Nihonkai Math. J.
Vol.1(1990),43-53

A Discrete Analog of Laplace's Differential Equations

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1. Intoduction

A differential equation of the form

$$(a_2t+b_2)y''(t)+(a_1t+b_1)y'(t)+(a_0t+b_0)y(t)=0 \tag{1.1}$$

was studied by Laplace in his treatise "Theorie analytique des probabilities"(cf. Yosida [4,p.53]), so that (1.1) may be called the Laplace differential equations. In a recent monograph [4],Yosida gives a new treatment of (1.1) by his operational calculus method. In the present paper, we shall study a discrete analog of the Laplace differential equation (1.1), namely, the monodiffric difference equation of the form

$$K(z)\circ\frac{d^2f}{dz^2}+(a_1H(z)+b_1)\circ\frac{df}{dz}+(a_0G(z)+b_0)\circ f(z)=0. \tag{1.2}$$

We shall use the formal series method to find the general solution of (1.2). Our main result is Theorem 2. And Theorem 2 can be applied to solve Bessel, Laguerre and Gauss monodffric difference equations which will be introduced in Section 6.

2. Definition and Notation

For the sake of convenience,we give some definitions and notations which are mentioned in [3]. Let $\mathbf{C}$ be the complex plane,

$$D=\{z\in\mathbf{C}\mid z=x+iy,\ x \text{ and } y \text{ are integers }\}.$$

DEFINITION 1. The function $f:D\to\mathbf{C}$ is said to be monodiffric at z if

$$(i-1)f(z)+f(z+i)-if(z+1)=0. \tag{2.1}$$

The function f is said to be monodiffric in D if it is monodiffric at any point in D.

DEFINITION 2. The monodiffric derivative f' of f is defined by

$$f'(z)=\frac{1}{2}[(i-1)f(z)+f(z+1)-if(z+i)]. \tag{2.2}$$

We also use the symbol $\frac{df}{dz}$ to represent f'.

DEFINITION 3. Suppose that f and g are complex-valued functions defined on D. Let $z\in D$ and $h\in\{1,i,-1,-i\}$. The line integrals from z to $z+h$ are defined respectively by

$$\int_z^{z+h} f(t)dt = hf(z) \qquad \text{if } h = 1 \text{ or } i$$
$$= -\int_{z+h}^{z} f(t)dt \quad \text{if } h = -1 \text{ or } -i \tag{2.3}$$

$$\int_z^{z+h} f(t) : g(t)dt = f(z+h)[g(z+h) - g(z)] \qquad \text{if } h = 1 \text{ or } i$$
$$= -\int_{z+h}^{z} f(t) : g(t)dt \qquad \text{if } h = -1 \text{ or } -i. \tag{2.4}$$

DEFINITION 4. For every $z \in D$,the *-convolution product of monodiffric functions f and g is defined by

$$(f * g)(z) = \int_0^z f(z-t) : g(t)dt. \tag{2.5}$$

DEFINITION 5. A sequence of monodiffric polynomial $\{z^{(n)}\}$ is defined by

$$z^{(n)} = n\int_0^z z^{(n-1)}dz \qquad n = 1, 2, ...$$
$$= 1 \qquad n = 0. \tag{2.6}$$

The following definitions and notations of a formal power series will be needed in our discussion of the solutions of (1.2).

The class of formal power series

$$R_{xy} = \{\sum_{\substack{m=0\\n=0}}^{\infty} a_{mn}x^m y^n ; a_{mn} \in C\}$$

endowed with the usual addition and multiplication

$$(\sum a_{mn}x^m y^n) + (\sum b_{mn}x^m y^n) = \sum(a_{mn} + b_{mn})x^m y^n$$

$$(\sum a_{mn}x^m y^n)(\sum b_{mn}x^m y^n) = \sum_{\substack{m=0\\n=0}}^{\infty}(\sum_{k=0}^{m}\sum_{r=0}^{n} a_{k,r}b_{m-k,n-r})x^m y^n$$

is a ring. More precisely R_{xy} is an integral domain. The classes of formal power series of one variable

$$R_x = \{\sum_{m=0}^{\infty} a_m x^m ; a_m \in \mathbf{C}\}, \quad R_y = \{\sum_{n=0}^{\infty} b_n y^n ; b_n \in \mathbf{C}\}$$

are subrings of R_{xy}.

Let $f : z^+ \times z^+ \rightarrow \mathbf{C}$ and let f determined by the formal power series $F(x,y) = \sum_{m,n=0}^{\infty} f(m,n)x^m y^n$. The relation between a formal power series and a monodiffric function,we have the following.

THEOREM 1 [3]. *Suppose*

(1) $f(m,n)$ *is monodiffric at* $z = m + ni$

(2) $\phi_f(x) = \sum_{m=0}^{\infty} f(m,0)x^m$

(3) $\psi_f(y) = \sum_{n=0}^{\infty} f(0,n)y^n$.

Then

$$F(x,y) = \frac{x\phi_f(x) - iy\psi_f(y)}{x - xy + ixy - iy}.$$

Moreover,

$$F(x,y) = \frac{x\phi_f(x) - \frac{iy}{1+(i-1)y}\phi_f(\frac{iy}{1+(i-1)y})}{x - xy + ixy - iy}. \tag{2.7}$$

Since a monodiffric function f is uniquely determined by ϕ_f and evidently each monodiffric function determines such a ϕ_f,there is a one-to-one correspondence between monodiffric functins and the elements of the set R_x.

We also need the following table (cf. Tu [3]).

Table I

Monodiffric Functions	Formal Power series
1. $f(x,y) = c$	$\phi_c = \frac{c}{1-x}$
2. f'	$\phi_{f'} = \frac{(1-x)\phi_f - f(\hat{0})}{x}$
3. $f^{(n)}$	$\phi_{f^{(n)}} = \frac{1}{x^n}\{(1-x)^n\phi_f - \sum_{k=0}^{n-1} x^k f^{(k)}(\hat{0})(1-x)^{n-1-k}\}$
4. $f * g$	$\phi_{f*g} = (1-x)\phi_g\phi_f - g(\hat{0})\phi_f$
5. $z^{(n)}$	$\phi_{z^{(n)}} = \frac{n!x^n}{(1-x)^{n+1}} \quad n = 0,1,2,\ldots$
6. $e^{a,z} = (1+a)^x(1+ia)^y$	$\phi_{e^{a,z}} = \frac{1}{1-(1+a)x}$

where $z = x + iy, a \in \mathbf{C}$ and $\hat{0} = (0,0)$.

3. o-convolution product of monodiffric functions

Before we solve the generalized Laplace monodiffric difference equation (1.2), we need the following definition and results of Berzsenyi [2]. For the completeness, here, we give different proof which is much simpler than Berzsenyi's proof by use of the method introduced by the author [3].

DEFINITION 6 [2]. If $f, g \in M(z^+ \times z^+)$, then the function$f \circ g$ is given by $(f \circ g)(z) = (f * g)(z) + g(\hat{0}) f(z)$, where $\hat{0} = (0,0)$, and $f \circ g$ is called the o-convolution product of f and g. It is easy to see that $(f \circ g)(z)$ is monodiffric if $f(z)$ and $g(z)$ are monodiffric.

LEMMA 1. *The formal power series* $\phi_{f \circ g}$ *relative to monodiffric function* $f \circ g$ *is given by*

$$\phi_{f \circ g} = (1 - x)\phi_f \phi_g.$$

PROOF. By definition,

$$\phi_{f \circ g} = \phi_{(f*g)+g(\hat{0})f} = \sum_{m=0}^{\infty} [(f * g)(m) + g(\hat{0}) f(m)] x^m$$
$$= (1 - x)\phi_f \phi_g - g(\hat{0})\phi_f + g(\hat{0})\phi_f = (1 - x)\phi_f \phi_g.$$

PROPOSITION 1 [2]. *Let* $f, g, h \in M(z^+ \times z^+)$ *and* $c \in \mathbf{C}$. *Then*
(a) $(f + g) \circ h = (f \circ h) + (g \circ h)$,
(b) $f \circ (g + h) = (f \circ g) + (f \circ h)$,
(c) $(cf) \circ g = c(f \circ g) = f \circ (cg)$.

PROOF. $\phi_{(f+g) \circ h} = (1 - x)\phi_{f+g}\phi_h = (1 - x)[\phi_f + \phi_g]\phi_h = \phi_{f \circ h} + \phi_{g \circ h}$. On the other hand, $\phi_{(f \circ h)+(g \circ h)} = \phi_{f \circ h} + \phi_{g \circ h}$. Thus we have $\phi_{(f+g) \circ h} = \phi_{(f \circ h)+(g \circ h)}$. Similarly, we can prove (b) and (c).

PROPOSITION 2 [2]. *If* $f, g \in M(z^+ \times z^+)$ *then* $f \circ g = g \circ f$.

PROOF. By Lemma 1,

$$\phi_{f \circ g} = (1 - x)\phi_f \phi_g \text{ and } \phi_{g \circ f} = (1 - x)\phi_g \phi_f.$$

PROSITION 3 [2]. *If* f, g *and* $h \in M(z^+ \times z^+)$ *then* $(f \circ g) \circ h = f \circ (g \circ h)$.

PROOF. $\phi_{(f \circ g) \circ h} = (1 - x)\phi_{f \circ g}\phi_h = (1 - x)[(1 - x)\phi_f \phi_g]\phi_h = (1 - x)^2 \phi_f \phi_g \phi_h$,
$\phi_{f \circ (g \circ h)} = (1 - x)\phi_f \phi_{g \circ h} = (1 - x)\phi_f (1 - x)\phi_g \phi_h = (1 - x)^2 \phi_f \phi_g \phi_h$.

PROPOSITION 4 [2]. *If* $f, g \in M(z^+ \times z^+)$ *and* $f \circ g = 0$, *then either* $f = 0$ *or* $g = 0$.

PROOF. Since $\phi_{f \circ g} = (1 - x)\phi_f \phi_g = 0$, we have $\phi_f = 0$ or $\phi_g = 0$. It follows from Theorem 1 that $f = 0$ or $g = 0$.

4. Cauchy formula in monodiffric function

Proposition 5. *The monodiffric polynomia $z^{(n)}$ has the following relation*

$$\frac{z^{(n)}}{n!} \circ \frac{z^{(k)}}{k!} = \frac{z^{(n+k)}}{(n+k)!} \text{ for every } n, k = 1, 2, \ldots.$$

Proof.

$$\phi_{\frac{z^{(n)}}{n!} \circ \frac{z^{(k)}}{k!}} = (1-x)\phi_{\frac{z^{(n)}}{n!}} \phi_{\frac{z^{(k)}}{k!}} = (1-x)\frac{x^n}{(1-x)^{n+1}} \frac{x^k}{(1-x)^{k+1}}$$

$$= \frac{x^{n+k}}{(1-x)^{n+k+1}} = \phi_{\frac{z^{(n+k)}}{(n+k)!}}.$$

An analog of the well-known Cauchy's formula in the continuous case

$$\int_0^t \cdots \int_0^t f(t)dt \cdots dt = \frac{1}{(n-1)!} \int_0^t (t-z)^{n-1} f(z)dz.$$

We have

Proposition 6. *If $f \in M(z^+ \times z^+)$, then for n=0,1,2,...,*

$$\int_0^z \int_0^{t_1} \cdots \int_0^{t_n} f(t_{n+1})dt_{n+1} \cdots dt_1 = \frac{1}{(n+1)!}(f \circ z^{(n+1)}).$$

Proof. From (4.1) in [4],

$$\phi_{\int_0^z f(t)dt} = \frac{x}{1-x}\phi_f.$$

Since $\phi_{z^{(1)}} = \frac{x}{(1-x)^2}$, we obtain $\phi_{f \circ z^{(1)}} = (1-x)\phi_f \phi_{z^{(1)}} = \frac{x}{1-x}\phi_f$. Thus, we have $\int_0^z f(t)dt = f \circ z^{(1)}$.

Using of Propositions 3 and 5,

$$\int_0^{t_{n-1}} \int_0^{t_n} f(t_{n+1})dt_{n+1}dt_n = \int_0^{t_{n-1}} (f \circ z^{(1)})(t_n)dt_n = (f \circ z^{(1)}) \circ z^{(1)}$$

$$= f \circ (z^{(1)} \circ z^{(1)}) = f \circ \frac{z^{(2)}}{2!}.$$

$$\int_0^{t_{n-2}} \int_0^{t_{n-1}} \int_0^{t_n} f(t_{n+1})dt_{n+1}dt_n dt_{n-1} = (f \circ \frac{z^{(2)}}{2!}) \circ z^{(1)} = f \circ (\frac{z^{(2)}}{2!} \circ z^{(1)})$$

$$= f \circ \frac{z^{(3)}}{3!}.$$

By induction we conclude the proof.

5. Generalized Laplace monodiffric difference equation

We are now in position to discuss our main results.

THEOREM 2. *Let $k, H, g \in M(z^+ \times z^+)$. Corresponding to the solution of the generalized Laplace monodiffric difference equations of the form*

$$k(z) \circ \frac{d^2 f}{dz^2} + (aH(z) + b) \circ \frac{df}{dz} + (cg(z) + d) \circ f(z) = 0 \tag{5.1}$$

with $f(\hat{0}) = c_1$ and $f'(\hat{0}) = c_2$, its formal power series ϕ_f is given by

$$\phi_f = \frac{[(1-x)^2\phi_k + a(1-x)x\phi_H + bx]c_1 + (1-x)x\phi_k c_2}{(1-x)^3\phi_k + a(1-x)^2 x\phi_H + bx(1-x) + cx^2(1-x)\phi_g + dx^2}. \tag{5.2}$$

PROOF. By use of Lemma 1 and relation 3 in Table I, (5.1) becomes the following formal power series of the form

$$\phi_{k\circ f''} + a\phi_{H\circ f'} + b\phi_{f'} + c\phi_{g\circ f} + d\phi_f = 0,$$

$$(1-x)\phi_k \frac{1}{x^2}\{(1-x)^2\phi_f - c_1(1-x) - c_2 x\} + a(1-x)\phi_H \frac{1}{x}\{(1-x)\phi_f - c_1\}$$
$$+\frac{b}{x}\{(1-x)\phi_f - c_1\} + c(1-x)\phi_g\phi_f + d\phi_f = 0,$$

i.e.

$$[(1-x)^3\phi_k + a(1-x)^2 x\phi_H + bx(1-x) + cx^2(1-x)\phi_g + dx^2]\phi_f$$
$$= c_1[(1-x)^2\phi_k + a(1-x)x\phi_H + bx] + (1-x)x\phi_K c_2.$$

Thus we obtain (5.2).

6. Bessel, Laguerre, Laplace and Gauss equation

We shall apply Theorem 2 to the cases of Bessel, Laguerre, Laplace and Gauss equation as Corollaries.

COROLLARY 1. (The Bessel monodiffric difference equation) The equation is of the form

$$z^{(2)} \circ \frac{d^2 f}{dz^2} + z^{(1)} \circ \frac{df}{dz} + (z^{(2)} - \alpha^2) \circ f(z) = 0 \tag{6.1}$$

with $f(\hat{0}) = c_1$ and $f'(\hat{0}) = c_2$, where α is a complex number.

Corresponding to its solution of (6.1),the formal power series is

$$\phi_f = \frac{3(1-x)c_1 + 2xc_2}{(3-\alpha^2)(1-x)^2 + 2x^2} \tag{6.2}$$

PROOF. In (5.1), we take $k(z) = z^{(2)}, H(z) = z^{(1)}, g(z) = z^{(2)}, a = c = 1, b = 0$ and $d = -\alpha^2$. Then (5.2) becomes

$$\phi_f = \frac{[(1-x)^2\phi_{z^{(2)}} + (1-x)x\phi_{z^{(1)}}]c_1 + (1-x)x\phi_{z^{(2)}}c_2}{(1-x)^2\phi_{z^{(2)}} + (1-x)^2x\phi_{z^{(1)}} + x^2(1-x)\phi_{z^{(2)}} - \alpha^2x^2}.$$

Since $\phi_{z^{(1)}} = \frac{x}{(1-x)^2}$ and $\phi_{z^{(2)}} = \frac{2x^2}{(1-x)^3}$, we obtain

$$\begin{aligned}\phi_f &= \frac{[2(1-x)^2x^2 + (1-x)^2x^2]c_1 + 2(1-x)x^3c_2}{2(1-x)^3x^2 + (1-x)^3x^2 + 2x^4(1-x) - \alpha^2x^2(1-x)^3}\\ &= \frac{3(1-x)c_1 + 2xc_2}{(3-\alpha^2)(1-x)^2 + 2x^2}.\end{aligned}$$

COROLLARY 2. (The Laguerre monodiffric diference equation) The equation is of the form

$$z^{(1)} \circ \frac{d^2f}{dz^2} - (z^{(1)} + \alpha - 1) \circ \frac{df}{dz} + (\alpha + \lambda) \circ f(z) = 0 \tag{6.3}$$

with $f(\hat{0}) = c_1$ and $f'(\hat{0}) = c_2$, where α and λ are complex numbers. Corresponding to its solution of (6.3) the formal power series is

$$\phi_f = \frac{[(\alpha-3)x + (2-\alpha)]c_1 + c_2x}{(1-x)[2-\alpha+(2\alpha+\lambda-3)x]} \tag{6.4}$$

PROOF. In (5.1), we take $k(z) = H(z) = z^{(1)}, a = -1, b = 1-\alpha, c = 0$ and $D = \alpha + \lambda$. Then (5.2) becomes

$$\phi_f = \frac{[(1-x)^2\phi_{z^{(1)}} - (1-x)x\phi_{z^{(1)}} + (1-\alpha)x]c_1 + (1-x)x\phi_{z^{(1)}}c_2}{(1-x)^3\phi_{z^{(1)}} - (1-x)^2x\phi_{z^{(1)}} + (1-\alpha)x(1-x) + (\alpha+\lambda)x^2}.$$

Since $\phi_{z^{(1)}} = \frac{x}{(1-x)^2}$, we have

$$\begin{aligned}\phi_f &= \frac{[(1-x)^2x - (1-x)x^2 + (1-\alpha)x(1-x)^2]c_1 + (1-x)x^2c_2}{(1-x)^3x - (1-x)^2x^2 + (1-\alpha)x(1-x)^3 + (\alpha+\lambda)x^2(1-x)^2}\\ &= \frac{[(\alpha-3)x + 2 - \alpha]c_1 + c_2x}{(1-x)[2-\alpha+(2\alpha+\lambda-3)x]}.\end{aligned}$$

Thus, we obtain (6.4).

COROLLARY 3. (The Laplace monodiffric difference equation) The equation is of the form

$$z^{(1)} \circ \frac{d^2f}{dz^2} + (az^{(1)} + b) \circ \frac{df}{dz} + (cz^{(1)} + d) \circ f(z) = 0 \tag{6.5}$$

with $f(\hat{0}) = c_1$ and $f'(\hat{0}) = c_2$, where a, b, c and d are complex numbers. Corresponding to its solution of (6.5), the formal power series is

$$\phi_f = \frac{[b+1+(a-b-1)x]c_1 + c_2 x}{(1-x)[b+1+(a+d-b-1)x]+cx^2}. \tag{6.6}$$

Proof. In (5.1), we take $k(z) = H(z) = g(z) = z^{(1)}$. Then (5.2) becomes

$$\phi_f = \frac{[(1-x)^2\phi_{z^{(1)}} + a(1-x)x\phi_{z^{(1)}} + bx]c_1 + (1-x)x\phi_{z^{(1)}}c_2}{(1-x)^3\phi_{z^{(1)}} + a(1-x)^2x\phi_{z^{(1)}} + bx(1-x) + cx^2(1-x)\phi_{z^{(1)}} + dx^2}.$$

Since $\phi_{z^{(1)}} = \frac{x}{(1-x)^2}$, we have

$$\begin{aligned}\phi_f &= \frac{[(1-x)^2x + a(1-x)x^2 + bx(1-x)^2]c_1 + (1-x)x^2c_2}{(1-x)^3x + a(1-x)^2x^2 + bx(1-x)^3 + cx^3(1-x) + dx^2(1-x)^2}\\ &= \frac{[b+1+(a-b-1)x]c_1 + c_2x}{(1-x)[b+1+(a+d-b-1)x]+cx^2}.\end{aligned}$$

Thus, Corollary 3 is proved.

Corollary 4. (The Gauss monodiffric difference equation) The equation is of the form

$$z^{(1)} \circ (1 - ez^{(1)}) \circ \frac{d^2f}{dz^2} + (h - jz^{(1)}) \circ \frac{df}{dz} - mf(z) = 0 \tag{6.7}$$

with $f(\hat{0}) = c_1$ and $f'(\hat{0}) = c_2$, where e, h, j and m are complex numbers. Corresponding to its solution of (6.7),the formal power series is

$$\phi_f = \frac{(1-x)[(1+h)-(1+e+j+h)x]c_1 + x[1-(1+e)x]c_2}{(1-x)^2[(1+h)-(1+e+m+j+h)x]}. \tag{6.8}$$

Proof. In (5.1), we take $k(z) = z^{(1)} \circ (1 - ez^{(1)}), a = -j, H(z) = z^{(1)}, b = h, c = 0$ and $d = -m$. Then (5.2) becomes

$$\phi_f = \frac{[(1-x)^2\phi_k - j(1-x)x\phi_H + hx]c_1 + (1-x)x\phi_k c_2}{(1-x)^3\phi_k - j(1-x)^2x\phi_H + hx(1-x) - mx^2}.$$

Since $k(z) = z^{(1)} - \frac{e}{2}z^{(2)}$, we have

$$\begin{aligned}\phi_k &= \frac{x}{(1-x)^2} - \frac{ex^2}{(1-x)^3},\\ \phi_H &= \frac{x}{(1-x)^2},\\ \phi_f &= \frac{[(1-x)^3x - ex^2(1-x)^2 - j(1-x)^2x^2 + h(1-x)^3x]c_1 + [x^2(1-x)^2 - (1-x)x^3e]c_2}{(1-x)^4x - (1-x)^3x^2e - j(1-x)^3x^2 + hx(1-x)^4 - mx^2(1-x)^3}\\ &= \frac{(1-x)[(1+h)-(1+e+j+h)x]c_1 + x[1-(1+e)x]c_2}{(1-x)^2[(1+h)-(1+e+j+h+m)x]}.\end{aligned}$$

∎

Thus, Corollary 4 is proved.

COROLLARY 5. (The confluent hypergeometric monodiffric difference equation) The equation is of the form

$$z^{(1)} \circ \frac{d^2 f}{dz^2} + (h - z^{(1)}) \circ \frac{df}{dz} - mf(z) = 0 \tag{6.9}$$

with $f(\hat{0}) = c_1$ and $f'(\hat{0}) = c_2$, where h and m are complex numbers. Corresponding to its solution of (6.9), the formal power series is

$$\phi_f = \frac{[1 + h - (2 + h)x]c_1 + c_2 x}{(1 - x)[1 + h - (2 + h + m)x]}. \tag{6.10}$$

PROOF. This equation is obtained from the Gauss monodiffric difference equation (6.7) by confluence, i.e., by letting $e \to 0$ and $j \to 1$. So, (6.10) is obtained immediately from (6.8). Or, in another way, it is essentially the same as the Laguerre monodiffric difference equation (6.3). That is, in (6.4) if we take $h = 1 - \alpha$ and $m = -(\alpha + \lambda)$ then (6.4) becomes (6.10).

7. Applications

As for its applications to our main result, we shall define monodiffric exponential, sine and cosine functions as follows;

DEFINITION 6. Monodiffric exponential, sine and cosine functions are defined respectively as follows.

$$e^{a,z} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{(n)}$$

$$\sin az = \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n+1} z^{(2n+1)}}{(2n+1)!}$$

$$\cos az = \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n} z^{(2n)}}{(2n)!},$$

where a is a complex number.

PROPOSITION 7. *Corresponding to monodiffric exponential, sine and cosine functions, their formal power series are*

$$\phi_{e^{a,z}} = \frac{1}{1 - (1 + \alpha)x} \tag{7.1}$$

$$\phi_{\sin az} = \frac{ax}{(1 - x)^2 + a^2 x^2} \tag{7.2}$$

$$\phi_{\cos az} = \frac{1 - x}{(1 - x)^2 + a^2 x^2}. \tag{7.3}$$

PROOF.

$$\phi_{e^{a,z}} = \phi_{\sum_{n=0}^{\infty} \frac{a^n}{n!} z^{(n)}} = \sum_{n=0}^{\infty} \frac{(ax)^n}{(1-x)^{n+1}} = \frac{1}{1-(1+a)x}.$$

Thus (7.1) is proved.

$$\begin{aligned}\phi_{\sin az} &= \phi_{\sum_{n=0}^{\infty}(-1)^n \frac{(az)^{2n+1}}{(2n+1)!}} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{(ax)^{2n+1}}{(1-x)^{2n+2}}\right) \\ &= \frac{ax}{(1-x)^2+a^2x^2}.\end{aligned}$$

Thus we obtain (7.2). Similarly we have (7.3).

This monodiffric exponential function $e^{a,z} = (1+a)^x(1+ia)^y$ for $z = x+iy$ and $a \in C$ was introduced by Issacs [1].

EXAMPLE 1. The solution of the equation

$$\frac{df}{dz} - mf(z) = 0 \qquad \text{with } f(\hat{0}) = 1 \tag{7.4}$$

is give by $f(z) = e^{m,z}$.

PROOF. In (5.1),we take $c_1 = 1, c_2 = 0, k(z) = 0, H(z) = 0, g(z) = 0, b = 1, c = 0$ and $d = -m$. Then (5.2) becomes

$$\phi_f = \frac{x}{x(1-x)-mx^2} = \frac{1}{1-(1+m)x} = \phi_{e^{m,z}}.$$

Therefore, we have $f(z) = e^{m,z}$.

EXAMPLE 2. The solution of the equation

$$\frac{d^2f}{dz^2} + m^2 f(z) = 0 \qquad \text{with } f(\hat{0}) = 0 \text{ and } f'(\hat{0}) = m \tag{7.5}$$

is given by $f(z) = \sin mz$.

PROOF. In (5.1), we take $k(z) = 1, a = b = c = 0, d = m^2, c_1 = 0$ and $c_2 = m$. Then (5.2) becomes

$$\phi_f = \frac{m(1-x)x\phi_1}{(1-x)^3\phi_1 + m^2x^2}.$$

Since $\phi_1 = \frac{1}{1-x}$, we have $\phi_f = \frac{mx}{(1-x)^2+m^2x^2}$. By (7.2), $\phi_f = \phi_{\sin mz}$ i.e. $f(z) = \sin mz$. Similarly, we have example 3.

EXAMPLE 3. The solution of the equation

$$\frac{d^2f}{dz^2} + m^2 f(z) = 0 \qquad \text{with } f(\hat{0}) = 1 \text{ and } f'(\hat{0}) = 0$$

is given by $f(z) = \cos mz$.

PROPOSITION 8.

$$(1) \qquad \cos az = \frac{e^{ia,z} + e^{-ia,z}}{2}$$

$$(2) \qquad \sin az = \frac{e^{ia,z} - e^{-ia,z}}{2i}$$

$$(3) \qquad \frac{d}{dz}\cos az = -a \sin az$$

$$(4) \qquad \frac{d}{dz}\sin az = a \cos az$$

$$(5) \qquad e^{ia,z} = \cos az + i \sin az.$$

PROOF. These facts follow directly from Proposition 7.

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