A class of homogeneous Riemannian manifolds

By

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1. Introduction

R. L. Bishop and B. O'Neill [1] constructed a wide class of Riemannian manifolds of negative curvature by warped product using convex functions. For two Riemannian manifolds B and F, a warped product is denoted by $B\times_f F$ where f is a positive C^∞ function on B. The purpose of this paper is to prove

THEOREM. *Let (F, g) be a Riemannian manifold of constant curvature $K\leqq 0$. Let E^n be an n-dimensional Euclidean space and let f be a positive C^∞ function on E^n. If either $E^n\times_f F$ is homogeneous (Riemannian) or the Ricci tensor of $E^n\times_f F$ is parallel, then $E^n\times_f F$ is locally symmetric.*

The proof of the last theorem is motivated by [2], in which S. Tanno deals with some related problems.

2. The curvature tensor of $\mathbf{E}^n\times_f\mathbf{F}$

Let (F, g) be a Riemannian manifold and let E^n be a Euclidean n-space. We consider the product manifold $E^n\times F$. For vector fields A, B, C, etc. on E^n, we denote vector fields $(A, 0)$, $(B, 0)$, $(C, 0)$, etc. on $E^n\times F$ by also A, B, C, etc. Likewise, for vector fields X, Y, etc. on F, we denote vector fields $(0, X)$, $(0, Y)$, etc. on $E^n\times F$ by X, Y, etc.

We denote the inner product of A and B on E^n by $\langle A, B\rangle$. Let f be a positive C^∞-function on E^n. Then the (Riemannian) inner product $\langle\,,\rangle$ for $A+X$ and $B+Y$ on the warped product $E^n\times_f F$ at (a, x) is given by (cf. [1].)

$$\langle A+X, B+Y\rangle_{(a,x)}=\langle A, B\rangle_{(a)}+f^2(a)g_x(X, Y).$$

We extend the function f on E^n to that on $E^n\times_f F$ by $f(a,x)=f(a)$. The Riemannian connections defined by $\langle\,,\rangle$ on E^n and $E^n\times_f E$ are denoted by ∇^o and ∇, respectively. The Riemannian connection defined by g on F is denoted by D. Then we have the identities (cf. Lemma 7.3, [1].)

$$\nabla_A B = \nabla^o{}_A B, \tag{2. 1}$$

$$\nabla_A X = \nabla_X A = (Af/f)X,$$

$$\nabla_X Y = D_X Y - (\langle X, Y\rangle/f)\ grad\,f. \tag{2. 2}$$

By (2. 1) we identify ∇^o with ∇ in the sequel. In (2. 2) $grad\,f$ on E^n is identified with $grad\,f$ on $E^n\times_f F$ and we have

$$\langle grad\,f,\ A\rangle = df\,(A) = Af.$$

The Riemannian curvature tensors defined by ∇ and D are denoted by R and S respectively. We use both notations $R(X, Y)$ and R_{XY}, etc. :

$$R(X,Y) = R_{XY} = \nabla_{[X,\ Y]} - [\nabla_X, \nabla_Y], \text{ etc.}$$

Then, noticing that E^n is flat, we have (cf. Lemma 4.4, [1])

$$R_{AB}C = 0,$$

$$R_{AX}B = +(1/f)\langle\nabla_A grad\,f,\ B\rangle X,$$

$$R_{AB}X = R_{XY}A = 0,$$

$$R_{AX}Y = (1/f)\langle X,\ Y\rangle\nabla_A grad\,f,$$

$$R_{XY}Z = S_{XY}Z - (\langle grad\,f,\ grad\,f\rangle/f^2)(\langle X, Z\rangle Y - \langle Y, Z\rangle X). \tag{2. 3}$$

From now on we assume that (F, g) is of constant curvature $K\leqq 0$. Then we have

$$S_{XY}Z = K(g(X, Z)Y - g(Y, Z)X) = (K/f^2)(\langle X, Z\rangle Y - \langle Y, Z\rangle X).$$

In this case, (2. 3) is written as

$$R_{XY}Z = P(\langle X, Z\rangle Y - \langle Y, Z\rangle X)$$

where we have put

$$P = (K - \langle grad\,f,\ grad\,f\rangle)/f^2 \leqq 0. \tag{2. 4}$$

Then we have the following

LEMMA 2. 1. *(cf. Lemma 4. 1, [2]) On* $E^n\times_f F$, $\nabla R=0$ *if and only if*

$$fP\ grad\,f + \nabla_{grad\ f}\ grad\,f = 0, \tag{2. 5}$$

$$f\,\nabla_A\,\nabla_B\ grad\,f - f\nabla_T grad\,f - Af\,\nabla_B\ grad\,f = 0,\quad T = \nabla_A B \tag{2. 6}$$

and

$$Bf\,\nabla_A\ grad\,f - \langle\nabla_A\ grad\,f,\ B\rangle\ grad\,f = 0. \tag{2. 7}$$

Let $A_\alpha(\alpha=1, 2, \cdots, n)$ be unit vector fields on some open set on $E^n\times_f F$ such that they are mutually orthogonal and are tangent to E^n at each point of the open set. We denote by R_1 the Ricci curvature tensor. Then we have (cf. §5, [2])

$$(2.8)\qquad \begin{cases} R_1(Y, Z)=[(r-1)P-(1/f)\sum_\alpha \langle\nabla_{A_\alpha} grad\, f, A_\alpha\rangle]\langle Y, Z\rangle \\ R_1(B, Y)=0 \\ R_1(B, C)=-(r/f)\langle\nabla_B\, grad\, f, C\rangle, \quad r=dim.\ F. \end{cases}$$

3. Lemmas

LEMMA 3. 1. *Let R_1 be the Ricci tensor field of a Riemennian manifold (M, g). Let R^1 be a field of symmetric endomorphism which corresponds to R_1, that is, $g(R^1 X, Y)=R_1(X, Y)$ for all vector fields X and Y on M. If either*

a) M is homogeneous (Riemannian)

or

b) the Ricci tensor of M is parallel,

then the characteristic roots of R^1 are constant in value and multiplicity on M.

PROOF. a) Since $R_1(\varphi_* X, \varphi_* Y)=R_1(X, Y)$ for every isometry φ of M, it follows that ${\varphi_*}^{-1} R^1 \varphi_*=R^1$ on M. Since M is homogeneous, this proves the first of the lemma.

b) In this case R^1 is also parallel and the result is immediate. q. e. d.

Returning to an argument of $E^n\times_f F$, we have

LEMMA 3. 2. *(cf. Lemma 6. 1, [2]) On $E^n\times_f F$, (2. 5) is equivalent to $P=constant$.*

PROOF. By (2. 4) and (2. 5) we have

$$(1/f)(K-\langle grad\, f, grad\, f\rangle)\, grad\, f+\nabla_{grad\, f}\, grad\, f=0.$$

Since this equation is an equation on E^n, we introduce the natural coordinate system $(x^\alpha;\ \alpha=1, \cdots, n)$ on E^n. Then the last equation is nothing but

$$(K-\sum_\alpha \frac{\partial f}{\partial x^\alpha}\frac{\partial f}{\partial x^\alpha})\frac{\partial f}{\partial x^\beta}+f\sum_\alpha \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}\frac{\partial f}{\partial x^\alpha}=0.$$

The last equation multiplied by $2f$ is

$$(K-\sum_\alpha(\frac{\partial f}{\partial x^\alpha})^2)\frac{\partial f^2}{\partial x^\beta}-f^2\frac{\partial}{\partial x^\beta}(K-\sum_\alpha(\frac{\partial f}{\partial x^\alpha})^2)=0,$$

which implies that each partial derivative of

$$(3.1)\qquad P=(K-\sum_\alpha(\frac{\partial f}{\partial x^\alpha})^2)/f^2$$

vanishes. Thus, P is constant. The converse is clear. q. e. d.

4. Proof of theorem

In (2. 8), we may put $A_\alpha=\frac{\partial}{\partial x^\alpha}$, where $x^\alpha(\alpha=1, \cdots, n)$ are natural coordinates of E^n. Then the characteristic roots of R^1 at a point $(a, x)\in E^n\times_f F$ consist of

$$(r-1)P(a)-(1/f(a))\sum_\alpha\frac{\partial^2 f}{\partial x^\alpha\partial x^\alpha}(a) \qquad (n\text{-multiplicity})$$

and the roots $\lambda_1(a), \lambda_2(a), \cdots, \cdots\lambda_r(a)$ of

$$det\ (-(r/f(a))\frac{\partial^2 f}{\partial x^\beta\partial x^\alpha}(a)-\lambda\delta_{\beta\alpha})=0.$$

Since $E^n\times_f F$ is homogeneous, we have

$$(r-1)P-(1/f)\sum_\alpha\frac{\partial^2 f}{\partial x^\alpha\partial x^\alpha}=constant$$

and

$$\lambda_1+\cdots+\lambda_n=-(r/f)\sum_\alpha\frac{\partial^2 f}{\partial x^\alpha\partial x^\alpha}=constant$$

by lemma 3. 1 and by the continuity of the characteristic roots of R^1. Therefore P is constant and (2. 5) is satisfied by lemma 3. 2.

Now, we solve (3. 1) with P=constant and show that f satisfies (2. 6) and (2. 7). Then $E^n\times_f F$ is locally symmetric. (3. 1) is

$$K-\sum_\alpha(\frac{\partial f}{\partial x^\alpha})^2-Pf^2=0.$$

S. Tanno [2] solved the last partial differential equation by Lagrange-Charpit method to get a solution

$$f=(\frac{1}{2\sqrt{-P}})((K/b)\ exp(c_\beta x^\beta)-b\ exp\ (-c_\beta x^\beta))$$

where b and $c_1, \cdots, c_n$ are some constant. Consequently, we see that f satisfies (2. 6) and (2. 7) which are written as

$$f\frac{\partial^3 f}{\partial x^\alpha\partial x^\beta\partial x^\gamma}-\frac{\partial f}{\partial x^\alpha}\frac{\partial^2 f}{\partial x^\beta\partial x^\gamma}=0$$

$$\frac{\partial f}{\partial x^\beta}\frac{\partial^2 f}{\partial x^\alpha\partial x^\gamma}-\frac{\partial^2 f}{\partial x^\alpha\partial x^\beta}\frac{\partial f}{\partial x^\gamma}=0.$$

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References

1. R. L. BISHOP and B. ONEILL: *Manifolds of negative curvature,* Trans. Amer. Math. Soc., 145 (1969), 1–49
2. S. TANNO: *A class of Riemannian manifolds satisfying* $R\ (X,\ Y)\cdot R=0$, to appear.