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BOHMAN-KOROVKIN-WULBERT OPERATORS ON C[0,1] for $\{1, x, x^2, x^3, x^4\}$

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Abstract. A class of operators on the Banach space C[0,1] of all complex-valued functions on [0,1] satisfying a Bohman-Korovkin-Wulbert type theorem is investigated. Under the test function $\{1, x, x^2, x^3, x^4\}$ the sum of two homomorphisms on C[0,1] is an example.

In 1952, H. Bohman [2] proved that the sequene of interpolation operators

$$B_n = \sum_{k=0}^{n} b_{k,n} \otimes \delta_{t_{k,n}}$$

$$\left(\begin{array}{l} 0 \leqq t_{k,n} \leqq 1, \ t_{j,n} < t_{k,n} \ (j < k), \ 0 \leqq b_{k,n} \in C[0,1], \\ \delta_t \text{ is the evaluation at } t \end{array} \right)$$

on C[0,1] converges strongly to the identity operator if $\lim_{n\to\infty} \| B_n(f) - f \|_\infty = 0$ for $f = 1, x, x^2$, where $\| \ \|_\infty$ denotes the supremum norm on C[0,1]. Such functions $\{1, x, x^2\}$ are called test functions.

In 1959, P. P. Korovkin [3] proved that Bohman's theorem is true even if the interpolation operators B_n are replaced with positive linear operators on C[0,1]. In 1968, D. E. Wulbert [5] proved that

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Korovkin's theorem is true even if the positivity assumption on the operators is replaced with the operator norm condition on which the sequence of operator norms converges to one. Then given a subset F of $C[0,1]$, it will be valuable to investigate a class of operators T on $C[0,1]$ such that if $\{T_\lambda\}$ is any net of bounded linear operators on $C[0,1]$ with $\lim_\lambda \| T_\lambda \| = \| T \|$ and $\lim_\lambda \| T_\lambda f - f \| = 0$ for all $f \in F$, then $\{T_\lambda\}$ converges strongly to T, i.e., $\lim_\lambda \| T_\lambda f - Tf \| = 0$ for all $f \in C[0,1]$. We call such T a Bohman-Korovkin-Wulbert (BKW) operator on $C[0,1]$ for the test functions F. Therefore the Korovkin-Wulbert approximation theorem asserts that the identity operator on $C[0,1]$ is BKW for the test functions $\{1, x, x^2\}$. The author [4] showed that every homomorphism of $C[0,1]$ into itself is BKW for the test functions $\{1, x, x^2\}$. However the sum of two homomorphisms of $C[0,1]$ into itself is not in general BKW even for test functions $\{1, x, x^2, x^3\}$. In fact the sum of one-dimensional homomorphisms $1\otimes\delta_{1/3}$ and $1\otimes\delta_{2/3}$ is not BKW for $\{1, x, x^2, x^3\}$. This can be observed from the following facts:

$$\left\| \frac{1}{9}(1\otimes\delta_0) + \frac{16}{9}(1\otimes\delta_{1/2}) + \frac{1}{9}(1\otimes\delta_1) \right\| = \| 1\otimes\delta_{1/3} + 1\otimes\delta_{2/3} \|$$
$$= 2$$

and

$$\left\{\frac{1}{9}(1\otimes\delta_0) + \frac{16}{9}(1\otimes\delta_{1/2}) + \frac{1}{9}(1\otimes\delta_1)\right\}(x^m) = (1\otimes\delta_{1/3} + 1\otimes\delta_{2/3})(x^m)$$
$$= (1/3)^m + (2/3)^m$$

for each $m = 0, 1, 2, 3$.

Then what is a family of test functions for which the sum of two homomorphisms is always BKW ? The following result gives an answer of this question.

Theorem. If T_1 and T_2 are two homomorphisms of $C[0,1]$ into itself, then $T_1 + T_2$ is a BKW-operator for the test functions $\{1, x, x^2, x^3, x^4\}$.

To show this we need the following lemma which is observed in [4, Lemma 1] or [1, Corollary 1.2].

Lemma. Let $M[0,1]$ be the space of all bounded regular Borel measure on $[0,1]$, F a subset of $C[0,1]$ and T a bounded linear operator on $C[0,1]$. Suppose that for any $t \in [0,1]$, $\| T^*\delta_t \| = \| T \|$ and $T^*\delta_t$ is the only measure in $M[0,1]$ which extends $T^*\delta_t | F$ and which norm is less than $\| T^*\delta_t \|$. Then any net $\{T_\lambda\}$ of bounded linear operators on $C[0,1]$ such that $\lim_\lambda \| T_\lambda \| = \| T \|$ and $\lim_\lambda \| T_\lambda f - Tf \| = 0$ $(f \in F)$ converges strongly to T.

Proof of Theorem. We can assume that $T_1 \neq 0$ and $T_2 \neq 0$. Then it suffices from the above lemma to show that the following properties hold for any $t \in [0,1]$:

(1) $\| (T_1 + T_2)^*\delta_t \| = \| T_1 + T_2 \|$.

(2) If $\mu \in M[0,1]$, $\| \mu \| \leqq \| (T_1 + T_2)^*\delta_t \|$ and $\mu(x^m) = (T_1x^m)(t) + (T_2x^m)(t)$ $(0 \leqq m \leqq 4)$, then $\mu = (T_1 + T_2)^*\delta_t$.

Here T^* denotes the adjoint operator of T, δ_t the Dirac measure concentrated at t. To do this let $t \in [0,1]$ and set $t_1 = \varphi_1(t)$, $t_2 = \varphi_2(t)$, where φ_1 and φ_2 denote the representation functions of T_1 and T_2, respectively. Then $(T_1 + T_2)^*\delta_t = \delta_{t_1} + \delta_{t_2}$ and hence $\| (T_1 + T_2)^*\delta_t \| = \| T_1 + T_2 \| = 2$, and (1) was shown. Now suppose that $\mu \in M[0,1]$, $\| \mu \| \leqq 2$, $\mu(1) = 2$ and $\mu(x^m) = t_1^m + t_2^m$ $(1 \leqq m \leqq 4)$. Then μ is positive and $\mu((x - t_1)^2(x - t_2)^2) = 0$. Hence the support of μ is contained in $\{t_1, t_2\}$. Therefore a simple calculation implies that $\mu = \delta_{t_1} + \delta_{t_2}$ and so (2) was shown. Q. E. D.

Remark 1. By a similar method used in the proof of Theorem, we have that $1\otimes\delta_0 + 1\otimes\delta_1$ is BKW for $\{1, x, x^2\}$ and that for any homomorphism T of $C[0,1]$ into itself, $1\otimes\delta_0 + T$, $1\otimes\delta_1 + T$ are BKW for $\{1, x, x^2, x^3\}$. Then it will be valuable to investigate a pair of two homomorphisms whose sum is BKW for $\{1, x, x^2\}$ or $\{1, x, x^2, x^3\}$.

Remark 2. For the n-case, we have that if $T_1, \ldots, T_n$ are homomorphisms of $C[0,1]$ into itself, then $T_1 + \ldots + T_n$ is a BKW-operator for the test functions $\{1, x, \ldots, x^{2n}\}$. In fact note that if $t_1, \ldots, t_n \in [0,1]$ and $\mu \in M[0,1]$ such that $\mu(x^m) = \sum_{k=0}^{n} t_k^m$ $(m = 0, 1, \ldots, 2n)$, then

$\mu(\prod_{k=0}^{n}(x - t_k)^2) = 0$ and hence $\mu = \sum_{k=0}^{n} \delta_{t_k}$ is the only measure in $M[0,1]$ which extends $\mu \mid \{1, x, \ldots, x^{2n}\}$ and which norm is less than n $(= \|\mu\|)$. Therefore the desired result follows from the similar method used in the proof of Theorem.

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