A note on the cut loci and the first conjugate loci of riemannian manifolds

By

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1. Introduction

Let M be an n-dimensional complete riemannian manifold and $d(M)$ (resp. K_M) its diameter (resp. sectional curvature), C_p (resp. Q_p^1) the cut locus (resp. the first conjugate locus) in the tangent space M_p to M at $p \in M$. It is well known that $K_M \geq \delta > 0$ implies $\pi/\sqrt{\delta} \geq d(M)$ (Myers [1]). On the other hand, the following is a classical problem concerning the infimum of the diameter $d(M)$ of a riemannian manifold M.

PROBLEM 1. Does $\delta \geq K_M$ imply $d(M) \geq \pi/\sqrt{\delta}$ for a simply connected riemannian manifold?

In this problem, we can not remove the assumption of simple connectivity for M. In fact, an n-dimensional real projective space $\boldsymbol{RP}^n$ with the canonical metric of constant curvature δ has the diameter $\pi/2\sqrt{\delta}$. Furthermore, if M is non-compact, then the diameter of M is infinite by Hopf-Rinow's theorem. Therefore, in the sequel, we may assume that M is compact. Now it may be easily understood that if the answer to the following problem 2 is affirmative, we have the same answer to the above problem 1 by virtue of Morse-Schoenberg's theorem.

PROBLEM 2. (Weinstein [2]). Is there a point $p \in M$ such that $C_p \cap Q_p^1 \neq \emptyset$ for a compact simply connected riemannian manifold M?

In this paper, we give some partial answers to problem 2. In §2, we introduce the notion of flaps of a riemannian manifold and prove proposition 1 by making use of the properties of flaps. Here "flap" intuitively means the pasting part in making a cylinder from a rectangle. In §3 of this note, the following theorem is proved.

THEOREM. *Let M be a compact simply connected riemannian manifold with a point $p \in M$ satisfying the condition that if $\Omega(p, q)$ is non-degenerate, there is at most one geodesic of index* 1 *in $\Omega(p, q)$. Then C_p and Q_p^1 have an intersection.*

This statement includes a result of Warner [3] that if there are no geodesics of index 1 in a simply connected manifold M, the cut locus and first conjugate locus coincide.

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2. Definitions of flaps and results

Let M be a compact riemannian manifold and $\exp_p : M_p \longrightarrow M$ an exponential map.

DEFINITION 1. For $\varepsilon > 0$ and $p \in M$, the *ε-flap* $F_{\varepsilon p}$ of p in M_p denotes the subset

$$F_{\varepsilon p} := \{ (1 + t\varepsilon / \|v\|) v ;\ v \in C_p,\ 0 < t < 1 \}.$$

And moreover, the *ε-flap* $F_\varepsilon(p)$ of p in M denotes the subset

$$F_\varepsilon(p) := \exp_p F_{\varepsilon p}.$$

First, we have the following

LEMMA 1. *$F_{\varepsilon p}$ is homeomorphic to a cylinder* $S^{n-1} \times (0, 1)$ (cf. [4] Lemma 5. 4, p. 16).

From this lemma, we have

LEMMA 2. *If $C_p \cap Q_p^1 = \emptyset$, then there exists a positive number ε such that the map $\exp_{p|F_{\varepsilon p}}$ is non-singular.*

The converse to Lemma 2 is false in general. In fact, an n-dimensional sphere S^n with canonical metric of constant curvature has $C_p = Q_p^1 = S^{n-1}$.

DEFINITION 2. We say that M has k-fold flaps of p, if for all $\varepsilon > 0$, there exist k points $v_1, \dots, v_k \in F_{\varepsilon p}$ such that $v_i \neq v_j$, $\exp_p v_i = \exp_p v_j$ for all $i, j\ (i \neq j)$. Clearly, if $s > h$, then "s-fold" implies "h-fold." If M has 2-fold flaps of p, then we say that M has double flaps of p. If M has no 2-fold flaps of p, then we say that M has non-double flaps of p.

We shall prove Proposition 1 which shows that even if $\pi_1(M) \not\cong \boldsymbol{Z}_2$, the answer to Problem 1 is affirmative for M with non-double flaps.

PROPOSITION 1. *Let M be a compact riemannian manifold with $\pi_1(M) \not\cong \boldsymbol{Z}_2$ and have non-double flaps of p. Then there is a point $p \in M$ such that $C_p \cap Q_p^1 \neq \emptyset$.*

Proof. Assume that $C_p \cap Q_p^1 = \emptyset$. Then, by [5, Theorem B], we have distinct vectors $v_j \in C_p$ ($j = 1, 2, 3$) such that $\exp_p v_i = \exp_p v_j$ for all i, j. Let $q = \exp_p v_i$, and ε be an arbitrary positive number. Then there exist mutually disjoint neighbourhoods V_j of v_j in M_p ($j = 1, 2, 3$) such that for each j, $V_j \subset F_{\varepsilon p} \cup B_p$, $\exp_{p|V_j}$ is a diffeomorphism, and further, C_p divides V_j into two parts. Here we put $B_p = \{tv;\ v \in C_p,\ 0 \leq t \leq 1\}$. Now let $IV_i = V_i \cap (B_p - C_p)$, $EV_i = V_i \cap (M_p - B_p)$, and further, $V_i(p) = \exp_p V_i$, $IV_i(p) = \exp_p IV_i$, $EV_i(p) = \exp_p EV_i$. Then by the fundamental property of the cut locus (cf. [4] § 5. 4), we have

$$IV_i(p) \cap IV_j(p) = \emptyset \qquad \text{for all } i, j\ (i \neq j). \tag{3. 1}$$

Let γ_i be a geodesic determined by each v_i between p and q. Then there is a positive number δ such that $\gamma_1(s - \delta, s) \subset \bigcap_{i=1}^{3} V_i(p)$, where γ_1 is a normal geodesic and s is a distance between p and q. Now we put $\gamma_{1\delta} = \gamma_1(s - \delta, s)$ for simplicity. Then, $\gamma_{1\delta} \subset IV_1(p)$ and

$\gamma_{1\delta}\subset EV_j\,(p)\;(j=2,3)$ hold by (3. 1). $\tilde{\gamma}_j$ denotes the lifting $(\exp_{p|V_j})^{-1}\gamma_{1\delta}$ of $\gamma_{1\delta}$ to $M_p(j=2,3)$. Then $\tilde{\gamma}_j$ is a curve in EV_j with a boundary point v_j. Hence we have $\tilde{\gamma}_2\cap\tilde{\gamma}_3=\emptyset$ and $\exp_p\tilde{\gamma}_2(t)=\exp_p\tilde{\gamma}_3(t)=\gamma_1(t)$ for $s-\delta<t<s$. Since $EV_j\subset F_{\varepsilon p}$ $(j=1, 2\ 3)$, it follows that the map $\exp_{p|F_{\varepsilon p}}$ is not injective.

REMARK 1. In the following table, we shall give some examples of flaps.

M	$S^n(1)$	KP^n	$T^n(0)$	$L(3;1)$
flaps	non-double $(0<\varepsilon\leq\pi)$	non-double $(0<\varepsilon\leq\pi/2)$	double $(2^n-1$ fold)	double

,

where $S^n(1)$, $L(3;1)$ (resp. $T^n(0)$) in this table have the canonical metrics of constant curvature 1 (resp. 0) and projective spaces $\boldsymbol{K}P^n$ for $\boldsymbol{K}=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$ have the canonical structure (cf. [6] 3.30, 3.33).

REMARK 2. Concerning the existence of double flaps, the following assertion holds by the main theorem in [2] and Proposition 1.

Let M be a compact differentiable manifold not homeomorphic to S^2, and $\pi_1(M)\not\cong Z_2$. Then there is a riemannian metric g on M and a point $p\in M$ such that (M, g) has double flaps of p.

PROPOSITION 2. *Let M be a compact riemannian manifold with $\pi_1(M)\not\cong Z_2$. If C_p is smooth in M_p, then there is a point $p\in M$ such that $C_p\cap Q^1_p\neq\emptyset$.*

Proof. Assume that $C_p\cap Q^1_p=\emptyset$. There are v_j, V_j $(j=1, 2, 3)$ in the proof of Proposition 1. Hereafter we use the same symbols as Proposition 1. By the assumption concerning the cut locus C_p, we have the tangent space $C^{n-1}_{v_j}$ to C_p at v_j in M_p $(j=1, 2, 3)$. Putting $C(v_j)=d\exp_p(C^{n-1}_{v_j})$, we have $\dim C^{n-1}_{v_j}=\dim C(v_j)=n-1$. We put $f=(\exp_{p|V_1})^{-1}$, $C_f(v_j)=df(C(v_j))$. Then we have easily

(3. 2) $\qquad C_f(v_1)=C^{n-1}_{v_1},\ \dim C_f(v_j)=n-1 \qquad$ for all j.

Now let $I_1V_j(p)=IV_j(p)\cap V_1(p)$, $I_fV_j=f(I_1V_j(p))$ $(j=1, 2, 3)$. Then (3. 2) implies that there are i, j $(i\neq j)$ such that $I_fV_i\cap I_fV_j\neq\emptyset$. Since the map f is an imbedding, we have $I_1V_i(p)\cap I_1V_j(p)\neq\emptyset$ for some $i, j (i\neq j)$. Namely this contradicts (3. 1) in Proposition 1.

3. A result on index of geodesics

First we shall prove inequalities on type numbers. These are easily obtained by Morse Inequalities (cf. [7], Theorem 4.89, Corollary 2).

LEMMA 3. *Let M be a compact $k(\geq 1)$-connected riemannian manifold with positive Ricci curvature. Suppose that the path space $\Omega(p, q)$ is non-degenerate.*

Then we have $\quad n_s\geq\sum_{i=1}^{s+1}(-1)^{i-1}n_{s-i} \qquad (0\leq s\leq k)$,

where $n_{-1}=1$, and $n_i(i\geq 0)$ is the number of all geodesics of index i in $\Omega(p, q)$.

Proof. We write Ω for $\Omega(p, q)$ to simplify the notation. By [8, Theorem 19.6], there is a positive number a such that each geodesic of *energy* $> a$ has *index* $> k$. Since M is k-connected, we have

$$H_j(\Omega^a; \boldsymbol{Z}) \simeq H_j(\Omega; \boldsymbol{Z})$$
$$\simeq \pi_j(\Omega) \simeq \pi_{j+1}(M)=0 \qquad \text{for } j=1,\dots, k-1,$$

where $\Omega^a = \{c \in \Omega;\ E(c) \leq a\}$.

We put β_j = the j-dimensional Betti number of Ω

β_j^a = the j-dimensional Betti number of Ω^a

$n_j^a = \#\{$geodesics of index j in $\Omega^a\}$.

Then we have $n_j^a = n_j$ $(j=0, 1,\dots, k)$ and $\beta_0^a = \beta_0 = 1$, $\beta_j^a = \beta_j = 0 (j=1,\dots, k-1)$. Hence, by [7], we have

$$\sum_{i=0}^{s} (-1)^{s-i} n_i \geq (-1)^s \beta_0^a + \beta_s^a \geq (-1)^s \qquad \text{for all integers } s (0 \leq s \leq k).$$

Lemma 4. *Let M be a simply connected riemannian manifold and Ω a non-degenerate path space, and $n_1 < +\infty$. Then we have $n_0 < +\infty$, $n_1 \geq n_0 - 1$.*

Proof. It suffices to show $n_0 < +\infty$. It is well known that the homology group of Ω is isomorphic to the celluar homology group of a CW-complex Λ, and each j-dimensional chain group $C_j(\Lambda)$ of Λ is identified with a free module with basis $\{e_\alpha^j\}_{\alpha=1,\dots,n_j}$, where $e_\alpha^j \in \Lambda$ is a cell of dimension j corresponding to each geodesic γ_α^j of index j from p to q. Hence $C_j(\Lambda) \simeq \underset{n_j}{\oplus} \boldsymbol{Z}$ holds. Since the $\boldsymbol{Z}$-module $\boldsymbol{Z}$ is projective, there is a homomorphism $\iota: \boldsymbol{Z} \longrightarrow C_0(\Lambda)$ such that the following diagram commutes:

$$\begin{tikzcd} & \boldsymbol{Z} \arrow[dl, "\iota"'] \arrow[d, "1_z"] \\ C_0(\Lambda) \arrow[r, "\pi"'] & \boldsymbol{Z} \simeq C_0(\Lambda)/\mathrm{Im}\partial_1, \end{tikzcd}$$

where $\partial_1: C_1(\Lambda) \longrightarrow C_0(\Lambda)$ is a boundary operator.
Then we have $C_0(\Lambda) \simeq \mathrm{Ker}\,\pi \oplus \boldsymbol{Z} = \mathrm{Im}\partial_1 \oplus \boldsymbol{Z}$ by the characterization of the one-sided direct sum diagram (cf. [9] Proposition 4.2, p. 16). Since $C_0(\Lambda)$ is free, $\mathrm{Im}\partial_1$ is also free. Hence $n_1 < +\infty$ implies $\mathrm{Im}\partial_1 \simeq \underset{\textit{finite}}{\oplus} \boldsymbol{Z}$. Namely $n_0 < +\infty$ holds.

We shall prove the main result by Lemma 4.

Proof of theorem in §1. Assume that $C_p \cap Q_p^1 = \emptyset$. Lemma 2 implies that there is an $\varepsilon > 0$ such that the map $\exp_{p|F_\varepsilon p}$ is non-singular. Then we have v_j, $V_j (j=1, 2, 3)$ in Proposition 1. By Sard's theorem, there is a point $r \in \bigcap_{j=1}^{3} V_j(p)$ such that $\Omega(p, r)$ is non-degenerate. Let $u_j = (\exp_{p|V_j})^{-1}(r)$ $(j=1, 2, 3)$. Then each $\bigcup_{0 \leq t \leq 1} \exp_p tu_j$ is a geodesic of index 0 from p to r. Hence we have $n_1 \geq n_0 - 1 \geq 2$ by Lemma 4. This contradicts the condition $n_1 \leq 1$.

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