mathbb{R}} x^2\nu_z(x,t)dx \leqslant C_7\int_{\mathbb{R}}(1+y^2)u_0(y+z(0))dy + C_7\int_0^t\int_{\mathbb{R}}(1+y^2)\mathcal{F}(y+z(\tau),\tau)dyd\tau \\ &= C_7\Big\{\int_{\mathbb{R}}[1+(\xi-z(0))^2]u_0(\xi)d\xi + \int_0^t\int_{\mathbb{R}}[1+(\xi-z(\tau))^2]\mathcal{F}(\xi,\tau)d\xi d\tau\Big\} \\ &\leqslant C_8(1+\|z\|_T^2)\Big\{\int_{\mathbb{R}}(1+\xi^2)u_0(\xi)d\xi + \int_0^t\int_{\mathbb{R}}(1+\xi^2)\mathcal{F}(\xi,\tau)d\xi d\tau\Big\}\end{aligned}$$

for all $t \in [0,T]$. Since u_0, $x^2u_0 \in L^1(\mathbb{R})$ and $\mathcal{F}$, $x^2\mathcal{F} \in L^1(0,T;L^1(\mathbb{R}))$ by (H4) and (H3 vi), we obtain (3.8). ■

If ν is a regular solution of $(P_z; 0, \omega)$ with $\omega = u_0(\cdot + z(0))$, we obtain the following

Lemma 3.4. *Let ν be a regular solution of $(P_z; 0, u_0(\cdot + z(0)))$ satisfying $0 \leqslant \nu \leqslant 1$ on $\mathbb{R} \times [0, T]$. Then $\nu \in L^\infty(0, T; L^1(\mathbb{R}))$ and*

$$\|\nu\|_{L^\infty(0,T;L^1(\mathbb{R}))} \leqslant \|u_0\|_{L^1(\mathbb{R})} + \|\mathcal{F}\|_{L^1(\mathbb{R}\times]0,T[)}.$$

Proof. Let $0 < \delta < t \leqslant T$. Multiply the equation by $\psi_m(x) := \exp(-x^2/m)$ $(x \in \mathbb{R}, m \in \mathbb{N})$ and then integrate the result to obtain

$$\int_{\mathbb{R}} \nu(x,t)\psi_m(x)dx - \int_{\mathbb{R}} \nu(x,\delta)\psi_m(x)dx - \varepsilon \int_\delta^t d\tau \int_{\mathbb{R}} \nu_{xx}(x,\tau)\psi_m(x)dx$$
$$= \int_\delta^t d\tau \int_{\mathbb{R}} \varphi_z^*(x,\tau,\nu(x,\tau))\psi_m(x)dx.$$

Integration by parts and (H3 vi) yields that

$$\int_{\mathbb{R}} \nu(x,t)\psi_m(x)dx \tag{3.9}$$
$$\leqslant \int_{\mathbb{R}} \nu(x,\delta)\psi_m(x)dx + \varepsilon \int_\delta^t d\tau \int_{\mathbb{R}} \nu(x,\tau)\frac{d^2\psi_m}{dx^2}(x)dx + \|\mathcal{F}\|_{L^1(\mathbb{R}\times]0,T[)}.$$

Since $\nu \in BUC(\mathbb{R} \times [0,T])$ and $\psi_m \in L^1(\mathbb{R})$,

$$\lim_{\delta\downarrow 0} \int_{\mathbb{R}} \nu(x,\delta)\psi_m(x)dx = \int_{\mathbb{R}} u_0(x+z(0))\psi_m(x)dx.$$

Thus letting $\delta \downarrow 0$ in (3.9) yields

$$\int_{\mathbb{R}} \nu(x,t)\psi_m(x)dx$$
$$\leqslant \int_{\mathbb{R}} u_0(x+z(0))\psi_m(x)dx + \varepsilon \int_0^t d\tau \int_{\mathbb{R}} \nu(x,\tau)\frac{d^2\psi_m}{dx^2}(x)dx + \|\mathcal{F}\|_{L^1(\mathbb{R}\times]0,T[)}$$
$$\leqslant \|u_0\|_{L^1(\mathbb{R})} + \int_0^T d\tau \int_{\mathbb{R}} \nu(x,\tau)\left|\frac{d^2\psi_m}{dx^2}(x)\right|dx + \|\mathcal{F}\|_{L^1(\mathbb{R}\times]0,T[)} \tag{3.10}$$

for all $t \in]0,T]$ and $m \in \mathbb{N}$.

Now we claim that there exists a constant $C_9 > 0$, independent of m, z, ε, such that

$$\int_0^T d\tau \int_{\mathbb{R}} \nu(x,\tau)\left|\frac{d^2\psi_m}{dx^2}(x)\right|dx \leqslant \frac{C_9}{\sqrt{m}}. \tag{3.11}$$

A simple calculation shows $|(d^2/dx^2)\psi_m(x)| \leqslant (2/m)\psi_m(x) - (2/m)x(d/dx)\psi_m(x)$. Thus by integration by parts in space variable, we have

$$\int_0^T d\tau \int_{\mathbb{R}} \nu(x,\tau)\left|\frac{d^2\psi_m}{dx^2}(x)\right|dx \leqslant \int_0^T d\tau \int_{\mathbb{R}}\left|\frac{d^2\psi_m}{dx^2}(x)\right|dx$$
$$\leqslant T\left\{\frac{2}{m}\int_{\mathbb{R}} \psi_m(x)dx - \frac{2}{m}\int_{\mathbb{R}} x\frac{d\psi_m}{dx}(x)dx\right\} = \frac{4T}{m}\int_{\mathbb{R}} \psi_m(x)dx = \frac{4T}{\sqrt{m}}\int_{\mathbb{R}} e^{-\xi^2}d\xi,$$

as claimed. From (3.10) and (3.11), we get

$$\int_{\mathbb{R}} \nu(x,t)\psi_m(x)dx \leqslant \|u_0\|_{L^1(\mathbb{R})} + \frac{C_9}{\sqrt{m}} + \|\mathcal{F}\|_{L^1(\mathbb{R}\times]0,T[)}$$

for all $t \in]0,T]$ and $m \in \mathbb{N}$. Since the positive function $\nu(x,t)\psi_m(x)$ converges increasingly to $\nu(x,t)$ as $m \to \infty$ for each $x \in \mathbb{R}$ and $t \in]0,T]$, by the Beppo-Levi Monotone Convergence Theorem, we have $\nu(\cdot,t) \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \nu(x,t)dx \leqslant \|u_0\|_{L^1(\mathbb{R})} + \|\mathcal{F}\|_{L^1(\mathbb{R}\times]0,T[)}$$

for all $t \in]0,T]$. This proves the lemma. ■

4. Equivalent problems

In view of the previous sections, we can reduce Problem (P_1) to the following problem (P_2). Actually, in the last of this section, we will find that Problems (P_1) and (P_2) are equivalent in conjunction with another problem (P_3).

Problem (P_2). Find $v \in BUC(\mathbb{R}\times[0,T]) \cap BUC^{2+\eta,1+\frac{\eta}{2}}(\mathbb{R}\times[\delta,T])$ for each $\delta \in]0,T]$ and $z \in C^{0,1}[0,T]$ such that $w_z^* v \in L^\infty(0,T;L^1(\mathbb{R}))$ and satisfying

$$a < \int_{\mathbb{R}} w_z^*(x,t)v(x,t)dx < b, \quad t \in [0,T], \tag{4.1}$$

$$z(t) = L\Big(\int_{\mathbb{R}} w_z^*(x,t)v(x,t)dx\Big), \quad t \in [0,T]; \tag{4.2}$$

$$v_t(x,t) - \varepsilon v_{xx}(x,t) = \varphi_z^*(x,t,v(x,t)), \quad (x,t) \in \mathbb{R}\times]0,T], \tag{4.3}$$

$$v(x,0) = u_0(x+z(0)), \quad x \in \mathbb{R}. \tag{4.4}$$

Here φ_z^* and w_z^* are the functions defined by (2.8).

Notice that Problem (P_2) has a meaning even if we only ask for $z \in C[0,T]$, while that is no longer true for Problem (P_1).

Proposition 4.1. *Suppose that a pair (v,z) solves Problem (P_2). Then setting*

$$u(x,t) := v(x-z(t),t), \quad (x,t) \in \mathbb{R}\times[0,T], \tag{4.5}$$

the pair (u,z) solves Problem (P_1).

Proof. It is easily checked that $u \in BUC^{2+\eta,\frac{\eta}{2}}(\mathbb{R}\times[\delta,T])$ for each $\delta \in]0,T]$. Since $v \in BUC^{2+\eta,1+\frac{\eta}{2}}(\mathbb{R}\times[\delta,T])$ and $z \in C^{0,1}[0,T]$, we have $u(x,\cdot) \in C^{0,1}[\delta,T]$ uniformly for $x \in \mathbb{R}$. Since z is differentiable *a.e.* on $]0,T[$,

$$u_t(x,t) = -z'(t)v_x(x-z(t),t) + v_t(x-z(t),t) \tag{4.6}$$

for all $x \in \mathbb{R}$, *a.e.* $t \in]0,T[$; and hence $u_t(\cdot,t) \in BUC(\mathbb{R})$ for *a.e.* $t \in]0,T[$ since $v_x(\cdot - z(t),t)$, $v_t(\cdot - z(t),t) \in BUC(\mathbb{R})$ for every $t \in]0,T]$. By changing variables,

$$\int_{\mathbb{R}} |w(x)u(x,t)|dx = \int_{\mathbb{R}} |w(x+z(t))u(x+z(t),t)|dx = \int_{\mathbb{R}} |w_z^*(x,t)v(x,t)|dx$$

for each $t \in [0,T]$, which leads to $wu \in L^\infty(0,T;L^1(\mathbb{R}))$ since $w_z^* v \in L^\infty(0,T;L^1(\mathbb{R}))$. Also, by changing variables, (2.1) and (2.2) hold from (4.1) and (4.2), respectively. From (4.5)–(4.6), and (4.3), one easily obtains

$$\begin{aligned} u_t(x,t) - \varepsilon u_{xx}(x,t) + z'(t)u_x(x,t) &= v_t(x-z(t),t) - \varepsilon v_{xx}(x-z(t),t) \\ &= \varphi_z^*(x-z(t),t,z(t),v(x-z(t),t)) = \varphi(x,t,z(t),u(x,t)) \end{aligned}$$

for all $x \in \mathbb{R}$, *a.e.* $t \in]0,T[$, and (2.3) follows. Finally, due to (4.4) and (4.5), $u(x,0) = v(x-z(0),0) = u_0(x)$ for $x \in \mathbb{R}$, and the proof is complete. ■

We now define a function μ_z for each $z \in C[0,T]$ by

$$\mu_z(x,t) := \nu_z(x-z(t),t), \quad (x,t) \in \mathbb{R} \times [0,T], \tag{4.7}$$

where ν_z is the mild solution of $(AP_z; 0, u_0(\cdot + z(0)))$ defined by Theorem 3.1. Then we have

Proposition 4.2. (i) *For any $z \in C^{0,1}[0,T]$ the function μ_z defined by (4.7) satisfies $\mu_z \in BUC(\mathbb{R}\times[0,T]) \cap BUC^{2+\eta,\frac{\eta}{2}}(\mathbb{R}\times[\delta,T])$ for all $\delta \in]0,T]$, $\mu_z(x,\cdot) \in C^{0,1}[\delta,T]$ uniformly for $x \in \mathbb{R}$ and $\delta \in]0,T]$, $\partial\mu_z(\cdot,t)/\partial t \in BUC(\mathbb{R})$ for a.e. $t \in]0,T[$, and fulfills*

$$\begin{cases} \dfrac{\partial \mu_z}{\partial t}(x,t) - \varepsilon \dfrac{\partial^2 \mu_z}{\partial x^2}(x,t) + z'(t)\dfrac{\partial \mu_z}{\partial x}(x,t) = \varphi(x,t,z(t),\mu_z(x,t)), \ x \in \mathbb{R}, \ a.e.\ t \in]0,T[, \\ \mu_z(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{4.8}$$

(ii) *For $z \in C[0,T]$, the function μ_z satisfies $w\mu_z \in L^\infty(0,T;L^1(\mathbb{R}))$ and $0 \leqslant \mu_z \leqslant 1$ on $\mathbb{R}\times[0,T]$. Moreover, for every $R > 0$, there exists a constant $M_6(R) > 0$ such that for any $z \in C[0,T]$ with $\|z\|_T \leqslant R$ and for any $\alpha \in [0,2]$, the inequality*

$$\int_{\mathbb{R}} |x|^\alpha \mu_z(x,t)dx \leqslant M_6(R)$$

holds for all $t \in [0,T]$.

Proof. (i) is obtained quite similarly to Proposition 4.1. The assertion (ii) is an easy consequence of Corollary 3.1. ■

We introduce the following problem which is shown to be mutually equivalent to Problems (P_1) and (P_2).

Problem (P_3). Find $z \in C^{0,1}[0,T]$ satisfying

$$a < \int_{\mathbb{R}} w(x)\mu_z(x,t)dx < b, \quad t \in [0,T], \tag{4.9}$$

$$z(t) = L\Big(\int_{\mathbb{R}} w(x)\mu_z(x,t)dx\Big), \quad t \in [0,T], \tag{4.10}$$

where μ_z is the one defined by (4.7).

Theorem 4.1. *Problems* (P_1), (P_2) *and* (P_3) *are mutually equivalent in the following sense:*

(i) *if* (u,z) *solves* (P_1), *then* z *solves* (P_3) *and* $\mu_z \equiv u$;

(ii) *if* z *solves* (P_3), *then* (ν_z, z) *solves* (P_2);

(iii) *if* (v,z) *solves* (P_2), *then* (u,z) *defined by* (4.5) *solves* (P_1) *and* $\nu_z \equiv v$,

where μ_z *is defined by* (4.7) *and* ν_z *is the regular solution of* $(P_z; 0, \omega)$ *with* $\omega = u_0(\cdot + z(0))$ *defined in Proposition* 3.5 (ii).

Proof. (i) Let (u,z) solves (P_1). Then it is clear that z solves (P_3). Let us prove that $\mu_z \equiv u$. Set

$$v(x,t) := u(x+z(t),t), \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{4.11}$$

Then, as in the proof of Proposition 4.1, we find that $v \in BUC(\mathbb{R}\times[0,T]) \cap BUC^{2+\eta,\frac{\eta}{2}}(\mathbb{R} \times [\delta,T])$ for all $\delta \in \,]0,T]$, $v(x,\cdot)$ is differentiable almost everywhere on $]0,T[$ for each $x \in \mathbb{R}$, and $v_t(\cdot,t) \in BUC(\mathbb{R})$ for *a.e.* $t \in \,]0,T[$; and

$$\begin{aligned} v_t(x,t) - \varepsilon v_{xx}(x,t) &= \varphi_z^*(x,t,v(x,t)), \quad x \in \mathbb{R}, \ a.e.\ t \in \,]0,T[, \\ v(x,0) &= u_0(x+z(0)), \quad x \in \mathbb{R}. \end{aligned}$$

Since $v_t(\cdot,t), v_{xx}(\cdot,t) \in X_0$ for *a.e.* $t \in \,]0,T[$ and $v \in C([0,T];X_0)$, v is a strong solution of $(AP_z; 0, u_0(\cdot + z(0)))$. Hence v is also a mild solution by Proposition 3.3 (ii). Since ν_z defined by Theorem 3.1 is a unique mild solution of $(AP_z; 0, u_0(\cdot + z(0)))$, we find $v \equiv \nu_z$; and so $u \equiv \mu_z$ by (4.7) and (4.11).

(ii) This is clear from (4.7), Proposition 3.5 (ii) and Corollary 3.1 (ii).

(iii) This is a consequence of Proposition 4.1, part (i) of this proposition and (4.7). ■

Remark. Like Problem (P_2), Problem (P_3) has a meaning even when $z \in C[0,T]$, since μ_z is defined for $z \in C[0,T]$ by (4.7).

5. Proof of Main Theorem

First, we prove the uniqueness result.

Proposition 5.1. *Problem* (P_i) $(i = 1,\ 2,\ 3)$ *has at most one solution.*

Proof. By virtue of Theorem 4.1, it suffices to prove only for (P_3). Note that if z is a solution of (P_3), then (ν_z, z) is a solution of (P_2).

Suppose that ζ and θ are solutions of (P_3). Put $\hat{K} := \max\{\|\zeta\|_T, \|\theta\|_T\}$. If $\zeta(t) \leqslant \theta(t)$ $(0 \leqslant t \leqslant T)$, then it follows from (H1)–(H2) that

$$\begin{aligned} &M_1(\hat{K})^{-1}|\zeta(t) - \theta(t)| \leqslant \lambda(\zeta(t)) - \lambda(\theta(t)) \\ &= \int_{\mathbb{R}} [w(x+\zeta(t)) - w(x+\theta(t))]\nu_\zeta(x,t)dx + \int_{\mathbb{R}} w(x+\theta(t))[\nu_\zeta(x,t) - \nu_\theta(x,t)]dx \\ &\leqslant \int_{\mathbb{R}} |w(x+\theta(t))\{\nu_\zeta(x,t) - \nu_\theta(x,t)\}|dx \\ &\leqslant C_w(1+\hat{K}) \int_{\mathbb{R}} (1+|x|)|\nu_\zeta(x,t) - \nu_\theta(x,t)|dx. \end{aligned} \tag{5.1}$$

Notice that ν_ζ and ν_θ are mild solutions of $(AP_\zeta; 0, u_0(\cdot + \zeta(0)))$ and $(AP_\theta; 0, u_0(\cdot + \theta(0)))$ respectively (defined by Theorem 3.1) and $\zeta(0) = L\Big(\int_{\mathbb{R}} w(x)u_0(x)dx\Big) = \theta(0)$. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}} (1+|x|)|\nu_\zeta(x,t) - \nu_\theta(x,t)|dx \\
&\leqslant \int_0^t \int_{\mathbb{R}} |\varphi(y+\zeta(\tau),\tau,\zeta(\tau),\nu_\zeta(y,\tau)) - \varphi(y+\theta(\tau),\tau,\theta(\tau),\nu_\theta(y,\tau))| \\
&\qquad \times \int_{\mathbb{R}} (1+|x|)K_\varepsilon(x-y,t-\tau)dxdyd\tau
\end{aligned}$$

for $t \in [0,T]$. By changing variables, it is easily checked that

$$\int_{\mathbb{R}} (1+|x|)K_\varepsilon(x-y,t)dx \leqslant C_{10}(1+|y|), \quad y \in \mathbb{R},\ t \in [0,T],$$

where C_{10} is independent of y, t (see the proof of Lemma 3.2). Hence, due to Corollary 3.1, (H3 viii) and Remark 2.1 (3), we have

$$\begin{aligned}
&\int_{\mathbb{R}} (1+|x|)|\nu_\zeta(x,t) - \nu_\theta(x,t)|dx \\
&\leqslant C_{10} \int_0^t \int_{\mathbb{R}} (1+|y|)|\varphi(y+\zeta(\tau),\tau,\zeta(\tau),\nu_\zeta(y,\tau)) - \varphi(y+\theta(\tau),\tau,\theta(\tau),\nu_\zeta(y,\tau))|dyd\tau \\
&\quad + C_{10} \int_0^t \int_{\mathbb{R}} (1+|y|)|\varphi(y+\theta(\tau),\tau,\theta(\tau),\nu_\zeta(y,\tau)) - \varphi(y+\theta(\tau),\tau,\theta(\tau),\nu_\theta(y,\tau))|dyd\tau \\
&\leqslant C_{10} \int_0^t h(\tau)|\zeta(\tau)-\theta(\tau)|\left\{1 + \int_{\mathbb{R}} (1+|y|)\nu_\zeta(y,\tau)dy\right\}d\tau \\
&\quad + C_{10}M_4(\hat{K}+1)\int_0^t \int_{\mathbb{R}} (1+|y|)|\nu_\zeta(y,\tau) - \nu_\theta(y,\tau)|dyd\tau.
\end{aligned}$$

By the Gronwall Lemma, we have

$$\begin{aligned}
&\int_{\mathbb{R}} (1+|x|)|\nu_\zeta(x,t) - \nu_\theta(x,t)|dx \\
&\leqslant C_{10}\exp(C_{11}T)\int_0^t h(\tau)|\zeta(\tau)-\theta(\tau)|\left\{1 + \int_{\mathbb{R}} (1+|x|)\nu_\zeta(x,\tau)dx\right\}d\tau
\end{aligned}$$

for $t \in [0,T]$, where $C_{11} = C_{10}M_4(\hat{K}+1)$. Returning to (5.1), we have

$$\begin{aligned}
|\zeta(t) - \theta(t)| &\leqslant M_1(\hat{K})C_w(1+\hat{K})C_{10}\exp(C_{11}T) \\
&\qquad \times \int_0^t h(\tau)\left\{1 + \int_{\mathbb{R}} (1+|x|)\nu_\zeta(x,\tau)dx\right\}|\zeta(\tau)-\theta(\tau)|d\tau
\end{aligned}$$

for each $t \in [0,T]$. Again, by the Gronwall Lemma, we conclude that $\zeta \equiv \theta$ on $[0,T]$. ■

Remark 5.1. Notice that we use only the continuity of θ and ζ and they need not be Lipschitz continuous.

In order to prove the existence result, we need the following four technical lemmas, whose proofs are done in the next section.

Lemma 5.1 (A PRIORI ESTIMATE). *There exists a constant $K > 0$, independent of ε, such that any solution (u, z) of Problem (P_1) satisfies*

$$\|z\|_T \leqslant K, \tag{5.2}$$

$$a < \lambda(K) \leqslant \int_{\mathbb{R}} w(x)u(x,t)dx \leqslant \lambda(-K) < b, \quad t \in [0,T], \tag{5.3}$$

where λ is the inverse function of L (see (H1)).

Lemma 5.2. *Let $\theta, \zeta \in C[0,T]$ satisfy*

$$\max\{\|\theta\|_T,\ \|\zeta\|_T\} \leqslant 2K, \tag{5.4}$$

where K is defined in Lemma 5.1. Suppose that for some $0 \leqslant r < T$, $\theta(t) = \zeta(t)$ for $t \in [0,r]$. Then there are constants $\sigma \in\ ]0,1[$ and $C_{12} > 0$, depending on K but not on $\varepsilon, r, \theta, \zeta$, such that

$$\int_r^t \int_{\mathbb{R}} (1+|x|)|\varphi_\zeta^*(x,\tau,\nu_\theta(x,\tau)) - \varphi_\zeta^*(x,\tau,\nu_\zeta(x,\tau))|dxd\tau \leqslant C_{12}(t-r)^\sigma \|\zeta-\theta\|_t$$

for each $t \in [r,T]$, where φ_ζ^* is defined by (2.8), ν_θ and ν_ζ are the mild solutions of $(AP_\theta; 0, u_0(\cdot + \theta(0)))$ and $(AP_\zeta; 0, u_0(\cdot + \zeta(0)))$ respectively defined by Theorem 3.1.

For any $\zeta \in C[0,T]$, define a function $\Gamma_\zeta :]0,T[\times]0,T[\to \mathbb{R}$ by

$$\Gamma_\zeta(t,\tau) := \begin{cases} \int_{\mathbb{R}} w_\zeta^*(x,t) \int_{\mathbb{R}} K_\varepsilon(x-y,t-\tau)\varphi_\zeta^*(y,\tau,\nu_\zeta(y,\tau))dydx & \text{if } 0<\tau<t<T; \\ 0 & \text{otherwise,} \end{cases}$$

where ν_ζ is a mild solution of $(AP_\zeta; 0, u_0(\cdot + \zeta(0)))$ defined in Theorem 3.1. Then we have

Lemma 5.3. *Let $\zeta \in C[0,T]$. Then the function Γ_ζ is well-defined and has the following properties:*

(i) *If $\|\zeta\|_T \leqslant 2K$, then there exists a constant $C_{13} > 0$, depend only on K but not on ε and ζ, such that*

$$|\Gamma_\zeta(t,\tau)| \leqslant C_{13} \quad \textit{for all } t,\ \tau \in\ ]0,T[.$$

(ii) *For any $c \in [0,T[$, the function $t \mapsto \int_c^t \Gamma_\zeta(t,\tau)d\tau$ is continuous on $[c,T]$.*

Lemma 5.4. *Let θ and ζ be the same as in Lemma 5.2. Then there exists a constant $C_{14} > 0$, depending on K but not on ε, r, θ, ζ, such that*

$$\int_r^t |\Gamma_\zeta(t,\tau)|d\tau \leqslant C_{14}(t-r),$$

$$\int_r^t |\Gamma_\zeta(t,\tau) - \Gamma_\theta(t,\tau)|d\tau \leqslant C_{14}(t-r)^\sigma \|\zeta-\theta\|_t$$

for each $t \in [r,T]$, where σ is the constant defined in Lemma 5.2.

We shall show the existence of the solution z of (P_3) by applying the contraction mapping principle step by step in time as done in [3]. Since each time interval will have the same width $d > 0$ (to be specified later), we can conclude that the solution exists on the whole given interval $[0, T]$.

Let $r \in [0, T[$ be arbitrary and assume only the continuous function z satisfying (4.9)–(4.10) to be known in $[0, r]$. We introduce the function space

$$X_r := \{\zeta \in C[0, r+d] : \zeta = z \text{ in } [0, r],\ \|\zeta\|_{r+d} \leqslant 2K\}$$

as a complete metric space, where K is the constant defined by Lemma 5.1. The equation (4.10) is rewritten as

$$\lambda(z(t)) = \int_{\mathbb{R}} w(x + z(t)) \int_{\mathbb{R}} K_\varepsilon(x - y, t - r)\nu_z(y, r)dydx$$
$$+ \int_{\mathbb{R}} w(x + z(t)) \int_r^t \int_{\mathbb{R}} K_\varepsilon(x - y, t - \tau)\varphi_z^*(y, \tau, \nu_z(y, \tau))dyd\tau dx \qquad (5.5)$$

since ν_z satisfies

$$\nu_z(x, t) = \int_{\mathbb{R}} K_\varepsilon(x - y, t - r)\nu_z(y, r)dy$$
$$+ \int_r^t \int_{\mathbb{R}} K_\varepsilon(x - y, t - \tau)\varphi_z^*(y, \tau, \nu_z(y, \tau))dyd\tau \qquad (5.6)$$

for $t \in [r, T]$ (see Definition 3.2 and Proposition 3.2). We now define

$$\lambda^K(\xi) := \begin{cases} \lambda(-2K) - \xi - 2K & \text{if } \xi < -2K\ ; \\ \lambda(\xi) & \text{if } |\xi| \leqslant 2K\ ; \\ \lambda(2K) - \xi + 2K & \text{if } \xi > 2K, \end{cases}$$

$$\lambda_r^K(\xi, t) := \lambda^K(\xi) - \int_{\mathbb{R}} \mu_z(y, r) \int_{\mathbb{R}} \frac{e^{-x^2}}{\sqrt{\pi}} w\big(2x\sqrt{\varepsilon(t - r)} + y + \xi - z(r)\big)dxdy$$

for $(\xi, t) \in \mathbb{R} \times [r, T]$. Then, by the definition of Γ_z and (5.5), we have

$$\lambda_r^K(z(t), t) = \int_r^t \Gamma_z(t, \tau)d\tau, \quad t \in [r, T].$$

Lemma 5.5. *The function $\lambda_r^K(\xi, t)$ is continuous and strictly decreasing in ξ for fixed t and satisfies*

$$|\xi_1 - \xi_2| \leqslant M|\lambda_r^K(\xi_1, t) - \lambda_r^K(\xi_2, t)|, \quad \xi_i \in \mathbb{R}\ (i = 1,\ 2),\ t \in [r, T], \qquad (5.7)$$

where $M = \max\{1, M_1(2K)\}$.

Proof. Since λ is continuous, λ^K is obviously continuous. We put

$$g(\xi, t) := \int_{\mathbb{R}} \mu_z(y, r) \int_{\mathbb{R}} \frac{e^{-x^2}}{\sqrt{\pi}} w\big(2x\sqrt{\varepsilon(t - r)} + y + \xi - z(r)\big)dxdy.$$

By using the Lipschitz continuity of w and Proposition 4.2 (ii), $g(\xi, t)$ is shown to be continuous, and so $\lambda_r^K(\xi, t) = \lambda^K(\xi) - g(\xi, t)$ is also continuous. Since λ is strictly decreasing and w is increasing, $\xi \mapsto \lambda_r^K(\xi,t)$ is strictly decreasing. One can easily check that (5.7) holds from (H1). ■

We denote by $L_{r,t}^K$ the inverse function of $\xi \mapsto \lambda_r^K(\xi,t)$. Notice that (5.7) implies the Lipschitz continuity of $L_{r,t}^K$. We define an operator $S_r^K : X_r \to X_r$ as follows:

$$[S_r^K(\zeta)](t) := \begin{cases} z(t) & \text{for } t \in [0,r[; \\ L_{r,t}^K\Big(\int_r^t \Gamma_\zeta(t,\tau)d\tau\Big) & \text{for } t \in [r, r+d], \end{cases}$$

for $\zeta \in X_r$. We shall show that for sufficiently small $d > 0$, which does not depend on r, S_r^K is well-defined and is a contraction mapping in X_r. Then S_r^K has a unique fixed point in X_r, which is shown to be the solution of (P_3).

Proposition 5.2. *For sufficiently small $d > 0$, $S_r^K : X_r \to X_r$ is well-defined and is a contraction mapping.*

Proof. First, let us ascertain that S_r^K is well-defined. Let $\zeta \in X_r$. Then it is obvious from the definition that $[S_r^K(\zeta)](t) = z(t)$ is continuous in $t \in [0,r[$. Using (5.7) and Lemma 5.3 (ii), we obtain that for $t_0 \in [r, r+d]$,

$$\begin{aligned} &M^{-1}\big|[S_r^K(\zeta)](t) - [S_r^K(\zeta)](t_0)\big| \leqslant \big|\lambda_r^K([S_r^K(\zeta)](t),t) - \lambda_r^K([S_r^K(\zeta)](t_0),t)\big| \qquad (5.8)\\ &\leqslant \big|\lambda_r^K([S_r^K(\zeta)](t),t) - \lambda_r^K([S_r^K(\zeta)](t_0),t_0)\big| + \big|\lambda_r^K([S_r^K(\zeta)](t_0),t_0) - \lambda_r^K([S_r^K(\zeta)](t_0),t)\big| \\ &= \left|\int_r^t \Gamma_\zeta(t,\tau)d\tau - \int_r^{t_0} \Gamma_\zeta(t_0,\tau)d\tau\right| + \big|\lambda_r^K([S_r^K(\zeta)](t_0),t_0) - \lambda_r^K([S_r^K(\zeta)](t_0),t)\big| \longrightarrow 0 \end{aligned}$$

as $t \to t_0$, which implies that $[S_r^K(\zeta)](t)$ is continuous on $[r, r+d]$. From the definition of λ_r^K and (4.10), we find that $\lambda_r^K(z(r),r) = 0 = \int_r^r \Gamma_\zeta(r,\tau)d\tau$, and hence,

$$L_{r,r}^K\Big(\int_r^r \Gamma_\zeta(r,\tau)d\tau\Big) = z(r).$$

Thus, we see that $S_r^K(\zeta) \in C[0, r+d]$ and $S_r^K(\zeta) = z$ on $[0,r]$. Let $r \leqslant t \leqslant r+d$. Letting $t_0 = r$ in (5.8), we have

$$\begin{aligned} M^{-1}\big|[S_r^K(\zeta)](t) - z(r)\big| &= M^{-1}\big|[S_r^K(\zeta)](t) - [S_r^K(\zeta)](r)\big| \\ &\leqslant \left|\int_r^t \Gamma_\zeta(t,\tau)d\tau\right| + \big|\lambda_r^K([S_r^K(\zeta)](r),r) - \lambda_r^K([S_r^K(\zeta)](r),t)\big| \\ &= \left|\int_r^t \Gamma_\zeta(t,\tau)d\tau\right| + \big|\lambda_r^K(z(r),r) - \lambda_r^K(z(r),t)\big|. \qquad (5.9) \end{aligned}$$

Since $dw/dx \in W^{1,\infty}(\mathbb{R})$ (see (H2)), from Taylor's formula we get

$$\begin{aligned} &w(2x\sqrt{\varepsilon(t-r)} + y) - w(y) \\ &= \frac{dw}{dx}(y) \cdot 2x\sqrt{\varepsilon(t-r)} + \int_0^{2x\sqrt{\varepsilon(t-r)}} (2x\sqrt{\varepsilon(t-r)} - \xi)\frac{d^2w}{dx^2}(y+\xi)d\xi. \end{aligned}$$

Thus, noting that $\int_{\mathbb{R}} x\exp(-x^2)dx = 0$ and $\int_{\mathbb{R}} x^2\exp(-x^2)dx = \sqrt{\pi}/2$, we have

$$\begin{aligned}
&\left|\lambda_r^K(z(r),r) - \lambda_r^K(z(r),t)\right| \\
&= \left|\int_{\mathbb{R}} \mu_z(y,r)\int_{\mathbb{R}} \frac{e^{-x^2}}{\sqrt{\pi}}\left\{w\left(2x\sqrt{\varepsilon(t-r)}+y\right) - w(y)\right\}dxdy\right| \\
&= \left|\int_{\mathbb{R}} \mu_z(y,r)\int_{\mathbb{R}} \frac{e^{-x^2}}{\sqrt{\pi}}\int_0^{2x\sqrt{\varepsilon(t-r)}}\left(2x\sqrt{\varepsilon(t-r)}-\xi\right)\frac{d^2w}{dx^2}(y+\xi)d\xi dxdy\right| \\
&\leqslant \frac{1}{\sqrt{\pi}}\left\|\frac{d^2w}{dx^2}\right\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\mu_z(y,r)dy\cdot\int_{\mathbb{R}} e^{-x^2}\int_0^{2x\sqrt{\varepsilon(t-r)}}\left(2x\sqrt{\varepsilon(t-r)}-\xi\right)d\xi dx \\
&= \frac{2}{\sqrt{\pi}}\left\|\frac{d^2w}{dx^2}\right\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\mu_z(y,r)dy\cdot\int_{\mathbb{R}} x^2e^{-x^2}dx\cdot\varepsilon(t-r) \leqslant \left\|\frac{d^2w}{dx^2}\right\|_{L^\infty(\mathbb{R})} M_6(2K)\varepsilon(t-r).
\end{aligned}$$

So, putting $C_{15} := \|d^2w/dx^2\|_{L^\infty(\mathbb{R})}M_6(2K)$, we have

$$\left|\lambda_r^K(z(r),r) - \lambda_r^K(z(r),t)\right| \leqslant C_{15}(t-r). \tag{5.10}$$

It then follows from (5.9)–(5.10) and Lemma 5.4 that

$$\left|[S_r^K(\zeta)](t) - z(r)\right| \leqslant C_{16}(t-r) \tag{5.11}$$

for each $t \in [r, r+d]$, where $C_{16} := M(C_{14}+C_{15})$. Choose $d > 0$ such that

$$d \leqslant C_{16}^{-1}K.$$

Then it follows (5.11) and (5.2) that

$$\left|[S_r^K(\zeta)](t)\right| \leqslant \left|[S_r^K(\zeta)](t) - z(r)\right| + |z(r)| \leqslant C_{16}d + |z(r)| \leqslant 2K$$

for all $t \in [r, r+d]$. Hence $S_r^K(\zeta) \in X_r$ for every $\zeta \in X_r$, which shows that S_r^K is well-defined.

Next, we show that S_r^K is a contraction. Let $\zeta, \theta \in X_r$. For $t \in [r, r+d]$, using (5.7), the definitions of S_r^K and $L_{r,t}^K$, and Lemma 5.4, we obtain that

$$\begin{aligned}
&\left|[S_r^K(\zeta)](t) - [S_r^K(\theta)](t)\right| \leqslant M\left|\lambda_r^K([S_r^K(\zeta)](t),t) - \lambda_r^K([S_r^K(\theta)](t),t)\right| \\
&= M\left|\int_r^t \Gamma_\zeta(t,\tau)d\tau - \int_r^t \Gamma_\theta(t,\tau)d\tau\right| \leqslant MC_{14}(t-r)^\sigma\|\zeta-\theta\|_t \leqslant MC_{14}d^\sigma\|\zeta-\theta\|_t.
\end{aligned}$$

Noting that $S_r^K(\zeta) = S_r^K(\theta) = z$ on $[0, r]$, take the supremum over $[0, r+d]$ to obtain

$$\|S_r^K(\zeta) - S_r^K(\theta)\|_{r+d} \leqslant MC_{14}d^\sigma\|\zeta-\theta\|_{r+d}.$$

Consequently, choosing d so small that

$$0 < d < \min\{C_{16}^{-1}K,\ (MC_{14})^{-1/\sigma}\}, \tag{5.12}$$

the operator S_r^K is well-defined and a contraction mapping in X_r. ■

Now, we are ready to prove our main theorem.

Proof of Main Theorem. Since the uniqueness is shown in Proposition 5.1, we have only to show the existence result. In what follows, fix $d > 0$ as (5.12) is fulfilled. Then by Proposition 5.2, S_r^K has a unique fixed point $\tilde{z} \in X_r$. Note that $\tilde{z}$ satisfies $\tilde{z} \in C[0, r+d]$, $\tilde{z} = z$ in $[0, r]$, $\|\tilde{z}\|_{r+d} \leqslant 2K$ and

$$\tilde{z}(t) = [S_r^K(\tilde{z})](t) = L_{r,t}^K\Big(\int_r^t \Gamma_{\tilde{z}}(t,\tau)d\tau\Big), \quad t \in [r, r+d].$$

Since $L_{r,t}^K$ is the inverse function of $\lambda_r^K(\cdot, t)$, we see that

$$\lambda_r^K(\tilde{z}(t), t) = \int_r^t \Gamma_{\tilde{z}}(t,\tau)d\tau, \quad t \in [r, r+d].$$

Therefore, by the definitions of λ_r^K and $\Gamma_{\tilde{z}}$,

$$\begin{aligned}\lambda(\tilde{z}(t)) = \int_{\mathbb{R}} w(x+\tilde{z}(t))\Big\{&\int_{\mathbb{R}} K_\varepsilon(x-y, t-r)\nu_{\tilde{z}}(y,r)dy \\ &+ \int_r^t \int_{\mathbb{R}} K_\varepsilon(x-y, t-\tau)\varphi_{\tilde{z}}^*(y,\tau,\nu_{\tilde{z}}(y,\tau))dyd\tau\Big\}dx \end{aligned} \tag{5.13}$$

for $t \in [r, r+d]$. Note that $\tilde{z} = z$ and $\nu_{\tilde{z}} = \nu_z$ in $[0, r]$. Since $\nu_{\tilde{z}}$ satisfies (5.6) in $[r, r+d]$, it follows from (5.13) that

$$\lambda(\tilde{z}(t)) = \int_{\mathbb{R}} w(x+\tilde{z}(t))\nu_{\tilde{z}}(x,t)dx, \quad t \in [r, r+d].$$

Hence, $\tilde{z}$ satisfies (4.9)–(4.10) in $[r, r+d]$. On the other hand, z satisfies (4.9)–(4.10) in $[0, r]$ from our assumptions. Consequently, $\tilde{z}$ satisfies (4.9)–(4.10) in $[0, r+d]$.

Next, we show that $\tilde{z}$ is Lipschitz continuous in $[r, r+d]$. For each $s \in [r, r+d[$, let $z_s \in X_s = \{\zeta \in C[0, s+d] : \zeta = \tilde{z} \text{ in } [0,s],\ \|\zeta\|_{s+d} \leqslant 2K\}$ be the fixed point of S_s^K. Note that z_s satisfies (4.9)–(4.10) in $[0, s+d]$ and

$$\big|[S_s^K(z_s)](t) - \tilde{z}(s)\big| \leqslant C_{16}(t-s)$$

for each $t \in [s, s+d]$ (see (5.11)). Since continuous function satisfying (4.9)–(4.10) is unique (see Proposition 5.1 and Remark 5.1), we have $z_s = \tilde{z}$ in $[0, r+d]$. Consequently, for each t, $s \in [r, r+d]$ with $s < t$,

$$|\tilde{z}(t) - \tilde{z}(s)| = |z_s(t) - \tilde{z}(s)| = \big|[S_s^K(z_s)](t) - \tilde{z}(s)\big| \leqslant C_{16}(t-s).$$

This implies that $\tilde{z}$ is Lipschitz continuous in $[r, r+d]$. Since r is arbitrary, we can construct step by step a Lipschitz continuous function z on $[0, T]$ satisfying (4.9)–(4.10), i.e., a solution of (P_3). ■

Remark 5.2. Note that the Lipschitz constant of the solution z of (P_3) is independent of ε because

$$|z(t) - z(s)| \leqslant C_{16}|t-s|$$

for all t, $s \in [0, T]$, and C_{16} is independent of ε.

6. Proofs of technical lemmas 5.1–5.4

In this section we give proofs of Lemmas 5.1–5.4.

Proof of Lemma 5.1. The proof is split into five steps.

Step 1: Set

$$W(x) := \int_0^x w(\xi)d\xi, \quad x \in \mathbb{R}; \quad \Lambda(y) := \int_0^y \lambda(\xi)d\xi, \quad y \in \mathbb{R}.$$

In view of (H2) and (H1), it is easily seen that there exists a constant $C_{17} > 0$ such that

$$0 \leqslant W(x) \leqslant C_{17}x^2, \quad x \in \mathbb{R} \tag{6.1}$$

and that Λ is nonpositive and satisfies

$$-\Lambda(y) \to +\infty \quad \text{as } |y| \to +\infty. \tag{6.2}$$

Let (u, z) be a solution of Problem (P_1). Then z solves (P_3) and $\mu_z \equiv u$ satisfies (4.8) by Proposition 4.2 (i) and Theorem 4.1. Let $0 < \delta < t \leqslant T$ and set $\psi_m(x) := \exp(-x^2/m)$ and $\phi_m(x) := W(x)\psi_m(x)$ $(x \in \mathbb{R}, m \in \mathbb{N})$. Then from (6.1), one can find that $\phi_m \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Since $\partial\mu_z/\partial x$, $\partial^2\mu_z/\partial x^2 \in L^\infty(\mathbb{R}\times]\delta, T[)$ (by Proposition 4.2 (i)), and $z' \in L^\infty(0,T)$, we have $z'(\partial\mu_z/\partial x)\phi_m$, $(\partial^2\mu_z/\partial x^2)\phi_m \in L^1(\mathbb{R}\times]\delta, T[)$. Let $\varphi_z(x,t,\zeta) = \varphi(x,t,z(t),\zeta)$. Then $\varphi_z(\cdot,\cdot,\mu_z)\phi_m \in L^1(\mathbb{R}\times]0,T[)$. In fact, since $0 \leqslant \mu_z \leqslant 1$, we have by (H3 vi–vii) that

$$|\varphi_z(x,\tau,\mu_z(x,\tau))\phi_m(x)| \leqslant \mathcal{F}(x,\tau)\phi_m(x) + M_3(\|z\|_T)(1+|x|)\mu_z(x,\tau)\phi_m(x)$$

for *a.e.* $(x,\tau) \in \mathbb{R}\times]0,T[$. Since $(1+|x|)\mu_z\phi_m \in L^1(\mathbb{R}\times]0,T[)$ (see Proposition 4.2 (ii)), the function $\varphi_z(\cdot,\cdot,\mu_z)\phi_m$ is integrable on $\mathbb{R}\times]0,T[$. Multiplying the equation (4.8) by ϕ_m and then integrating over $\mathbb{R}\times]\delta,t[$, we have

$$\begin{aligned}\int_{\mathbb{R}} \phi_m(x)\mu_z(x,t)dx &= \int_{\mathbb{R}} \phi_m(x)\mu_z(x,\delta)dx + \varepsilon\int_\delta^t d\tau \int_{\mathbb{R}} \frac{\partial^2\mu_z}{\partial x^2}(x,\tau)\phi_m(x)dx \\ &- \int_\delta^t z'(\tau)\int_{\mathbb{R}} \frac{\partial\mu_z}{\partial x}(x,\tau)\phi_m(x)dxd\tau + \int_\delta^t d\tau\int_{\mathbb{R}} \varphi(x,\tau,z(\tau),\mu_z(x,\tau))\phi_m(x)dx.\end{aligned} \tag{6.3}$$

Step 2 (letting $\delta \downarrow 0$): First note that $0 \leqslant \phi_m(x) \to 0$ and $|d\phi_m(x)/dx| \to 0$ as $|x| \to +\infty$. Then by integrating by parts, (6.3) becomes

$$\begin{aligned}\int_{\mathbb{R}} \phi_m(x)\mu_z(x,t)dx &= \int_{\mathbb{R}} \phi_m(x)\mu_z(x,\delta)dx + \varepsilon\int_\delta^t d\tau \int_{\mathbb{R}} \frac{d^2\phi_m}{dx^2}(x)\mu_z(x,\tau)dx \\ &+ \int_\delta^t z'(\tau)\int_{\mathbb{R}} \frac{d\phi_m}{dx}(x)\mu_z(x,\tau)dxd\tau + \int_\delta^t d\tau\int_{\mathbb{R}} \varphi(x,\tau,z(\tau),\mu_z(x,\tau))\phi_m(x)dx.\end{aligned} \tag{6.4}$$

Since $\mu_z \in BUC(\mathbb{R}\times[0,T]) \subset C([0,T];X_0)$, $X_0 = BUC(\mathbb{R})$, and $\mu_z(\cdot,0) = u_0$, we obtain

$$\int_{\mathbb{R}} \phi_m(x)\mu_z(x,\delta)dx \longrightarrow \int_{\mathbb{R}} \phi_m(x)u_0(x)dx \quad \text{as } \delta \downarrow 0.$$

Taking the limit in (6.4) as $\delta \downarrow 0$, we have

$$\int_{\mathbb{R}} \phi_m(x)\mu_z(x,t)dx = \int_{\mathbb{R}} \phi_m(x)u_0(x)dx + \varepsilon \int_0^t d\tau \int_{\mathbb{R}} \frac{d^2\phi_m}{dx^2}(x)\mu_z(x,\tau)dx \tag{6.5}$$
$$+ \int_0^t z'(\tau) \int_{\mathbb{R}} \frac{d\phi_m}{dx}(x)\mu_z(x,\tau)dxd\tau + \int_0^t d\tau \int_{\mathbb{R}} \varphi(x,\tau,z(\tau),\mu_z(x,\tau))\phi_m(x)dx.$$

Step 3: A simple calculation shows that

$$\sup_{x\in\mathbb{R}}\left|\frac{d\psi_m}{dx}(x)\right| = \sqrt{\frac{2}{me}}, \quad \sup_{x\in\mathbb{R}}\left|\frac{d^2\psi_m}{dx^2}(x)\right| \leqslant \frac{4}{me} + \frac{2}{m}.$$

Since $d^2\phi_m/dx^2 = (dw/dx)\psi_m + 2w \cdot d\psi_m/dx + W \cdot d^2\psi_m/dx^2$, we have from (6.5) that

$$\int_{\mathbb{R}} \phi_m(x)\mu_z(x,t)dx \leqslant \int_{\mathbb{R}} \phi_m(x)u_0(x)dx + \int_0^t d\tau \int_{\mathbb{R}} \frac{dw}{dx}(x)\psi_m(x)\mu_z(x,\tau)dx$$
$$+ \frac{C_{18}}{\sqrt{m}} \int_0^t d\tau \int_{\mathbb{R}} |w(x)|\mu_z(x,\tau)dx + \frac{C_{19}}{m}\int_0^t d\tau \int_{\mathbb{R}} x^2\mu_z(x,\tau)dx$$
$$+ \int_0^t z'(\tau)\int_{\mathbb{R}} \frac{d\phi_m}{dx}(x)\mu_z(x,\tau)dxd\tau + \int_0^t d\tau \int_{\mathbb{R}} \mathcal{F}(x,\tau)\phi_m(x)dx \tag{6.6}$$

for $t \in [0,T]$, $m \in \mathbb{N}$, where C_{18} and C_{19} are independent of m. Here we have used (6.1) and (H3 vi).

Step 4 (letting $m \to \infty$): By using (6.1) and Proposition 4.2 (ii), we find

$$\int_{\mathbb{R}} \phi_m(x)\mu_z(x,t)dx \leqslant C_{17}\int_{\mathbb{R}} x^2\mu_z(x,t)dx \leqslant C_{17}M_6(\|z\|_T)$$

for each $t \in [0,T]$, $m \in \mathbb{N}$, where $C_{17}M_6(\|z\|_T)$ is independent of m. Besides the nonnegative function $\phi_m(x)\mu_z(x,t) = W(x)\exp(-x^2/m)\mu_z(x,t)$ converges increasingly to $W(x)\mu_z(x,t)$ as $m \to \infty$ for each $x \in \mathbb{R}$, $t \in [0,T]$. Then by the Beppo-Levi Monotone Convergence Theorem,

$$\int_{\mathbb{R}} \phi_m(x)\mu_z(x,t)dx \longrightarrow \int_{\mathbb{R}} W(x)\mu_z(x,t)dx \quad \text{as } m \to \infty$$

for each $t \in [0,T]$. By (6.1), (H4) and the Monotone Convergence Theorem,

$$\int_{\mathbb{R}} \phi_m(x)u_0(x)dx \longrightarrow \int_{\mathbb{R}} W(x)u_0(x)dx \quad \text{as } m \to \infty.$$

Recalling (H2) and Proposition 4.2 (ii), we can use again the Monotone Convergence Theorem to obtain

$$\int_0^t d\tau \int_{\mathbb{R}} \frac{dw}{dx}(x)\psi_m(x)\mu_z(x,\tau)dx \longrightarrow \int_0^t d\tau \int_{\mathbb{R}} \frac{dw}{dx}(x)\mu_z(x,\tau)dx \quad \text{as } m \to \infty.$$

Further, by (6.1) and the Monotone Convergence Theorem,

$$\int_0^t d\tau \int_{\mathbb{R}} \mathcal{F}(x,\tau)\phi_m(x)dx \longrightarrow \int_0^t d\tau \int_{\mathbb{R}} \mathcal{F}(x,\tau)W(x)dx \quad \text{as } m \to \infty.$$

Note that $|z'(d\phi_m/dx)\mu_z| \leqslant \|z'\|_{L^\infty(0,T)}(C_w|x|\mu_z + C_{20}x^2\mu_z)$ by (6.1), where C_{20} is independent of m. Recalling Proposition 4.2 (ii), we can use the Lebesgue Dominated Convergence Theorem to obtain

$$\int_0^t z'(\tau)\int_{\mathbb{R}} \frac{d\phi_m}{dx}(x)\mu_z(x,\tau)dxd\tau \longrightarrow \int_0^t z'(\tau)\int_{\mathbb{R}} w(x)\mu_z(x,\tau)dxd\tau \quad \text{as } m\to\infty.$$

Since $w\mu_z$, $x^2\mu_z \in L^1(\mathbb{R}\times]0,T[)$ (by Proposition 4.2 (ii)), taking the limit in (6.6) as $m\to\infty$ leads to

$$\begin{aligned}&\int_{\mathbb{R}} W(x)\mu_z(x,t)dx - \int_0^t z'(\tau)\int_{\mathbb{R}} w(x)\mu_z(x,\tau)dxd\tau \\ &\leqslant \int_{\mathbb{R}} W(x)u_0(x)dx + \int_0^t d\tau\int_{\mathbb{R}} \frac{dw}{dx}(x)\mu_z(x,\tau)dx + \int_0^t d\tau \int_{\mathbb{R}} \mathcal{F}(x,\tau)W(x)dx\end{aligned} \tag{6.7}$$

for each $t\in[0,T]$.

Step 5 (a priori estimate): It follows from (4.10) and the definition of Λ that

$$\Lambda(z(t)) = \int_0^t z'(\tau)\int_{\mathbb{R}} w(x)\mu_z(x,\tau)dxd\tau + \Lambda(z(0)), \quad t\in[0,T].$$

By (6.7), we get

$$\begin{aligned}0\leqslant -\Lambda(z(t)) \leqslant &\int_{\mathbb{R}} W(x)u_0(x)dx + \int_0^T d\tau\int_{\mathbb{R}}\frac{dw}{dx}(x)\mu_z(x,\tau)dx \\ &+\int_0^T d\tau\int_{\mathbb{R}}\mathcal{F}(x,\tau)W(x)dx - \Lambda(z(0)), \quad t\in[0,T].\end{aligned}$$

It follows from (4.7) and Lemma 3.4 that

$$\begin{aligned}\int_0^T d\tau\int_{\mathbb{R}}\frac{dw}{dx}(x)\mu_z(x,\tau)dx &\leqslant C_w\int_0^T d\tau\int_{\mathbb{R}}\nu_z(x-z(\tau),\tau)dx \\ &= C_w\int_0^T d\tau\int_{\mathbb{R}}\nu_z(y,\tau)dy \leqslant C_wT(\|u_0\|_{L^1}+\|\mathcal{F}\|_{L^1}).\end{aligned}$$

Noting that $\Lambda(z(0))$ is actually independent of z, we can take a constant C_{21}, independent of z and ε, satisfying

$$0\leqslant -\Lambda(z(t))\leqslant C_{21}, \quad t\in[0,T].$$

Hence by (6.2), we obtain (5.2). Finally, (5.3) is a direct consequence of (H1), (4.10) and (5.2). ■

Proof of Lemma 5.2. The proof is split into six steps.

Step 1 (approximation): Let $\frac{1}{2}\leqslant\beta\leqslant 1$ and take $\theta_n\in C^{0,\beta}[r,T]$ such that $\theta_n(r)=\theta(r)$ and $\theta_n\to\theta$ in $C[r,T]$ as $n\to\infty$.

Let ν_{θ_n} be a regular solution to $(AP_\theta; r, \nu_\theta(\cdot,r))$. By Theorem 3.1, we have $\nu_{\theta_n}\to\nu_\theta$ in $C([r,T];X_0)$ as $n\to\infty$. For ζ, the same fact holds, that is, there exists a sequence $\{\zeta_n\}$ in

$C^{0,\beta}[r,T]$ such that $\zeta_n(r) = \zeta(r)$, $\zeta_n \to \zeta$ in $C[r,T]$ as $n \to \infty$, and we have $\nu_{\zeta_n} \to \nu_\zeta$ in $C([r,T];X_0)$ as $n \to \infty$, where ν_{ζ_n} is a regular solution of $(AP_\zeta; r, \nu_\zeta(\cdot, r)))$.

Setting $\nu_n := \nu_{\zeta_n} - \nu_{\theta_n}$, it is easy to see that ν_n satisfies

$$\frac{\partial \nu_n}{\partial t} - \varepsilon \frac{\partial^2 \nu_n}{\partial x^2} + \varphi^*_{\zeta_n}(x,t,\nu_{\theta_n}) - \varphi^*_{\zeta_n}(x,t,\nu_{\zeta_n}) = \varphi^*_{\zeta_n}(x,t,\nu_{\theta_n}) - \varphi^*_{\theta_n}(x,t,\nu_{\theta_n}) \tag{6.8}$$

for $(x,t) \in \mathbb{R}\times]r,T]$. Furthermore, we have $\nu_n(\cdot, r) = 0$. Indeed, since $\theta = \zeta$ on $[0,r]$ and both ν_θ and ν_ζ are the mild solutions of $(AP_\theta; 0, u_0(\cdot + \theta(0)))$ on $[0,r]$, the uniqueness of the mild solution yields $\nu_\theta = \nu_\zeta$ on $[0,r]$; in particular, $\nu_{\theta_n}(r) = \nu_\theta(r) = \nu_\zeta(r) = \nu_{\zeta_n}(r)$.

Step 2: Set $\psi_m(x) := \exp(-x^2/m)$, $\phi(x) := 2 + x \tanh x$, and

$$\Psi(x,t) := \phi(x)\psi_m(x)\tanh(m\nu_n(x,t)), \quad \Phi_m(x) := \int_0^x \tanh(m\xi)d\xi$$

for $x \in \mathbb{R}$, $t \in [r,T]$, $m \in \mathbb{N}$. Then it follows that

$$1 + |x| \leqslant \phi(x) \leqslant 2(1+|x|), \tag{6.9}$$

$$0 \leqslant \Phi_m(x) \leqslant |x|. \tag{6.10}$$

Let $r < \delta < t < T$. Multiplying (6.8) by Ψ and then integrating over $\mathbb{R}\times]\delta, t[$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \phi\psi_m\Phi_m(\nu_n(t))dx - \int_{\mathbb{R}} \phi\psi_m\Phi_m(\nu_n(\delta))dx - \varepsilon\int_\delta^t d\tau \int_{\mathbb{R}} \Psi \frac{\partial^2 \nu_n}{\partial x^2}dx \\
&+ \int_\delta^t d\tau \int_{\mathbb{R}} \Psi(x,\tau)[\varphi^*_{\zeta_n}(x,\tau,\nu_{\theta_n}) - \varphi^*_{\zeta_n}(x,\tau,\nu_{\zeta_n})]dx \\
&= \int_\delta^t d\tau \int_{\mathbb{R}} \Psi(x,\tau)[\varphi^*_{\zeta_n}(x,\tau,\nu_{\theta_n}) - \varphi^*_{\theta}(x,\tau,\nu_{\theta_n})]dx.
\end{aligned} \tag{6.11}$$

Here we have used $(\partial/\partial\tau)\Phi_m(\nu_n) = (\partial\nu_n/\partial\tau)\tanh(m\nu_n)$. By integration by parts yields the third term of the left-hand side of (6.11) becomes

$$\varepsilon\int_\delta^t d\tau \int_{\mathbb{R}} \frac{m\phi\psi_m}{\cosh^2(m\nu_n)}\left|\frac{\partial \nu_n}{\partial x}\right|^2 dx - \varepsilon \int_\delta^t d\tau \int_{\mathbb{R}} \Phi_m(\nu_n)\frac{d^2}{dx^2}(\phi\psi_m)dx.$$

Thus we have

$$\begin{aligned}
&\int_{\mathbb{R}} \phi\psi_m\Phi_m(\nu_n(t))dx + \varepsilon\int_\delta^t d\tau \int_{\mathbb{R}} \frac{m\phi\psi_m}{\cosh^2(m\nu_n)}\left|\frac{\partial \nu_n}{\partial x}\right|^2 dx \\
&+ \int_\delta^t d\tau \int_{\mathbb{R}} \Psi(x,\tau)[\varphi^*_{\zeta_n}(x,\tau,\nu_{\theta_n}) - \varphi^*_{\zeta_n}(x,\tau,\nu_{\zeta_n})]dx \\
&= \int_{\mathbb{R}} \phi\psi_m\Phi_m(\nu_n(\delta))dx + \varepsilon\int_\delta^t d\tau \int_{\mathbb{R}} \Phi_m(\nu_n)\frac{d^2}{dx^2}(\phi\psi_m)dx \\
&\qquad + \int_\delta^t d\tau \int_{\mathbb{R}} \Psi(x,\tau)[\varphi^*_{\zeta_n}(x,\tau,\nu_{\theta_n}) - \varphi^*_{\theta_n}(x,\tau,\nu_{\theta_n})]dx.
\end{aligned} \tag{6.12}$$

Step 3: By (6.9), one can check that

$$\left|\frac{d^2}{dx^2}(\phi(x)\psi_m(x))\right| \leqslant C_{22}\Big(1+\frac{x^2}{m^2}\Big)\phi(x)\psi_m(x),$$

where C_{22} is independent of m. Hence the second term of the right-hand side of (6.12) is estimated by

$$C_{22}\int_\delta^t d\tau \int_{\mathbb{R}} \Phi_m(\nu_n)\phi\psi_m dx + \frac{C_{22}}{m^2}\int_\delta^t d\tau \int_{\mathbb{R}} \Phi_m(\nu_n)x^2\phi\psi_m dx.$$

Furthermore, by using (5.4), (6.9)–(6.10) and Corollary 3.1,

$$\begin{aligned}
&\int_\delta^t d\tau \int_{\mathbb{R}} \Phi_m(\nu_n(x,\tau))x^2\phi(x)\psi_m(x)dx \leqslant \int_\delta^t d\tau \int_{\mathbb{R}} |\nu_n(x,\tau)|\phi(x)x^2 e^{-x^2/m}dx\\
&\leqslant \frac{m}{e}\int_\delta^t d\tau \int_{\mathbb{R}} (1+|x|)|\nu_n(x,\tau)|dx\\
&\leqslant \frac{m}{e}\int_\delta^t d\tau \int_{\mathbb{R}} (1+|x|)(|\nu_{\zeta_n}(x,\tau)|+|\nu_{\theta_n}(x,\tau)|)dx \leqslant 4me^{-1}TM_5(2K+1).
\end{aligned}$$

Therefore, we find that the second term of the right-hand side of (6.12) is majorized by

$$C_{22}\int_\delta^t d\tau \int_{\mathbb{R}} \Phi_m(\nu_n)\phi\psi_m dx + \frac{C_{23}}{m},$$

for sufficiently large n, where C_{23} is independent of r, θ, ζ, ε ,m, n, and δ.

We next estimate the third term of the right-hand side of (6.12). In view of Corollary 3.1, it follows from (H3 viii) that for some $h \in L^q(0,T)$ with $1 < q < \infty$,

$$\begin{aligned}
&\int_\delta^t d\tau \int_{\mathbb{R}} \Psi(x,\tau)[\varphi^*_{\zeta_n}(x,\tau,\nu_{\theta_n}) - \varphi^*_{\theta_n}(x,\tau,\nu_{\theta_n})]dx\\
&\leqslant 2\int_\delta^t d\tau \int_{\mathbb{R}} (1+|x|)|\varphi(x+\zeta_n(\tau),\tau,\zeta_n(\tau),\nu_{\theta_n}(x,\tau))\\
&\qquad\qquad - \varphi(x+\theta_n(\tau),\tau,\theta_n(\tau),\nu_{\theta_n}(x,\tau))|dx\\
&\leqslant 2\int_\delta^t h(t)|\zeta_n(\tau)-\theta_n(\tau)|\Big\{1+\int_{\mathbb{R}}(1+|x|)\nu_{\theta_n}(x,\tau)dx\Big\}d\tau\\
&\leqslant 2\|\zeta_n-\theta_n\|_{C[r,t]}[1+2M_5(2K+1)]\int_\delta^t h(\tau)d\tau\\
&\leqslant C_{24}\|\zeta_n-\theta_n\|_{C[r,t]}(t-\delta)^\sigma,
\end{aligned}$$

where n is large enough, $\sigma := 1-q^{-1}$ and $C_{24} := 2\{1+2M_5(2K+1)\}\|h\|_{L^q(0,T)}$. Since the second term of the left-hand side of (6.12) is nonnegative, we get

$$\begin{aligned}
&\int_{\mathbb{R}} \phi\psi_m\Phi_m(\nu_n(t))dx + \int_\delta^t d\tau \int_{\mathbb{R}} \phi\psi_m \tanh(m\nu_n)[\varphi^*_{\zeta_n}(x,\tau,\nu_{\theta_n}) - \varphi^*_{\zeta_n}(x,\tau,\nu_{\zeta_n})]dx\\
&\leqslant \int_{\mathbb{R}} \phi\psi_m\Phi_m(\nu_n(\delta))dx + C_{22}\int_\delta^t d\tau \int_{\mathbb{R}} \Phi_m(\nu_n)\phi\psi_m dx\\
&\quad + \frac{C_{23}}{m} + C_{24}(t-\delta)^\sigma\|\zeta_n-\theta_n\|_{C[r,t]}. \qquad (6.13)
\end{aligned}$$

Step 4 (passage to the limit as $\delta \downarrow r$): Since $\nu_n(x,r) = 0$, we have

$$\left|\int_{\mathbb{R}} \phi(x)\psi_m(x)\Phi_m(\nu_n(x,\delta))dx\right| \leqslant \|\nu_n(\cdot,\delta)\|_{X_0} \int_{\mathbb{R}} \phi(x)\psi_m(x)dx \to 0 \quad \text{as } \delta \downarrow r.$$

Then letting $\delta \downarrow r$ in (6.13) leads to

$$\begin{aligned}
&\int_{\mathbb{R}} \phi\psi_m\Phi_m(\nu_n(t))dx + \int_r^t d\tau \int_{\mathbb{R}} \phi\psi_m \tanh(m\nu_n)[\varphi^*_{\zeta_n}(x,\tau,\nu_{\theta_n}) - \varphi^*_{\zeta_n}(x,\tau,\nu_{\zeta_n})]dx \\
&\leqslant C_{22}\int_r^t d\tau \int_{\mathbb{R}} \Phi_m(\nu_n)\phi\psi_m dx + \frac{C_{23}}{m} + C_{24}(t-r)^\sigma \|\zeta_n - \theta_n\|_{C[r,t]} \qquad (6.14)
\end{aligned}$$

for each $t \in [r,T]$ and m, n large enough.

Step 5 (passage to the limit as $m \to \infty$): Noting that the second term of the left-hand side of (6.14) is nonnegative since $\varphi(x,t,z,\cdot)$ is decreasing on $[0,+\infty[$ by (H3 iv), it follows in particular that

$$\begin{aligned}
\int_{\mathbb{R}} \Phi_m(\nu_n(t))\phi\psi_m dx \leqslant C_{22}\int_r^t d\tau \int_{\mathbb{R}} \Phi_m(\nu_n(\tau))\phi\psi_m dx + \frac{C_{23}}{m} \\
+ C_{24}(t-r)^\sigma \|\zeta_n - \theta_n\|_{C[r,t]}
\end{aligned}$$

for each $t \in [r,T]$. Put $f_m(t) := C_{23}m^{-1} + C_{24}(t-r)^\sigma\|\zeta_n - \theta_n\|_{C[r,t]}$. Noting that the mapping $t \mapsto f_m(t)$ is increasing, we apply the Gronwall Lemma to get

$$\int_{\mathbb{R}} \Phi_m(\nu_n(t))\phi\psi_m dx \leqslant f_m(t)\exp\{C_{22}(t-r)\}$$

for each $t \in [r,T]$. Therefore, returning to (6.14), we have

$$\begin{aligned}
&\int_r^t d\tau \int_{\mathbb{R}} \phi\psi_m \tanh(m\nu_n)[\varphi^*_{\zeta_n}(x,\tau,\nu_{\theta_n}) - \varphi^*_{\zeta_n}(x,\tau,\nu_{\zeta_n})]dx \\
&\leqslant C_{22}\int_r^t f_m(\tau)\exp\{C_{22}(\tau-r)\}d\tau + f_m(t) \\
&\leqslant f_m(t)(\exp\{C_{22}(t-r)\} - 1) + f_m(t) \leqslant \exp(C_{22}T)f_m(t) \qquad (6.15)
\end{aligned}$$

for $t \in [r,T]$. By using (H3 iii), it is known that

$$g_m(x,\tau) := \phi(x)\psi_m(x)\tanh(m\nu_n(x,\tau))[\varphi^*_{\zeta_n}(x,\tau,\nu_{\theta_n}) - \varphi^*_{\zeta_n}(x,\tau,\nu_{\zeta_n})]$$

converges increasingly to $\phi(x)|\varphi^*_{\zeta_n}(x,\tau,\nu_{\theta_n}(x,\tau)) - \varphi^*_{\zeta_n}(x,\tau,\nu_{\zeta_n}(x,\tau))|$ for *a.e.* $(x,\tau) \in \mathbb{R}\times]r,T[$ as $m \to \infty$. Thus we can apply the Monotone Convergence Theorem and we obtain from (6.15) that

$$\begin{aligned}
&\int_r^t d\tau \int_{\mathbb{R}} \phi(x)|\varphi^*_{\zeta_n}(x,\tau,\nu_{\theta_n}(x,\tau)) - \varphi^*_{\zeta_n}(x,\tau,\nu_{\zeta_n}(x,\tau))|dx \\
&= \lim_{m\to\infty}\int_r^t d\tau \int_{\mathbb{R}} g_m(x,\tau)dx \leqslant C_{25}(t-r)^\sigma\|\zeta_n - \theta_n\|_{C[r,t]}
\end{aligned}$$

for each $t \in [r, T]$, where C_{25} is independent of r, t, z, ε, n.

Step 6 (passage to the limit as $n \to \infty$): By Fatou's Lemma,

$$\begin{aligned}
&\int_r^t d\tau \int_{\mathbb{R}} \phi(x) |\varphi_\zeta^*(x, \tau, \nu_\theta(x, \tau)) - \varphi_\zeta^*(x, \tau, \nu_\zeta(x, \tau))| dx \\
&\leqslant \liminf_{n \to \infty} \int_r^t d\tau \int_{\mathbb{R}} \phi(x) |\varphi_{\zeta_n}^*(x, \tau, \nu_{\theta_n}(x, \tau)) - \varphi_{\zeta_n}^*(x, \tau, \nu_{\zeta_n}(x, \tau))| dx \\
&\leqslant C_{25}(t - r)^\sigma \|\zeta - \theta\|_t.
\end{aligned}$$

Noting (6.9), we arrive at the desired estimate. ■

Proof of Lemma 5.3. (i) Let $0 < \tau < t < T$. By changing variables,

$$\begin{aligned}
\Gamma_\zeta(t, \tau) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-\xi^2}}{\sqrt{\pi}} w(y + 2\xi\sqrt{\varepsilon(t - \tau)} + \zeta(t)) \varphi_\zeta^*(y, \tau, \nu_\zeta(y, \tau)) d\xi dy, \qquad (6.16)\\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-\xi^2}}{\sqrt{\pi}} w(y + 2\xi\sqrt{\varepsilon(t - \tau)} - \zeta(\tau) + \zeta(t)) \varphi(y, \tau, \zeta(\tau), \mu_\zeta(y, \tau)) d\xi dy,
\end{aligned}$$

where μ_ζ is defined by (4.7). Let us denote the integrand above by $f(\xi, y, t, \tau)$. Using (H2), (H3 vi–vii), and Proposition 4.2 (ii), we have for $R \geqslant \|\zeta\|_T$,

$$\begin{aligned}
&\int_{\mathbb{R}} d\xi \int_{\mathbb{R}} |f(\xi, y, t, \tau)| dy \\
&\leqslant \frac{C_w}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\xi^2} \int_{\mathbb{R}} (|y| + 2|\xi|\sqrt{\varepsilon(t - \tau)} + |\zeta(\tau)| + |\zeta(t)|) |\varphi(y, \tau, \zeta(\tau), \mu_\zeta(y, \tau))| dy d\xi \\
&\leqslant \frac{C_w}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\xi^2} d\xi \int_{\mathbb{R}} (|y| + 2R) |\varphi(y, \tau, \zeta(\tau), \mu_\zeta(y, \tau))| dy \\
&\quad + \frac{2C_w\sqrt{T}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\xi^2} |\xi| d\xi \int_{\mathbb{R}} |\varphi(y, \tau, \zeta(\tau), \mu_\zeta(y, \tau))| dy d\xi \\
&\leqslant C_{26} \int_{\mathbb{R}} (|y| + 2R + 1) \{\mathcal{F}(y, \tau) + M_3(R)(1 + |y|) \mu_\zeta(y, \tau)\} dy, \qquad (6.17)
\end{aligned}$$

where $C_{26} = C_w + 2C_w\sqrt{T}/\sqrt{\pi}$. Due to (H3 vi) and Proposition 4.2 (ii), the above inequality shows that (6.16) is well-defined and the assertion of (i) holds true by taking $R = 2K$.

(ii) Let $c \leqslant t_0 \leqslant T$. If $\tau \in \,]c, t_0[$, then $f(\xi, y, t, \tau)$ is continuous in $t \in [\tau, T]$ and in view of (6.17), one can use the Lebesgue Dominated Convergence Theorem to obtain

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(\xi, y, t, \tau) d\xi dy \longrightarrow \int_{\mathbb{R}} \int_{\mathbb{R}} f(\xi, y, t_0, \tau) d\xi dy \quad \text{as } t \to t_0.$$

Since $\Gamma_\zeta(t, \tau) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(\xi, y, t, \tau) d\xi dy$ if $\tau < t$, the above fact shows that

$$\lim_{t \to t_0} \Gamma_\zeta(t, \tau) = \Gamma_\zeta(t_0, \tau), \quad c < \tau < t_0.$$

This combined with (i) yields

$$\lim_{t \to t_0} \int_c^{t_0} \Gamma_\zeta(t, \tau) d\tau = \int_c^{t_0} \Gamma_\zeta(t_0, \tau) d\tau.$$

Consequently,

$$
\begin{aligned}
&\left|\int_c^t \Gamma_\zeta(t,\tau)d\tau - \int_c^{t_0} \Gamma_\zeta(t_0,\tau)d\tau\right| \\
&\leqslant \left|\int_c^t \Gamma_\zeta(t,\tau)d\tau - \int_c^{t_0} \Gamma_\zeta(t,\tau)d\tau\right| + \left|\int_c^{t_0} \Gamma_\zeta(t,\tau)d\tau - \int_c^{t_0} \Gamma_\zeta(t_0,\tau)d\tau\right| \\
&\leqslant C_{13}|t-t_0| + \left|\int_c^{t_0} \Gamma_\zeta(t,\tau)d\tau - \int_c^{t_0} \Gamma_\zeta(t_0,\tau)d\tau\right| \longrightarrow 0 \quad \text{as } t \to t_0.
\end{aligned}
$$

This completes the proof. ■

Proof of Lemma 5.4. The first estimate is an easy consequence of Lemma 5.3 (i). Let us show the second estimate. By (6.16) that

$$
\begin{aligned}
&\int_r^t |\Gamma_\zeta(t,\tau) - \Gamma_\theta(t,\tau)|d\tau \\
&\leqslant \int_r^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-\xi^2}}{\sqrt{\pi}} |w(y + 2\xi\sqrt{\varepsilon(t-\tau)} - \zeta(\tau) + \zeta(t)) \\
&\qquad\qquad - w(y + 2\xi\sqrt{\varepsilon(t-\tau)} - \zeta(\tau) + \theta(t))||\varphi_\zeta(y,\tau,\mu_\zeta(y,\tau))|dyd\xi d\tau \\
&\quad + \int_r^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-\xi^2}}{\sqrt{\pi}} |w(y + 2\xi\sqrt{\varepsilon(t-\tau)} + \theta(t))| \\
&\qquad\qquad \times |\varphi_\zeta^*(y,\tau,\nu_\theta(y,\tau)) - \varphi_\theta^*(y,\tau,\nu_\theta(y,\tau))|dyd\xi d\tau \\
&\quad + \int_r^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-\xi^2}}{\sqrt{\pi}} |w(y + 2\xi\sqrt{\varepsilon(t-\tau)} + \theta(t))| \\
&\qquad\qquad \times |\varphi_\zeta^*(y,\tau,\nu_\zeta(y,\tau)) - \varphi_\zeta^*(y,\tau,\nu_\theta(y,\tau))|dyd\xi d\tau
\end{aligned}
$$

for $t \in [r,T]$. We denote by $I_1(t)$, $I_2(t)$ and $I_3(t)$, the three terms in the right-hand side, respectively. Due to (H2), (H3 vi–vii), (5.4) and Proposition 4.2 (ii), we have

$$
\begin{aligned}
I_1(t) &\leqslant \frac{C_w}{\sqrt{\pi}} \int_r^t \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\xi^2} |\zeta(t) - \theta(t)|[\mathcal{F}(y,\tau) + M_3(2K)(1+|y|)\mu_\zeta(y,\tau)]dyd\xi d\tau \\
&\leqslant C_w \|\zeta - \theta\|_t \int_r^t \int_{\mathbb{R}} [\mathcal{F}(y,\tau) + M_3(2K)(1+|y|)\mu_\zeta(y,\tau)]dyd\tau \\
&\leqslant C_w[\|\mathcal{F}\|_{L^\infty(0,T;L^1(\mathbb{R}))} + 2M_3(2K)M_6(2K)]T^{1-\sigma}(t-r)^\sigma \|\zeta - \theta\|_t
\end{aligned}
$$

for $t \in [r,T]$. Next it follows from (H2), (H3 viii), (5.4), and Corollary 3.1 that

$$
\begin{aligned}
I_2(t) &\leqslant \frac{C_w}{\sqrt{\pi}} \int_r^t \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\xi^2} (|y| + 2|\xi|\sqrt{T} + 2K) \\
&\qquad\qquad \times |\varphi_\zeta^*(y,\tau,\nu_\theta(y,\tau)) - \varphi_\theta^*(y,\tau,\nu_\theta(y,\tau))|dyd\xi d\tau \\
&\leqslant C_{27} \int_r^t \int_{\mathbb{R}} (1+|y|)|\varphi(y+\zeta(\tau),\tau,\zeta(\tau),\nu_\theta(y,\tau)) \\
&\qquad\qquad - \varphi(y+\theta(\tau),\tau,\theta(\tau),\nu_\theta(y,\tau))|dyd\tau \\
&\leqslant C_{27} \int_r^t h(\tau)|\zeta(\tau) - \theta(\tau)| \left\{1 + \int_{\mathbb{R}} (1+|y|)\nu_\theta(y,\tau)dy\right\} d\tau \\
&\leqslant C_{27}\{1 + 2M_5(2K+1)\}\|h\|_{L^q(0,T)}(t-r)^\sigma \|\zeta - \theta\|_t
\end{aligned}
$$

for each $t \in [r,T]$, where $C_{27} := 2\pi^{-\frac{1}{2}}C_w(1+\sqrt{T}+K)\int_{\mathbb{R}}(1+|\xi|)\exp(-\xi^2)d\xi$. Finally, we obtain from Lemma 5.2 that

$$\begin{aligned} I_3(t) &\leqslant C_{27}\int_r^t\int_{\mathbb{R}}(1+|y|)|\varphi_\zeta^*(y,\tau,\nu_\zeta(y,\tau)) - \varphi_\zeta^*(y,\tau,\nu_\theta(y,\tau))|dyd\tau \\ &\leqslant C_{27}C_{12}(t-r)^\sigma\|\zeta-\theta\|_t \end{aligned}$$

for each $t \in [r,T]$. Consequently, we get the desired estimate. ■

APPENDIX

For the reader's convenience, we give definitions and notations of some function spaces used in this paper. Let M, N and L be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively. Let η, $\sigma \in]0,1]$. By $BUC(M)$ we denote the space of all bounded and uniformly continuous functions on M with the supremum norm $\|\cdot\|_\infty$. For any function $f: M \to \mathbb{R}$, put

$$[f]_{M,\eta} := \sup_{\substack{x,y\in M\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\eta}.$$

The space of all Hölder continuous functions with exponent $\eta \in]0,1[$ is denoted by $C^{0,\eta}(M)$, that is, f belongs to $C^{0,\eta}(M)$ if and only if f is continuous and $[f]_{M,\eta}$ is finite. And then we set $BUC^\eta(M) := BUC(M) \cap C^{0,\eta}(M)$ for $\eta \in]0,1[$. We denote by $C^{0,1}(M)$ the space of Lipschitz continuous functions on M.

For a function $f: M\times N \to \mathbb{R}$, we say that $f(\cdot,y) \in BUC^\eta(M)$ *uniformly* for y in a subset N' of N if $\sup_{y\in N'}\|f(\cdot,y)\|_\infty < \infty$ and $\sup_{y\in N'}[f(\cdot,y)]_{M,\eta} < \infty$. For $f: M\times N \to \mathbb{R}$, put

$$[f]_{M,N,\eta,\sigma} := \sup_{\substack{(x_i,y_i)\in M\times N\\ (x_1,y_1)\neq(x_2,y_2)}} \frac{|f(x_1,y_1)-f(x_2,y_2)|}{|x_1-x_2|^\eta+|y_1-y_2|^\sigma}$$

and define

$$BUC^{\eta,\sigma}(M\times N) := \{u \in BUC(M\times N) : [u]_{M,N,\eta,\sigma} < \infty\}.$$

Let L' be a subset of L. For a function $f: M\times N\times L \to \mathbb{R}$, we say that $f(\cdot,\cdot,z) \in BUC^{\eta,\sigma}(M\times N)$ *uniformly* for z in L' if

$$\sup_{z\in L'}\|f(\cdot,\cdot,z)\|_\infty < \infty \quad \text{and} \quad \sup_{z\in L'}[f(\cdot,\cdot,z)]_{M,N,\eta,\sigma} < \infty.$$

Let k_i be nonnegative integers and M_i subsets of $\mathbb{R}^{m_i}$, $i=1,2$. Denote by $C^{k_1,k_2}(M_1\times M_2)$ the space of all functions u having partial derivatives $\partial_i^{l_i}u$ up to the order k_i, $i=1,2$, which are continuous on $M_1\times M_2$. The space $BUC^{k_1,k_2}(M_1\times M_2)$ is defined as a subspace of $C^{k_1,k_2}(M_1\times M_2)$ consisting of all functions u having partial derivatives $\partial_i^{l_i}u$ up to the order k_i, $i=1,2$, in $BUC(M_1\times M_2)$. Finally, we define

$$\begin{aligned} BUC^{k_1+\eta,k_2+\sigma}(M_1\times M_2) := \{u \in BUC^{k_1,k_2}(M_1\times M_2) : \partial_i^{l_i}u \in BUC^{\eta,\sigma}(M_1\times M_2) \\ \text{whenever } 0\leqslant l_i\leqslant k_i\ (i=1,2)\}. \end{aligned}$$

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Nobuyuki Kato
Department of Mathematics
Shimane University
Matsue 690–8504, Japan
E-mail address: nkato@math.shimane-u.ac.jp

Toshiyuki Yamaguchi
Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 739–8526, Japan

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