Certain anti-holomorphic submanifolds of almost Hermitian manifolds*

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1. Introduction

Let $(\widetilde{M}, J, <, >)$ (or briefly $\widetilde{M}$) be an almost Hermitian manifold with the almost Hermitian structure $(J, <, >)$ and M be a Riemannian submanifold of $\widetilde{M}$. If $JT_x(M) = T_x(M)$ at each point x of M, $T_x(M)$ being the tangent space over M in $\widetilde{M}$, then M is called a *holomorphic* submanifold of $\widetilde{M}$. If $JT_x(M) \subset T_x^{\perp}(M)$ at each point x of M, $T_x^{\perp}(M)$ being the normal space over M in $\widetilde{M}$, then M is called a *totally real* submanifold of $\widetilde{M}$. If $JT_x^{\perp}(M) \subset T_x(M)$ for all point x of M, then M is called an *anti-holomorphic* (also known as a *generic*) submanifold of $\widetilde{M}$. If, in particular, $JT_x^{\perp}(M) = T_x(M)$, then an anti-holomorphic submanifold M is a totally real submanifold such that $\dim M = 1/2 \dim \widetilde{M}$. In this case, M is called an *anti-invariant* submanifold of $\widetilde{M}$. M is called a *CR-submanifold* of $\widetilde{M}$ if there exists a C^{∞}-holomorphic distribution $\mathfrak{D}$ (i.e., $J\mathfrak{D} = \mathfrak{D}$) on M such that its orthogonal complement $\mathfrak{D}^{\perp}$ is totally real (i.e., $J\mathfrak{D}^{\perp} \subset T_x^{\perp}(M)$). Especially, if $\dim \mathfrak{D}_x^{\perp} = 0$ (resp. $\dim \mathfrak{D}_x = 0$) for any $x \in M$, a CR-submanifold M is a holomorphic (resp. totally real) submanifold of $\widetilde{M}$. A *proper* CR-submanifold (resp. anti-holomorphic submanifold) of an almost Hermitian manifold is a CR-submanifold (resp. anti-holomorphic submanifold) with non-trivial holomorphic distribution and totally real distribution. If $\dim \mathfrak{D}^{\perp} = \mathrm{codim}\, M$ $(= \dim \widetilde{M} - \dim M)$, a CR-submanifold is an anti-holomorphic submanifold of $\widetilde{M}$. A CR-submanifold (or anti-holomorphic submanifold) of an almost Hermitian manifold is called a *CR-product* if it is locally the Riemannian product of a holomorphic submanifold and a totally real submanifold. We remark that every hypersurface of an almost Hermitian manifold is an anti-holomorphic submanifold. In this paper, we study the integrability conditions on anti-holomorphic submanifolds of nearly Kaehlerian manifolds (see [5]) and give some results with respect to CR-products of nearly Kaehlerian manifolds (see [4]). In particular, we study anti-holomorphic submanifolds in a 6-dimensional sphere S^6 and obtain that if a proper anti-holomorphic submanifold is mixed-totally geodesic in S^6 and the leaf of the totally real distribution is totally geodesic in S^6, then the holomorphc distribution is not integrable (THEOREM 4.2).

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2. Preliminaries

Let f be an isometric immersion of a Riemannian m-manifold M^m into a Riemannian n-manifold $\widetilde{M}^n$. For all local formulae we may consider as an imbedding and thus identify $x \in M$ with $f(x) \in \widetilde{M}$. The tangent space $T_x(M)$ is identified with a subspace of the tangent space $T_x(\widetilde{M})$. The normal space $T_x^{\perp}(M)$ is the subspace of $T_x(\widetilde{M})$ consisting of all $X \in T_x(M)$ which are orthogonal to $T_x(M)$ with respect to the Riemannian metric $<, >$. Let $\widetilde{\nabla}$ (resp. ∇) be the Riemannian connection on $\widetilde{M}$ (resp. M) and $\widetilde{R}$ be the Riemannian curvature for $\widetilde{\nabla}$. Moreover, we denote by σ the second fundamental form of M in $\widetilde{M}$. Then the Gauss formula and the Weingarten formula are given by

$$\sigma(X,Y)=\widetilde{\nabla}_X Y-\nabla_X Y, \qquad \text{for } X,\ Y \in T_x(M), \tag{2.1}$$

and

$$\widetilde{\nabla}_X \xi=-A_\xi X+\nabla_X^{\perp} \xi, \qquad \text{for } \xi \in T_X^{\perp}(M), \tag{2.2}$$

respectively, where $-A_\xi X$ (resp. $\nabla_X^{\perp} \xi$) denotes the tangential (resp. normal) component of $\widetilde{\nabla}_X \xi$. The tangential component $A_\xi X$ is related to the second fundamental form σ as follows:

$$\langle \sigma(X, Y), \xi \rangle=\langle A_\xi X, Y \rangle, \qquad \text{for } X,\ Y \in T_x(M).$$

The Codazzi equation is given by

$$\{\widetilde{R}(X, Y)Z\}^{\perp}=(\nabla'_X \sigma)(Y, Z)-(\nabla'_Y \sigma)(X, Z), \tag{2.3}$$

where $(\nabla'_X\sigma)(Y, Z)=\nabla^{\perp}_X(\sigma(Y, Z))-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z)$,
$\{\widetilde{R}(X, Y)Z\}^{\perp}$ is the normal component of $\widetilde{R}(X, Y)Z$, for $X, Y, Z \in T_x(M)$.

We now recall some fundamental notions of an almost Hermitian manifold. Let $\widetilde{M}$ be a 2n-dimensional manifold endowed with an almost Hermitian structure $(J, <, >)$. Let N_J be the Nijenhuis' tensor of J. Then by the definition, N_J is given by

$$N_J(U, V)=[JU, JV]-[U, V]-J[JU, V]-J[U, JV],$$

for vector fields U, V on $\widetilde{M}$. It is well known that the almost complex structure J is a complex structure on $\widetilde{M}$ if and only if N_J vanishes on $\widetilde{M}$. An almost Hermitian manifold $\widetilde{M}=(\widetilde{M}, J, <, >)$ is called a *nearly Kaehlerian* manifold (also known as *K-space* or *almost Tachibana* space) provided that its almost Hermitian structure $(J, <, >)$ satisfies the condition $(\widetilde{\nabla}_X J)X=0$ for all tangent vectors X on $\widetilde{M}$. Easily we get

LEMMA 2.1. *Let $(\widetilde{M}, J, <, >)$ be a nearly Kaehlerian manifold, then the Nijenhuis' tensor N_J takes the following form:*

$$N_J(U, V)=-4J(\widetilde{\nabla}_U J)V, \qquad \textit{for vector fields } U,\ V \textit{ on } \widetilde{M}.$$

Let τ and τ' be two J-invariant planes in $T_x(\widetilde{M})$. Then the holomorphic *bisectional curvature* $\widetilde{H}_B(X, Y)$ is given by

(2. 4) $\quad \widetilde{H}_B(X, Y)=\widetilde{R}(X, JX, Y, JY),$

where $\langle \widetilde{R}(X, Y)Z, W\rangle=\widetilde{R}(X, Y, W, Z)$, and X and Y are unit vectors in τ and τ' respectively.

3. Integrability conditions on an anti-holomorphic submaifold

Lat $\widetilde{M}$ be a 2n-dimensional nearly Kaehlerian manifold endowed with an almost Hermitian structure $(J, <, >)$ and M be an m-dimensional Riemannian manifold immersed in $\widetilde{M}$. For any vector field X tangent to M, we put

(3. 1) $\quad JX=FX+\omega X,$

where FX and ωX are the tangential and normal components of JX, respectively. If M is a holomorphic (resp. totally real) submanifold of $\widetilde{M}$, then ω (resp. F) in (3. 1) vanishes identically. Let M be an anti-holomorphic submanifold of $\widetilde{M}$. The tangent space $T_x(M)$ of M is decomposed in the following way:

$$T_x(M)=H_x(M)\oplus JT_x^{\perp}(M) \text{ at each point } x \text{ of } M,$$

where $H_x(M)$ denotes the orthogonal complement of $JT_x^{\perp}(M)$ in $T_x(M)$. Thus we see that

$$JH_x(M)=H_x(M)=T_x(M)\cap JT_x(M).$$

That is, $H_x(M)$ is a holomorphic subspace of $T_x(M)$. From now on, we assume that $N_J(X, Y)\in T_x(M)$ for $X, Y\in T_x(M)$.
From (2. 1), (2. 2) and Lemma 2. 1 we get

(3. 2) $\quad J\sigma(X, Y)=(\nabla_X F)Y+(\overline{\nabla}_X\omega)Y-A_{\omega Y}X+\sigma(X, FY)-(1/4)F(N_J(X, Y))$
$\quad -(1/4)\,\omega(N_J(X, Y)),$

where we have put $(\overline{\nabla}_X\omega)Y=\nabla_X^{\perp}(\omega Y)-\omega\nabla_X Y$.
Since σ is symmetric and Nijenhuis' tensor is skew-symmetric, from (3. 2) we get

(3. 3) $\quad (\nabla_X F)Y-A_{\omega Y}X+\sigma(X, FY)+(\overline{\nabla}_X\omega)Y-(1/4)F(N_J(X, Y))$
$\quad -(1/4)\,\omega(N_J(X, Y))$
$\quad =(\nabla_Y F)X-A_{\omega X}Y+\sigma(Y, FX)+(\overline{\nabla}_Y\omega)X+(1/4)F(N_J(X, Y))$
$\quad +(1/4)\,\omega(N_J(X, Y))$

Comparing the tangential and normal parts of the boht sides of (3. 3), we have respectively

(3. 4) $\quad (\nabla_X F)Y-(\nabla_Y F)X=A_{\omega Y}X-A_{\omega X}Y+(1/2)F(N_J(X, Y)),$

and

(3.5) $(\overline{\nabla}_X \omega) Y-(\overline{\nabla}_Y\omega)X=\sigma(FX, Y)-\sigma(X, FY)+(1/2)\,\omega(N_J(X, Y))$.

Applying J to the both sides of (3.1), we get

$$-X=JFX+J\omega X=F^2X+\omega FX+J\omega X.$$

Since $JT_x^\perp(M)\subset T_x(M)$, we see that $J\omega X\in T_x(M)$. Thus we have

(3.6) $\omega FX=0$.

(3.7) $F^2X=-X-J\omega X$.

Similarly, from (3.2) we have

(3.8) $J\sigma(X, Y)=(\nabla_XF) Y-A_{\omega Y}X-(1/4)F(N_J(X, Y))$,

and

(3.9) $\sigma(X, FY)=-(\overline{\nabla}_X\omega)Y+(1/4)\omega(N_J(X, Y))$.

LEMMA 3.1. *Let M be an anti-holomorphic submanifold of a nearly Kaehlerian manifold $\widetilde{M}$. If M satisfies*

(3.10) $N_J(X, Y)\in JT_x^\perp(M)$, *for* $X\in T_x(M)$ *and* $Y\in JT_x^\perp(M)$,

then we have

(3.11) $A_{\omega Y}Z=A_{\omega Z}Y$, *for* $Z\in JT_x^\perp(M)$.

The proof is similar to [5], LEMMA 2.1.
Applying F to the both sides of (3.7), we have

$$F^3X=-FX, \qquad \text{for } X\in T_x(M).$$

Thus we have $F^3+F=0$.

On the other hand, the rank of F is equal to dim M−codim $M=m-(2n-m)=2(m-n)$ everywhere on M. Consequently, F defines an f-structure of rank $2(m-n)$ ([5]). We now put

$$L=-F^2 \text{ and } T=F^2+I.$$

We can easily see that L and T are complementary projective operators. Thus there exist complementary distributions $\mathfrak{L}$ and $\mathfrak{T}$ corresponding to the projection operators L and T respectively. Since the rank of P is $2(m-n)$, $\mathfrak{L}$ is $2(m-n)$-dimensional and $\mathfrak{T}$ is $(2n-m)$-dimensional. The distributions $\mathfrak{L}$ and $\mathfrak{T}$ are defined also by

$$\mathfrak{L}_x=\{X\in T_x(M):\omega X=0\},$$

and

$$\mathfrak{T}_x=\{X\in T_x(M):FX=0\}.$$

Hence the distribution $\mathfrak{L}$ (resp. $\mathfrak{T}$) is holomorphic (resp. totally real).

In [4], we showed the following

THEOREM A. *Let M be a CR-submanifold of a nearly Kaehlerian manifold $(\widetilde{M}, J, <, >)$. Then a necessary and sufficient condition for the holomorphic distribution $\mathfrak{D}$ to be integrable is that the following conditions are satisfied*:

$$\sigma(X, JY)=\sigma(JX, Y)$$

and

$$N_J(X, Y)\in\mathfrak{D}, \qquad \textit{for all } X, Y\in\mathfrak{D}.$$

THEOREM B. *Let M be a CR-submanifold of an early Kaehlerian manifold $(\widetilde{M}, J, <, >)$. Then a necessary and sufficient condition for the totally real distribution $\mathfrak{D}^{\perp}$ to be integrable is that the following condition is satisfied:*

$$\langle N_J(X, Z), W\rangle=\langle N_J(X, Z), JW\rangle=0, \qquad \textit{for all } X\in\mathfrak{D},\ Z, W\in\mathfrak{D}^{\perp}.$$

Hence, as the integrability conditions of $\mathfrak{L}$ and $\mathfrak{T}$, we obtain

THEOREM 3.1. *Let M be an anti-holomorphic submanifold of a nearly Kaehlerian manifold $\widetilde{M}$. If M satisfies*

$$\sigma(X, FY)=\sigma(FY, Y) \tag{3.12}$$

and

$$N_J(X, Y)\in\mathfrak{L}, \qquad \textit{for all } X, Y\in\mathfrak{L}, \tag{3.13}$$

then the holomorphic distribution $\mathfrak{L}$ is integrable.

PROOF. From (3.5), we get

$$\omega[X, Y]=\omega\nabla_X Y-\omega\nabla_Y X=-(\nabla_X\omega)Y+(\nabla_Y\omega)X$$
$$=-\sigma(FX, Y)+\sigma(X, FY)-1/2\,\omega(N_J(X, Y)),$$

for all $X, Y\in\mathfrak{L}$. Thus the assertion is showed by (3.12) and (3.13).

Q.E.D.

Therefore the maximal integral submanifold M_1 of $\mathfrak{L}$ through a point of $\widetilde{M}$ is a $2(m-n)$-dimensional neary Kaehlerian submanifold of $\widetilde{M}$.

With respect to the totally real distributions $\mathfrak{T}$, we have

THEOREM 3.2. *Let M be an anti-holomorphic submanifold of a nearly Kaehlerian manifold $\widetilde{M}$. If M satisfies*

$$N_J(X, Y)\in JT_x^{\perp}(M), \qquad \textit{for all } X, Y\in\mathfrak{T}, \tag{3.14}$$

then the totally real distribution $\mathfrak{T}$ is integrable.

PROOF. *From* (3.4) *we get*

$$F[Y, Y]=F\nabla_X Y-F\nabla_Y X=-(\nabla_X F)Y+(\nabla_Y F)X$$

$$=A\omega_X Y - A\omega_Y X - (1/2)F(N_J(X, Y)),$$

for all $X, Y \in \mathfrak{T}$. From LEMMA 3.1 and (3.14), we have $F[X, Y]=0$, for all $X, Y \in \mathfrak{T}$.
Q.E.D.

Hence the maximal integral submanifold M_2 of $\mathfrak{T}$ through a point of $\widetilde{M}$ is a $(2n-m)$-dimensional totally real submanifold of $\widetilde{M}$.

REMARK Let M be an anti-holomorphic submanifold of a nearly Kaehlerian manifold $\widetilde{M}$. If $JT_x^{\perp}(M) = T_x(M)$, EJIRI obtained the following identity ((2.10) in [3]):

$$[(\widetilde{\nabla}_X J)Y]^T = 0, \qquad \text{for all } X, Y \in T_x(M),$$

where $[\quad]^T$ is a tangential component of $(\widetilde{\nabla}_X J)Y$. Thus from LEMMA 2.1 we have

$$N_J(X, Y) = -4J(\widetilde{\nabla}_X J)Y \in JT_x^{\perp} J(M), \qquad \text{for all } X, Y \in T_x(M).$$

4. CR-products

Let M be a CR-submanifold ofa nearly Kaehlerian manifold $\widetilde{M}$. We denote by ν the complementary orthogonal subbundle of $J\mathfrak{D}^{\perp}$ *in* $T^{\perp}M$. Hence we have

$$T^{\perp}M = J\mathfrak{D}^{\perp} \oplus \nu, \ J\mathfrak{D}^{\perp}\ \nu.$$

In [4], we showed that if M satisfies (*) $\sigma(X, Y) \in \nu$, (**)$\sigma(X, Z) \in \nu$ and (***) $N_J(X, Y) \in \mathfrak{D} \oplus T^{\perp} M$, for all $X, Y \in \mathfrak{D}$, $Z \in \mathfrak{D}^{\perp}$, then M is a CR-product in M. Hence we immediately see that if, in particular, M is atotally geodesic CR-submanifold of a Kaehlerian manifold $\widetilde{M}$, then M is a CR-product in $\widetilde{M}$. Let M be an anti-holomorphic submanifold of a nearly Kaehlerian manifold $\widetilde{M}$. Thus we remark the following

THEOREM 4.1. *Let M be a totally geodesic anti-holomorphic submanifold of a Kaehlerian manifold* $\widetilde{M}$. *If M satisfies*

$$(***) \qquad N_J(X, Y) \in \mathfrak{L} \oplus T^{\perp}M, \qquad \textit{for all } X, Y \in \mathfrak{L},$$

then M is a CR-product in $\widetilde{M}$.

COROLLARY 4.1. *Let M be a totally geodesic real hypersurface of a nearly Kaehlerian manifold* $\widetilde{M}$. *If M satisfies the condition* (***), *then M is a CR-product in* $\widetilde{M}$.

It is well known that a 6–dimensional unit sphere S^6 admits an almost complex structure. We see that a unit sphere S^5 is a totally geodesic real hypersurface in S^6 but S^5 is not a CR-product in S^6. We thus remark that we can not omit the condition (***).

We now consider an anti-holomorphic submanifold in a 6-dimensional sphere S^6. Let M_2 be the maximal integral submanifold of $\mathfrak{T}$ through a point of $\widetilde{M}$. Let σ''(resp. σ_2) be the second fundamental form of M_2 in $\widetilde{M}$ (resp. M). Then we have

$$(4.1) \qquad \sigma''(Z, W) = \sigma_2(Z, W) + \sigma(Z, W), \qquad \text{for } Z, W \in \mathfrak{T}.$$

A CR-submanifold is said to be *mixed-totally geodesic* if $\sigma(X, Z)=0$, for all $X \in \mathfrak{D}$, $Z \in \mathfrak{D}^{\perp}$. A CR-submanifold M of an almost Hermitian manifold $\widetilde{M}$ is said to be *mixed-foliate* if it is mixed-totally geodesic and if its holomorphic distribution is integrable.

From PROPOSITION 6.2 in [4], we have

LEMMA 4.1. *Let M be an anti-holomorphic submanifold of a nearly Kaehlerian manifold $(\widetilde{M}, J, <,>)$. Then a necessary and sufficient condition for the totally real submanifold M_2 to be totally geodesic in M is that M is mixed-totally geodesic in $\widetilde{M}$.*

From LEMMA 4.1 and (4.1) we get

LEMMA 4.2 *Let M be an anti-holomorphic submanifold of a nearly Kaehlerian manifold $(\widetilde{M}, J, <,>)$. If M is mixed-totally geodesic in $\widetilde{M}$ and M_2 is totally geodesic in $\widetilde{M}$, then we have*

$$\sigma(Z, W)=0, \qquad \text{for all } Z, W \in \mathfrak{T}.$$

In this paper we shall show the following THEOREM.

THEOREM 4.2. *Let M be a proper anti-holomorphic submanifold in S^6. If M is mixed-totally geodesic in S^6 and M_2 is totally geodesic in S^6, then the holomorphic distribution is not integrable.*

COROLLARY 4 2. *Under the assumption of* THEOREM 4.2, *S^6 has no mixed-foliate proper anti-holomorphic submanifolds.*

COROLLARY 4.3. *Under the assumption of* THEOREM 4.2, *S^6 has no proper CR-products.*

PROOF of THEOREM 4.2. The Codazzi equation (2.3) implies

$$(4.2) \qquad \{\widetilde{R}(X, JX)Z\}^{\perp}=\nabla_X^{\perp}(\sigma(JX, Z))-\sigma(\nabla_X(JX), Z)-\sigma(JX, \nabla_X Z)$$
$$-\nabla_{JX}^{\perp}(\sigma(X, Z))+\sigma(\nabla_{JX}X, Z)+\sigma(X, \nabla_{JX}Z),$$

for all $X \in \mathfrak{L}$, $Z \in \mathfrak{T}$. From (4.2) we get

$$(4.3) \qquad \langle\widetilde{R}(X, JX)Z, JZ\rangle=\langle\nabla_X^{\perp}(\sigma(JX, Z))-\nabla_{JX}^{\perp}(\sigma(X, Z)), JZ\rangle$$
$$-\langle\sigma(\nabla_X(JX), Z)-\sigma(\nabla_{JX}X, Z), JZ\rangle$$
$$-\langle\sigma(JX, \nabla_X Z), JZ\rangle+\langle\sigma(X, \nabla_{JX}Z), JZ\rangle,$$

for all $X \in \mathfrak{L}$, $Z \in \mathfrak{T}$.

By the assumption of THEOREM, LEMMA 4.2 and (4.3) we have

$$(4.4) \qquad \langle\widetilde{R}(X, JX)Z, JZ\rangle=-\langle\sigma(JX, \nabla_X Z), JZ\rangle+\langle\sigma(X, \nabla_{JX}Z), JZ\rangle$$
$$=-\langle A_{JZ}(JX), \nabla_X Z\rangle+\langle A_{JZ}X, \nabla_{JX}Z\rangle.$$

By LEMMA 2.1 we get

$$(4.5) \qquad -\langle A_{JZ}(JX), \nabla_X Z\rangle=-\langle A_{JZ}(JX), \widetilde{\nabla}_X Z\rangle$$
$$=-\langle JA_{JZ}(JX), \widetilde{J\nabla}_X Z\rangle$$
$$=-\langle JA_{JZ}(JX), -A_{JZ}X+\nabla_X^{\perp}(JZ)$$
$$-(1/4)JN_J(X, Z)\rangle,$$

for all $X \in \mathfrak{L}$, $Z \in \mathfrak{T}$. From the assumption we get

$$\langle A_{JZ}X, W\rangle=\langle \sigma(X, W), JZ\rangle=0, \qquad \text{for all } X\in \mathfrak{L},\ Z, W\in \mathfrak{T}.$$

Thus we get

$$(4.6) \qquad A_{JZ}X\in \mathfrak{L} \qquad \text{for all } X\in \mathfrak{L},\ Z\in \mathfrak{T}.$$

From (4.5) and (4.6) we have

$$(4.7) \qquad -\langle A_{JZ}(JX), \nabla_X Z\rangle=-\langle JA_{JZ}(JX), -A_{JZ}X-(1/4)JN_J(X, Z)\rangle$$
$$=(1/4)\langle A_{JZ}(JX), N_J(X, Z)\rangle-\langle A_{JZ}(JX), JA_{JZ}X)\rangle.$$

Similarly we get

$$(4.8) \qquad \langle A_{JZ}X, \nabla_{JX}Z\rangle=-(1/4)\langle A_{JZ}X, N_J(JX, Z)\rangle+\langle A_{JZ}X, JA_{JZ}(JX)\rangle.$$

From (4.4), (4.7) and (4.8) we get

$$(4.9) \qquad \langle\widetilde{R}(X, JX)Z, JZ\rangle$$
$$=-2\langle A_{JZ}(JX), JA_{JZ}X\rangle+(1/4)\langle A_{JZ}(JX), N_J(X, Z)\rangle$$
$$-(1/4)\langle A_{JZ}X, N_J(JX,Z)\rangle, \qquad \text{for all } X\in \mathfrak{L},\ Z\in \mathfrak{T}.$$

A Nijenhuis' tensor of $\widetilde{M}$ satisfies the following identity:

$$(4.10) \qquad \langle N_J(U, V), W\rangle=\langle N_J(V, W), U\rangle, \qquad \text{for all } U, V, W\in T_x(\widetilde{M}).$$

From (4.6), (4.9) and (4.10) we have

$$(4.11) \qquad \widetilde{H}_B(X, Z)=2\langle A_{JZ}(JX), JA_{JZ}X\rangle+(1/4)\langle N_J(X, A_{JZ}(JX)), Z\rangle$$
$$-(1/4)\langle N_J(JX, A_{JZ}X), Z\rangle.$$

For an orthonormal basis $\{Ei\}$ (i$=1,\dots, m$) at $T_x(M)$, we get

$$(4.12) \qquad 2\langle A_{JZ}(JX), JA_{JZ}X\rangle=2\sum_{i=1}^{m}\langle A_{JZ}(JX), E_i\rangle\langle JA_{JZ}X, E_i\rangle$$
$$=-2\sum_{i=1}^{m}\langle \sigma(JX, E_i), JZ\rangle\langle \sigma(X, JE_i), JZ\rangle.$$

If the holomorphic distribution is integrable, then THEOREM 3.2, (4.6), (4.11) and (4.12) we have

$$\widetilde{H}_B(X, Z) = -2\sum_{i=1}^{m}\langle \sigma(JX, E_i), JZ\rangle^2 \leqq 0.$$

This is a contradiction since S^6 has positive holomorphic bisectional curvature.

Q.E.D.

Finally we give some remarks with respect to an anti-holomorphic submanifold of a complex projective space CP^n.

LEMMA ([1]). *Let M be an anti-holomorphic submanifold of a Kaehlerian manifold $\widetilde{M}$. Then a necessary and sufficient condition for totally real submanifold M_2 to be totally* *geodesic in $\widetilde{M}$ is M is mixed-totally geodesic in $\widetilde{M}$.*

Thus from LEMMA we get as THEOREM 4. 2

THEOREM ([2]). *Let M be a proper anti-holomorphic submanifold in CP^n. Jf M_2 is totally geodesic in CP^n, then the holomorphic distribution is not integrable.*

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