A note on compact connected transformation groups on spheres with codimension two principal orbit

By

Fuichi UCHIDA and Tsuyoshi WATABE

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Introduction

In his paper [2] Bredon has classified compact connected transformation groups on spheres with codimension two principal orbit and only one type of singular orbit. In this paper we shall consider compact connected differentiable transformation groups on spheres with codimension two principal orbit and only one isolated singular orbit.

We shall prove the following two results;

THEOREM A. *Let φ: $G\times S^n \longrightarrow S^n$ ($n\geqq 3$) be a differentiable action of a compact connected Lie group G on sphere S^n with codimension two principal orbit and only one isolated singular orbit. Then φ has the same orbit structure as one of the following actions, unless $n=11, 23$.*

I) *Consider the group $U(2)$. Let $\bar{\psi}_1$ be the homomorphism $U(2)\longrightarrow SO(3)$ with the center of $U(2)$ as kernel and ψ_1 the action of $U(2)$ on R^3 obtained from the standard action of $SO(3)$ on R^3 by $\bar{\psi}_1$. Let ψ_2 be the canonical action of $U(2)$ on $R^4=C^2$. Thus we obtain an action φ_1 of $U(2)$ on $S^6\subset R^3\times R^4$ defined by $\psi_1\times\psi_2$.*

II) *Consider the group $Sp(2)\times Sp(1)$. Let ψ_1 be the action of $Sp(2)\times Sp(1)$ on H^2 defined by*

$$\psi_1\left\{\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, e\right), \begin{pmatrix} x \\ y \end{pmatrix}\right\}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}^{-1}$$

and ψ_2 the action of $Sp(2)\times Sp(1)$ on R^5 defined by the natural homomorphism $Sp(2)\times Sp(1)\xrightarrow{proj.} Sp(2)\xrightarrow{cov.} SO(5)$. Thus we obtain an action φ_2: $Sp(2)\times Sp(1)\times S^{12}\longrightarrow S^{12}$. Note that the action obtained by restriction of $Sp(2)\times Sp(1)$ to $Sp(2)\times T^1$ or $Sp(2)$ has the same orbit structure as φ_2. We denote these actions by the same notation φ_2.

III) *Consider the group Spin (9). Let Δ: Spin (9)$\longrightarrow SO(16)$ be the spin representation and π: Spin (9)$\longrightarrow SO(9)$ the canonical double covering. Then we obtain an action φ_3: Spin (9) $\times S^{24}\longrightarrow S^{24}$.*

THEOREM B. *The action of a compact connected Lie group G on S^n with the same orbit structure as φ_1 or φ_2 is continuously equivalent to φ_1 or φ_2, respectively.*

In this paper, we shall consider only differentiable actions and use the following notations;

Z ; the ring of integers

Q ; the field of rational numbers

R ; the field of real numbers

C ; the field of complex numbers

H ; the field of quaternions

A_n, B_n, C_n, D_n; the classical groups of rank n

G_2, F_4, E_6, E_7, E_8; the exceptional groups

$G\sim G'$; two groups G and G' are locally isomorphic

G°; the identity component of the group G

$K\circ L$; the essentially direct product of K and L.

1. Preliminary results

Let φ: $G\times S^n\longrightarrow S^n$ be an action of a compact connected Lie group G on S^n with codimension two principal orbit G/H and with two types (L) and (K) of singular isotropy subgroups. Let G/L be non-isolated singular orbit and G/K the only one isolated singular orbit. It is well known that the orbit space is 2-dimensional disk and dim G/K is strictly smaller than dim G/L ([1], chap. IV section 8). It is easy to see that S^n is equivariantly diffeomorphic to a G-manifold $M_1\underset{f}{\cup}M_2$, where M_1 is a G-equivariant l-disk bundle over G/L, M_2 is a G-equivariant k-disk bundle over G/K and f: $bM_1\longrightarrow bM_2$ is an equivariant diffeomorphism (bM_i is the boundary of M_i). Note that bM_1 (respectively bM_2) is a sphere bundle over G/L (resp. over G/K). Since $2<l<k$, we see that the simply connectedness of S^n implies that G/K and G/L are both simply connected. In particular K and L are both connected.

We identify bM_1 and bM_2 by f and put $M_0=bM_1=bM_2$. From Mayer-Vietoris exact sequence, it follows that $H^i(M_0; Z)$ is isomorphic to $H^i(G/L; Z)\oplus H^i(G/K; Z)$ for $0<i<n-1$. In particular, the projections p_K; $M_0\longrightarrow G/K$ and p_L: $M_0\longrightarrow G/L$ induce isomorphisms p^*_K and p^*_L. Hence we have $M_0\underset{Z}{\sim}G/K\times S^{k-1}$ and $M_0\underset{Z}{\sim}G/L\times S^{l-1}$, where $X\underset{Z}{\sim}Y$ means that spaces X and Y have the same graded cohomology modules.

We have the following

PROPOSITON 1. *The Poincare polynomials of G/K and G/L are given by*

$$P(G/K)=(1+t^{l-1})\sum_{i=0}^{N}t^{i(k+l-2)}$$

$$P(G/L)=(1+t^{k-1})\sum_{i=0}^{N}t^{i(k+l-2)},\text{ where } n-1=(N+1)(k+l-2).$$

PROOF. We have already noted that

(1) $H^i(G/K; Q)\oplus H^i(G/L; Q)\cong H^i(M_0; Q)$ for $0<i<n-1$

and

(2) $G/K\times S^{k-1}\underset{Q}{\sim}M_0\underset{Q}{\sim}G/L\times S^{l-1}$.

Thus we have

(3) $P(M_0)=P(G/L)+P(G/K)+t^{n-1}-1$
$\qquad =P(G/K)(1+t^{k-1})=P(G/L)(1+t^{l-1})$.

It follows from (1) and (3) that $P(G/K)t^{k-1}=P(G/L)+t^{n-1}-1$ and $P(G/L)t^{l-1}=P(G/K)+t^{n-1}-1$ and hence we have

(4) $P(G/K)(1-t^{k+l-2})=(1+t^{l-1})(1-t^{n-1})$

and

(5) $P(G/L)(1-t^{k+l-2})=(1+t^{k-1})(1-t^{n-1})$.

Multiply both hand sides of (4) by $\sum_{i=0}^{N'}t^{i(k+l-2)}$. Then we have $n-1\equiv 0$ or $l-1$ (mod $k+l-2$) and $n+l-2\equiv 0$ or $l-1$ (mod $k+l-2$), because every terms of the left hand side is of positive degree mod $t^{(N'+1)(k+l-2)}$. Assume $n-1\equiv l-1$ and $n+l-2\equiv 0$ (mod $k+l-2$). Then $2(l-1)\equiv 0$ (mod $k+l-2$), which is impossible, because $2<l<k\leqq n$. Hence we have $n-1\equiv 0$ (mod $k+l-2$). Thus we have shown that there is an integer N such that $n-1=(N+1)(k+l-2)$ and $P(G/K)=(1+t^{l-1})\sum_{i=0}^{N}t^{i(k+l-2)}$ and $P(G/L)=(1+t^{k-1})\sum_{i=0}^{N}t^{i(k+l-2)}$. This completes the proof of the proposition.

The following propositions are useful for determination of the pair (G, K) of compact Lie groups with given Poincare polynomial $P(G/K)$.

PROPOSITION 2. *Let $U=U_1\times\cdots\times U_t$ be the product of compact simple Lie groups and V a semi-simple closed connected subgroup of U such that rank $V=$rank $U-1$. Then we have*

(1) *$V=V_1\times\cdots\times V_t$, where V_i is a subgroup of U_i*

or

(2) *$V=(V_1\times\cdots\times V_t)\circ V_0$, where $V_i\subset U_i$, rank $V_0=1$, rank V_i+rank $p_i(V_0)=$rank U_i for every i and the number of i such that $p_i(V_0)\neq 1$ is just two and p_i is the projection $U\longrightarrow U_i$.*

PROOF. We may assume that there is a simple normal factor V_0 of V such that the number n of i such that $p_i(V_0)\neq 1$ is greater than 1. In fact, if there is no such V_0, then the case (1) must occur. Put $V=V'\circ V_0$. Since V is semi-simple, we see that $p_i(V)=p_i(V')\circ p_i(V_0)$ for any i. Putting $V_i=p_i(V')$, we have

(i) rank V_i+rank $p_i(V_0)\leqq$rank U_i for every i

and hence

(ii) rank $(V_1\times\cdots\times V_t)+n$ rank $V_0\leqq$rank U.

Since

(iii) rank $V'\leqq$rank $(V_1\times\cdots\times V_t)$

and

(iv) rank $U=$rank $V+1=$ rank $V'+$rank V_0+1,

we have $(n-1)$ rank $V_0\leqq 1$ and hence $n=2$ and rank $V_0\leqq 1$. Thus we have rank $V'=$rank $(V_1\times\cdots\times V_t)$ and $V'=V_1'\times V_t'$, where $V_i'=V_i$, because $V_i=p_i(V')$. This completes the proof of the proposition.

By the same method, we can prove the following

PROPOSITION 3. *Let $U=U_1\times\cdots\times U_t$ be as in Proposition 2 and $V=V'\circ T^1$ be a closed connected subgroup of U, where V' is semi-simple and T^1 is a one-dimensional torus. Assume rank U=rank $V+1$. Then we have*

(1) $V=V_1\times\cdots\times V_t$, *where $V_i\subset U_i$ for every i*

(2) $V=(V_1\times\cdots\times V_t)\circ T^1$, *where V_i is semi-simple, the number of i such that $p_i(T^1)\neq 1$ is just 2 and rank V_i+rank $p_i(T^1)$=rank U_i for every i,*

or

(3) $V=(V_1\times\cdots\times V_t)\circ S$, *where $V_i\subset U_i$, S is locally isomophic to $Sp(1)$, the number of i such that $p_i(S)\neq 1$ is just 2 and rank V_i+rank $p_i(S)$=rank U_i for every i.*

Consider the action of K on S^{k-1} induced by the slice representation. Since G/K is the isolated singular orbit, this action has codimension one principal orbit K/H and two singular orbits K/L and K/L', where L and L' are conjugate each other in G. Let W be the identity component of the ineffective kernel of the action of K on S^{k-1}.

From results in [5] ((5.2), (7.4) (11.9)), it follows that there are following cases;

Case 1. l is even.

Subcase 1. $K/L=K/L'=S^{k-l}$, $K/H\approx K/L\times K/L'$ and $H=L\cap L'$.

Subcase 2. K/W is a simple proup of rank 2, $L/W\sim A_1\times T^1$ and $H/W\sim T^1\times T^1$, where $\sim$means "locally isomorphic"

Subcase 3. $K/W\sim C_3$, $L/W\sim C_1\times C_2$ and $H/W\sim C_1\times C_1\times C_1$.

Subcase 4. $K/W\sim F_4$, $L/W\sim B_4$ and $H/W\sim D_4$.

Case II. l is odd.

Subcase 1. $K/L=K/L'=S^{k-l}$, $K/H\approx K/L\times K/L'$ and $H=L\cap L'$.

Subcase 2. $l=3$. $P(K/H)=(1+t^3)(1+t)$, $P(K/L)=1+t^3$, $P(K/L')=1+t$ and K/L' is non-orientable.

Subcase 3. $l=3$, $P(K/H)=1+t^3$, $P(K/L')=P(K/L')=1$ and K/L, K/L' are non-orientable.

Subcase 4. $l=3$, $P(K/H)=(1+t^3)^2$ and $P(K/L')=P(K/L')=1+t^3$ and K/L, K/L' are non-orientable.

Since K and L are connected, K/L and K/L' are orintable. Hence subcaces 2, 3 and 4 of case II cannot occur.

2. The case n even

In this section and in next section we assume that G acts almost effectively on S^n. Note that the ineffective kernel of the action is precisely (center $G)\cap H$, where H is a principal isotropy subgroup.

For the case n even we shall prove the following

PROPOSITION 4. *$G/L=S^{k-1}$, $G/K=S^{l-1}$ and $n+1=k+l$.*

PROOF. It is clear that $X(S^n)=X(G/L)+X(G/K)$, where $X(\ \ )$ is the Euler characteristic. Since $L\subsetneq K$, we see that $X(G/K)=2$ and $X(G/L)=0$. Hence we have $G/K\underset{Z}{\sim}S^{n-k}$ and $G/L\underset{Z}{\sim}S^{n-l}$ from Proposition 1. Since G/L and G/K are simply connected, G/L and G/K are standard spheres (see [3]). This completes the proof of Proposition.

It is clear that l is odd and k is even. Hence the action of K on S^{k-1} induced by the slice representation has the following orbit structure;

(i) K/L and K/L' are $(k-l)$-spheres

(ii) $K/H=K/L\times K/L'$

and

(iii) $H=L\cap L'$.

Put $l=2r+1$. Then we have $k=4r$ and $n=6r$.

Let V be the identity component of the ineffective kernel of the action of G on G/L by the left translation. Note that V is a normal subgroup of G and is contained in L. Since L' is conjugate to L, V is contained in L' and hence contained in $H=L\cap L'$. Thus V acts on G/H trivially by the left translation, which means that $V=1$, since the action of G on G/H by the left translation is almost effective. Thus we have obtained the following

PROPOSITION 5. *The natural action of G on G/L by the left translation is almost effective.*

REMARK. If the action of G on S^n is assumed to be effective, then the action of G on G/L by the left translation is also effective.

Since $G/L=S^{4r-1}$, G is one of the following; D_{2r}, $A_{2r-1}\times T^1$, C_r, $C_r\times T^1$, $C_r\times C_1$, $B_4(r=4)$ and $B_3(r=2)$. Moreover, since G acts transitively on S^{2r}, G must be one of the followings;

Case 1. $G=A_1$ $r=1$

Case 2. $G=A_1\times A_1$ $r=1$

Case 3. $G=A_1\times T^1$ $r=1$

Case 4. $G=C_2\times C_1$, $C_2\times T^1$, C_2 $r=2$

Case 5. $G=B_4$ $r=4$.

We shall consider the above five cases separately.

Case 1. In this case, we have $\dim L=0$ and $\dim H=0$, which contradicts (ii).

Case 2. In this case, we have $K=T^1\times A_1$. Since $K/L=S^1$ and T^1 is the only Lie group which acts on S^1 transitively and effectively, L must be a normal subgroup of G, which contradicts to the almost effectivity of the action of G on G/L. Thus this case does not occur.

Case 3. In this case we have $G/L=S^3$, $G/K=S^2$ and $K/L=L/H=S^1$. It is clear that $K=T^1\times T^1$, $L=T^1$ and $H^0=1$. Suppose the action of G on S^6 be effective and $G=U(2)$.

Then $K=U(1)\times U(1)$ and $L=U(1)\times 1$. It is easy to see that $L\cap L'=1$ for any subgroups L' of K such that L' is conjugate to L and dim $L\cap L'=$dim $H=0$, which implies that $H=1$.

Now we shall examine more precisely the action φ_1 in Theorem in Introduction. The action ϕ_1 on R^3 is transitive on S^2 with isotropy subgroup a maximal torus $U(1)\times U(1)$ of $U(2)$. The action φ_2 is transitive on S^3 with isotropy subgroup $U(1)=U(1)\times 1$. Since $G_{(x,y)}=G_x\cap G_y$, we see that this is either finite or equal S^1 when $x\neq 0$. However G is transitive on $\{0\}\times S^3\subset S^6$ with isotropy subgroup $U(1)\times U(1)$. Since some conjugate of $U(1)$ can be seen to have trivial intersection with $U(1)\times U(1)$ we see that the principal isotropy subgroup is trivial (This is due to the arguments in [1]). Thus we have shown that in case 2 the action has the same orbit structure as φ_1.

Case 4. In this case we have $G/L=S^7$, $G/K=S^4$ and $K/L=L/H=S^3$. Suppose $G=Sp(2)\times Sp(1)$. Then we have $K=Sp(1)\times Sp(1)\times Sp(1)$, $L=Sp(1)\circ Sp(1)$ and $H=Sp(1)$. Assume G act on S^{12} effectively. It follows from the remark below Proposition 5 that G must be $Sp(2)\times Sp(1)/Z_2$, where Z_2 is the subgroup generated by $(-Id., -1)$.

Now we shall examine the action φ_2 in Theorem in Introduction more precisely. It is easily seen that there are points $x\in H^2$ and $y\in R^5$ such that $G_x=\left\{\left[\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}, b\right];\ a, b\in Sp(1)\right\}\approx Sp(1)\times Sp(1)$ and $G_y=\left\{\left[\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}, c\right];\ a, b, c\in Sp(1)\right\}=Sp(1)\times Sp(1)\times Sp(1)$. For the element $g=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1\end{pmatrix}\times 1\in G$, we have $G_{(gx,y)}=G_{gx}\cap G_y=Sp(1)$. We show that φ_2 has codimension two principal orbit $G/Sp(1)$, non-isolated singular orbit G/G_x and only one isolated singular orbit G/G_y. In fact, it is clear that $G/Sp(1)$ is a principal orbit. Since this action has no fixed point, we see that G/G_y is the only isolated singular orbit.

Consider the case $G=Sp(2)\times T^1$ or $Sp(2)$. It is not difficult to see that for this case the action has the same orbit structure as the action obtained from the restriction of $Sp(2)\times Sp(1)$ to the $Sp(2)\times T^1$ or $Sp(2)$.

Thus we have shown that the case 3 the action has the same orbit structure as φ_2 of Theorem in Introduction.

Case 5. In this case we have $G=Spin(9)$, $L=Spin(7)$, $K=Spin(8)$ and $H=G_2$ (G_2 denotes the exceptional simple Lie group of rank 2). It is clear that $Spin(9)$ must act on S^{24} effectively.

Now we shall examine the action φ_3 of Theorem in Introduction more precisely. Note that there are points $x\in R^{16}$ and $y\in R^9$ such that $G_x=Spin(7)\subset Spin(8)=G_y$. Assume the representation $Spin(7)\longrightarrow Spin(8)\longrightarrow SO(8)$ has a one dimensional trivial summand. Then $Spin(9)/G_x=V_{9,2}$, which contradicts to the fact $G/L=S^{15}$. Hence we have seen that the representation $Spin(7)=L\longrightarrow Spin(9)\longrightarrow SO(9)$ is $\Delta_7\oplus\theta^1$, where Δ_7 is the spin representation of $Spin(7)$ and θ^1 is one dimensional trivial representation. By a result in [6] (see section 1), we have the following equation

$$(\Delta_9+\pi)/Spin(7)=(Ad_{Spin(9)}/Spin(7)-Ad_{Spin(7)})\oplus\nu,$$

where ν is the slice representation at (x, y). From this equation, it follows that $\nu=\Delta_7\oplus\theta^1$.

Thus there is a point $z \in R^{16} \oplus R^{9}$ such that $Spin(9)_z = G_2$. It is easy to see that φ_3 has codimension two principal orbit $Spin(9)/G_2$ and two types $Spin(9)/Spin(7)$ and $Spin(9)/Spin(8)$ of singular orbits. Clearly φ_3 is effective. Thus we have shown that the case 4 the action has the same orbit structure as φ_3.

3. The case n odd

In this section, we shall show that there is no compact connected differentiable transformation group of odd dimensional sphere with codimension two principal orbit and only one isolated singular orbit, unless n=11, 23.

We shall use the notations as in section 1. First we shall consider the case in which l is greater than 3. Let $\bar{G}$, $\bar{K}$ and $\bar{L}$ be semi-simple parts of G, K and L respectively, $G=\bar{G}\times T^a$, $K=\bar{K}\circ T^b$ and $L=\bar{L}\circ T^c$. We may assume that $\bar{G}$ is simply connected.

We have the following

LEMMA 6. *The restricted $\bar{G}$-action has codimension 2 principal orbit and only one isolated singular orbit.*

PROOF. We consider the following commutative diagram;

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
\pi_1(\bar{G}\cap L)\otimes Q & \longrightarrow & \pi_1(\bar{G})\otimes Q=0 \\
\downarrow & & \downarrow \\
\pi_1(L)\otimes Q & \xrightarrow{\approx} & \pi_1(G)\otimes Q \\
\downarrow & & \downarrow p_{\#} \\
\pi_1(p(L))\otimes Q & \longrightarrow & \pi_1(T^a)\otimes Q \\
 & & \downarrow \\
 & & 0
\end{array}$$

where the vertical sequences are exact and p denotes the projection $G\longrightarrow T^a$. It follows from this diagram that $a=c$ and $(\bar{G}\cap L)^0=\bar{L}$. Since $\bar{G}/\bar{L}\longrightarrow G/L$ is a finite covering, we have $\bar{G}/\bar{L}=G/L$ and $\bar{G}\cap L=\bar{L}$, because G/L is simply connected. Since $L/H=S^{l-2}$ and $l\geqq 4$, we have also $\bar{L}/H\cap\bar{L}=L/H$ and hence $G=\bar{G}L=\bar{G}H$. By the same arguments as above, we have $\bar{G}\cap K=\bar{K}$ and $\bar{G}/\bar{K}=G/K$. This completes the proof of Lemma.

It follows immeadiately from the Borel's formula that rank G=rank $K+1$. Let $\bar{G}=G_1\times G_2\times\cdots\times G_t$ be the decomposition into the product of simply connected simple Lie groups. It follows from Proposition 2 that

(i) $\bar{K}=K_1\times\cdots\times K_t$, where $K_i\subset G_i$

or

(ii) $\bar{K}=(K_1\times\cdots\times K_t)\circ K_0$, where $K_i\subset G_i$, $K_0\sim A_1$, the number of i such that $p_i(K_0)\neq 1$ is 2 and rank K_i+rank $p_i(K_0)$=rank G_i.

We shall consider the cases I and II in section 1 separately.

Case I (l; even). In this case, we have rank $\bar{K}$=rank $\bar{L}$. Hence we have

$$\bar{L}=L_1\times\cdots\times L_t \text{ where } L_i\subset K_i \text{ and rank } L_i=\text{rank } K_i$$

or

$$\bar{L}=(L_1\times\cdots\times L_t)\circ L_0 \text{ where } L_i\subset K_i \text{ and rank } L_i=\text{rank } K_i \text{ and } L_0=K_0$$

correspondingly to the decomposition of $\bar{K}$.

Note that all K_i, except one K_j acts trivially on S^n. In fact this is clear for subcases (ii), (iii) and (iv) and proved as follows for the subcase (i). Since L and L' are conjugate in G, there is an element $g=g_1\times\cdots\times g_t\in\bar{G}$ such that $\bar{L}'=g\bar{L}g^{-1}$. If $K_i\subset\bar{L}$, then $g_iK_ig_i^{-1}\subset g\bar{L}g^{-1}=\bar{L}'\subset\bar{K}$. Since $p_j(g_iK_ig_i^{-1})=1$ for $j\neq i$, we have $g_iK_ig_i^{-1}\subset K_i$ and hence $K_i\subset\bar{L}\cap\bar{L}'=H$. Thus we may assume that $K_i=L_i\subset H$ for every i $(2\leqq i\leqq t)$. This implies that the restricted action of G_i on S^n has a unique orbit type G_i/K_i for $2\leqq i\leqq t$. We shall show that this is impossible. It is well known that $S^n=G_i/K_i\underset{\Gamma_{K_i}}{\times}F(K_i, S^n)$, where $\Gamma_{K_i}=N(K_i, G_i)/K_i$. Assume rank K_i=rank G_i. Since Γ_{K_i} is a finite group and $\pi_1(S^n)=1$, we have $S^n=G_i/K_i\times F$, where F is a connected component of $F(K_i, S^n)$. This is a contradiction, because dim $G_i/K_i<n$ and dim $F<n$. Next assume rank K_i=rank G_i-1. Then Γ_{K_i} is finite or of rank 1. If this is finite, then the same argument as above concludes a contradiction. If Γ_{K_i} is of rank 1, then we see that $S^n=G_i/K_i\underset{W}{\times}F$, where W is the identity component of Γ_{K_i} and F is a connected component of $F(K_i, S^n)$. Since W is a rational homology sphere of dimension 1 or 3, we have an isomorphism $H^r(G_i/K_i\times F;\ Q)\cong H^r(W\times S^n;\ Q)$ and hence $H^r(G_i/K_i\times F;\ Q)=0$ for $2\leqq r<n$ or $H^r(G_i/K_i\times F;\ Q)=0$ for $4\leqq r<n$ according to dim $W=1$ or 3 respectively. This implies that dim $G_i/K_i\leqq 3$ and dim $F\leqq 3$ and hence $n\leqq 6-\dim W\leqq 5$, which is impossible, because it follows from the facts that $n-1=(N+1)(k+l-2)$ and $2<l<k\leqq n$ that $n\geqq 6$.

Thus we have proved that $\bar{G}$ is simple. It follows from the Poincare polynomial of $\bar{G}/\bar{L}=G/L$ that $\bar{L}$ is also simple.

Subcase 1. Since $\bar{L}$ is simple, $\bar{K}$ is also simple and possible pairs of $(\bar{K}, \bar{L})$ are (B_r, D_r) or (G_2, A_2) $(l=8)$, where G_2 is the exceptional group of rank 2. On the other hand we have $\bar{L}/H\cap\bar{L}=L/H=S^{l-2}$, which is impossible for $\bar{L}=D_r$ or A_2.

Subcase 2. Since $\bar{K}$ and $\bar{L}$ are semi-simple we have $H^2(\bar{K}/\bar{L};\ Q)=0$, which contradicts to the fact that $L/W\sim A_1\times T^1$.

Subcase 3. This case cannot occur, because $\bar{L}$ is simple.

Subcase 4. This case cannot occur, because there is no simple group of rank 5 which contains F_4 as proper subgroup.

Thus we have shown that the case I does not occur.

Next we shall consider the case II. We divide this case into two subcases; subcase 1 in which l is greatar that 3 and subcase 2 in which l is 3.

Subcase 1. $l\geqq 5$.

We note the following facts.

(1) $k=2l-2$. This follows from that $K/H=K/L\times K/L'$.

(2) Let t, s and u be the number of simple factors of $\overline{G}$, $\overline{K}$ and $\overline{L}$ respectively. Then we have $t=u$ and $s=t$ or $t+1$. In fact, since $\pi_1(G/L)\otimes Q=\pi_2(G/L)\otimes Q=\pi_3(G/L)\otimes Q=\pi_4(G/L)\otimes Q=0$, we have $t=u$. It is not difficult to see that $s=t$ ($l\geqq 7$) and $s=t+1$ ($l=5$).

(3) All simple factors but one simple factor or $Sp(r)\circ Sp(1)$ act on $\overline{K}/\overline{L}$ trivially. This follows immeadiately from that $\overline{K}$ is semi-simple and $\overline{K}/\overline{L}=S^{l-2}$.

From (3) and the same arguments as in case I, it may be assumed that t is at most 2. Thus the subcase 1 is divided into the following five cases.

a) $\overline{G}$ is simple.

a. 1) **Case i.** $\overline{K}$ and $\overline{L}$ are simple.

a. 2) $\overline{L}$ is simple and $\overline{K}=K_1\circ K_1'$ (K_1, K_1'; simple)

Case ii. $\overline{L}=K_1'$, $K_1=Sp(1)$

Case iii. $K_1=K_1'$, $\overline{L}=Sp(1)$

b) $\overline{G}=G_1\times G_2$ (G_i; simple)

b. 1) **Case iv.** $\overline{K}=K_1\times K_2$ ($K_i\subset G_i$), K_i acts on $\overline{K}/\overline{L}$ non-trivially.

b. 2) **Case v.** $\overline{K}=(K_1\times K_2)\circ K_0(K_i\subset G_i)$, K_i acts on $\overline{K}/\overline{L}$ non-trivially.

Case i. It follows from $\overline{K}/\overline{L}=S^{l-2}$ that possible pair of $(\overline{K}, \overline{L})$ is (A_r, A_{r-1}), (D_{r+1}, B_r), (C_r, C_{r-1}), (B_4, B_3) or (B_3, G_2). Since $\overline{L}/\overline{L}\cap H=S^{l-2}$, all pairs except (D_4, B_3) are inadequate. Consider the case $(\overline{K}, \overline{L})=(D_4, B_3)$. Since D_4 cannot be subgroup of A_5 and C_5, $\overline{G}$ must be B_5 or D_5 and $\overline{K}\longrightarrow\overline{G}$ is the standard inclusion up to automorphism of $\overline{K}$. Comparing the Poincare polynomials of G/K and B_5/D_4 or D_5/D_4, we conclude a contradiction. This implies the case i does not occur.

Case ii. In this case we see that $\overline{K}=Sp(1)\circ\overline{L}$ and $l=5$, $k=8$. Since $\overline{K}/\overline{L}=S^3=\overline{L}/\overline{L}\cap H$, we have $\overline{L}=Sp(1)$ and rank $\overline{G}=3$. It follows from $p(\overline{G}/\overline{K})=(1+t^4)(1+t^{11}+\cdots)$ that $\overline{G}=B_3$ or C_3. In this case we see $n=23$ and $(\overline{G}, \overline{K}, \overline{L})=(B_3, C_1\times C_1, B_1)$ or $(C_3, C_1\times C_1, C_1)$.

Case iii. By the same arguments as in case ii, we have $\overline{K}=C_1\circ C_1$, $\overline{L}\sim C_1$ and rank $\overline{G}=3$. Consulting the Poincare polynomial of G/K, we have $\overline{G}=B_3$ or C_3 and hence $n=23$.

Case iv. In this case we have $\overline{G}=G_1\times G_2$ (G_i; simple), $\overline{K}=C_r\times C_1$, $\overline{L}\sim C_{r-1}\times C_1$, where $C_r\subset G_1$, C_1 (=factor of $\overline{K}$)$\subset G_2$ and the factor C_1 of $\overline{L}$ is monomorphically mapped in both C_r and C_1. Since $S^{l-2}=K/L=\overline{C}_r/\overline{C}_{r-1}=S^{4r-1}$, we have $l=4r+1$, $k=8r$ and $p(\overline{G}/\overline{K})=(1+t^{4r})(1+t^{12r-1}+\cdots)$. Assume $G_2=C_1$. Then rank $G_1=r+1$ and $H^i(G_1; Q)=H^i(C_r; Q)$ for $i\leqq 4r-2$. It follows that G_1 is one of B_{r+1}, C_{r+1}, D_{r+1}, $G_2(r=1)$ and $A_2(r=1)$. By dimensional arguments we can show a contracdiction. Next assume $G_1=C_r$ and rank $G_2=2$. It is not difficult to see that this case does not occur. Thus we have proved that the case (iv) does not occur.

Case v. In this case $\overline{G}=G_1\times G_2$, $\overline{K}\sim(C_1\times C_1)\circ C_1$, $\overline{L}\sim(1\times C_1)\circ C_1$, rank $G_i=2$ ($i=1, 2$) and the second factor C_1 of $\overline{L}$ is monomorphically mapped in both the first and second factor of $\overline{K}$. It is clear that $G_i=C_2$ or G_2 and $l=5$ and $k=8$. Consider the principal fibre bundle:

$$C_1 \longrightarrow \overline{G}/\overline{K} \longrightarrow X = G_1/(C_1 \circ C_1) \times G_2/(C_1 \circ C_1),$$

where $G_i/(C_1 \circ C_1)$ is $C_2/(C_1 \circ C_1) \underset{Q}{\sim} S^4$ or $G_2/(C_1 \circ C_1)$. It follows from the spectral-sequence of the fibre bundle and the Poincare polynomial of $\overline{G}/\overline{K}$ that $H^4(X; Q) = H^8(X; Q) = 2Q$ and hence $X \underset{Q}{\sim} S^4 \times G_2/SO(4)$, in other words $\overline{G} = C_2 \times G_2$ and hence dim $\overline{G}/\overline{K} = 15$, which implies also $G/K \underset{Q}{\sim} S^4 \times S^{11}$.

Thus we have shown that possibilities of $(\overline{G}, \overline{K}, \overline{L})$ in case II are $(C_2 \times G_2, (C_1 \times C_1) \circ C_1, (1 \times C_1) \circ C_1)$, $(B_3, C_1 \times C_1, C_1)$ or $(C_3, C_1 \times C_1, C_1)$. Note that in these cases $n = 23$.

Subcase 2. $l = 3$ and $k = 4$.

In this case $P(G/K) = (1+t^2)(1+t^5+\cdots)$ and $P(G/L) = (1+t^3)(1+t^5+\cdots)$.

Let $G = \overline{G} \times T^a$, where $\overline{G}$ is semi-simple. We may assume $\pi_1(\overline{G}) = 1$. Put $\overline{K} = K \cap \overline{G}$, $\overline{L} = L \cap \overline{G}$, $\overline{L}' = L' \cap \overline{G}$ and $\overline{H} = H \cap \overline{G}$. By the same argument as in page 9, we see that $\overline{L}$ and $\overline{L}'$ are semi-simple, connected and $\overline{G}/\overline{L} = G/L$, $\overline{G}/\overline{L}' = G/L'$.

From the commutative diagram;

$$\begin{array}{ccccc} \overline{K}/\overline{L} & \longrightarrow & \overline{G}/\overline{L} & \longrightarrow & \overline{G}/\overline{K} \\ \downarrow & & \downarrow & & \downarrow \\ K/L & \longrightarrow & G/L & \longrightarrow & G/K, \end{array}$$

it follows that $\overline{G}/\overline{K} = G/K$ and $\overline{K}/\overline{L} = K/L$. In particular $\overline{K}$ is connected. Since $\overline{K}/\overline{L} = \overline{K}/\overline{L}' = S^1$, we have $\overline{K} = \overline{L} \circ T^1 = \overline{L}' \circ T^1$ and hence $\overline{L} = \overline{L}' = \overline{H}$. Let $\overline{G} = G_1 \times G_2 \times \cdots G_t$ be the decomposition into the product of simple groups. It follows from Prop. 3 that $\overline{K}$ is given by

(1) $\overline{K} = K_1 \times \cdots \times K_t$, $K_i \subset G_i$

(2) $\overline{K} = (K_1 \times \cdots \times K_t) \circ T$, $K_i \subset G_i$, K_i; semi-simple

or

(3) $\overline{K} = (K_1 \times \cdots \times K_t) \circ S$, $K_i \subset G_i$, $S \sim C_1$.

Since $\overline{K} = \overline{H} \circ T^1$, $\overline{H} = \overline{L}$ is semi-simple and $\overline{K}/\overline{H} = S^1$, we may assume $1 \times K_2 \times \cdots \times K_t \subset \overline{H}$. By the same argument as in page 10, we can clonclude that $\overline{G}$ is simple. It follows easily from the spectral sequence of the fibration $\overline{K} \longrightarrow \overline{G} \longrightarrow \overline{G}/\overline{K}$ that $\overline{K} = T^1$ and hence $\overline{L} = 1$, which implies $G/L = \overline{G}$. Since rank $\overline{G} = 2$, $\overline{G}$ must be A_2 because $P(G/L) = P(\overline{G}) = (1+t^3)(1+t^5+\cdots)$. It is clear $n = 11$. Thus we have shown that the possibility of $(\overline{G}, \overline{K}, \overline{L})$ in subcase 2 is $(SU(3), T^1, 1)$.

Thus we have proved the statement in Introduction of this section. Summing up the arguments in sections 2 and 3 we have proved the Theorem A.

4. Classification of actions

In this section we shall complete the proof of the Theorem B in Introduction; in other words, we shall prove that an effective action of a compact connected Lie group G on sphere with codimension two principal orbit and two types of singular orbit is continuously eqivalent to one of the actions φ_1 and φ_2, unless dimension of the sphere is 11, 23 or the orbit structure is the same as φ_3.

Let φ: $G\times S^n \longrightarrow S^n$ be an effective action of a compact connected Lie group on S^n with codimension two principal orbit G/H and two types G/L and G/K of singular orbit, G/L is non-isolated and G/K is isolated. We have shown that possibilities of pair (G, H, L, K) are one of the followings

Case 1. $(U(2), L, U(1)\times 1, U(1)\times U(1))$

Case 2. $(Sp(2), 1, Sp(1)\times 1, Sp(1)\times Sp(1))$

Case 3. $(Sp(2)\times S/Z_2, S, Sp(1)\times S\times Sp(1))$, where S or $Sp(1)\times S$ denotes subgroup

$$\left\{\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, a\right)\right\} \text{ or } \left\{\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, b\right)\right\} \text{ of } G \text{ resp.}$$

and

Case 4. $(Spin\ (9), G_2, Spin\ (7), Spin\ (8))$

unless $n=11$, 23, where $S=T^1$ or $Sp\ (1)$.

We shall show that if φ: $G\times S^n \longrightarrow S^n$ has the same orbit structure as case 1, case 2 and case 3, then φ is continuously equivalent to the action φ_1 or φ_2 in Introduction.

First we consider the cases 1 and 2. We shall omit the case 2, since the proof for this case is completely analogus to the case 1. Put $G=U(2)$ and $M=S^6$. We identify the orbit space M^* with the unit disk in the complex plane. Let M^*_+ and M^*_- be the subset of M^* consisting respectively of points with non-negative and non-positive imaginary part and let $M^*_+\cap M^*_-=A^*$. We can construct cross section φ_+: $M^*_+ \longrightarrow M$ and φ_-: $M^*_- \longrightarrow M$ such that

$$G_{\varphi_+(x)}=\begin{cases} H & |x|<1 \\ L & |x|=1 \end{cases}$$

and

$$G_{\varphi_-(x)}=\begin{cases} H & |x|<1 \\ L & |x|=1,\ \mathrm{Re}\ x\neq 0 \\ K & |x|=1\ \ \mathrm{Re}\ x=0 \end{cases}$$

In fact let V be the slice of G/K such that the action of K on V has codimension 1 principal orbit. By a result in [1] (Lemma 6. 1, Chap II), we see that the orbit map $V \longrightarrow V/K$ has a cross section and hence there is a cross section on M^*_- which is assumed to have the above property. The same arguments show the existence of a cross section on M^*_+. Note that $\varphi_+(-1)$ and $\varphi_-(-1)$ belong to the same orbit. For $x\in A^*$, $|x|<1$, there is a unique element $f(x)\in N(H)/H=G$ such that $f(x)\varphi_-(x)=\varphi_+(x)$. Thus we have a function $f: (-1, 1) \longrightarrow G$, which is easily seen to be continuous. Now there is a homotopy h_t: $U(2) \longrightarrow U(2)$ such that $h_0 = id.$ and h_1 (a neighborhood of $N_L) = N$, where N_L is the normalizer of L in G). In fact let U be closed tubular neighborhood of N_L in G. Then there is a homotopy $\bar{h}_t$: $U \longrightarrow (2)$ such that $\bar{h}_0 =$ the inclusion and $\bar{h}_1(U) = N_L$. Since the pair $(U(2), U)$ has the absolute homotopy extension property, there is a homotopy h_t of required property. Since $f(x) \longrightarrow N_L$ as $x \longrightarrow \pm 1$ and f is continuous, f maps $(-1, -1 + \varepsilon)$ and $(1-\varepsilon, 1)$ into a nbhd. of N_L, where ε is a small positive real number. Thus f is homotopic through homotopy g_t with $g_t(\pm 1) \in N_L$ to the restriction of a function f' on $[-1, 1]$ to G such that $f'(\pm 1) \in N_L$. We call f' the comparison function of cross section (φ_-, φ_+). We have the following

LEMMA 7. *Any two maps f_0, f_1: $[-1, 1] \longrightarrow G$ with $f_i(\pm 1) \in N_L$ $(i = 0, 1)$ are homotopic through homotopy f_t with $f_t(\pm 1) \subset N_L$.*

PROOF. f_i induces a map $\bar{f}_i$: $[-1, 1] \longrightarrow G \longrightarrow G/N_L = S^2$ and it is easy to see that f_0 is homotopic to f_1 if and only if $\bar{f}_0$ is homotopic to $\bar{f}_1$ rel. $\{*\}$. This completes the proof of the Lemma.

LEMMA 8. *Let f_0, f_1: $[-1, 1] \longrightarrow G$ be maps with $f_i(\pm) \in N_L$. If f_0 and f_1 are homotopic through homotopy f_t with $f_t(\pm 1) \in N_L$, then $f_0 \circ f_1^{-1}$: $[-1, 1] \longrightarrow G$; $x \longrightarrow f_0(x) f_1(x)^{-1}$ is homotophic to the constant map through homotopy g_t with $g_t(\pm 1) \in N_L$.*

PROOF. By assumption we have a map F: $[-1, 1] \times I \longrightarrow G$ such that $F(t, i) = f_i(t)$ for $i = 0, 1$ and $F(\pm 1, t) \in N_L$. Define map H: $[-1, 1] \times I \longrightarrow G$ by $H(x, t) = F(x, t) F(x, 1)^{-1}$. This map H gives a homotopy between $f_0 f_1^{-1}$ and the constant map 1. This completes the proof of the Lemma.

Let $(\varphi_+^0, \varphi_-^0)$ and $(\varphi_+^1, \varphi_-^1)$ be two cross sections with the comparison functions f_0 and f_1 respectively. Assume f_0 is homotopic to f_1 through homotopy f_t with $f_t(\pm 1) \in N_L$. We can show that there is a map ψ: $M^*_+ \longrightarrow G$ such that $\psi(M^*_+ \cap B^*) \subset N_L$, where B^* is the orbit space of union of all singular orbits. In fact, since $f_0 f_1^{-1} \simeq 1$ rel. $\{\pm 1\}$, there is a map H: $[-1, 1] \times I \longrightarrow G$ with $H(x, 0) = f_0(x) f_1(x)^{-1}$, $H(x, 1) = 1$ and $H(\pm 1, t) \in N_L$. Let θ: $M^*_+ \longrightarrow [-1, 1] \times I$ be a homeomorphism such that

$$\theta([x \in M^*_+;\ |x| = 1,\ -1 \leqq \mathrm{Re}\ x \leqq -1/2]) = \{-1\} \times I$$
$$\theta([x \in M^*_+;\ |x| = 1,\ 1/2 \leqq \mathrm{Re}\ x \leqq 1]) = \{1\} \times I$$
$$\theta([x \subset M^*_+;\ \mathrm{Im}\ x = 0]) = [-1, 1] \times \{0\}$$

and

$$\theta([x \in M^*_+;\ |x| = 1,\ -1/2 \leqq \mathrm{Re}\ x \leqq 1/2]) = [-1, 1] \times \{1\}.$$

Then $\psi = H \circ \theta$ is the required map. Clearly the map $\bar{\varphi}^+{}_1$: $M^*_+ \longrightarrow M$ defined by $\bar{\varphi}^1{}_+(x) = \psi(x) \varphi^1{}_+(x)$ is a cross section and the comparison function of $(\varphi^1{}_-, \bar{\varphi}^1{}_+)$ is f_0. In fact we have $\bar{\varphi}^1{}_+(x) = \psi(x) \varphi^1{}_+(x) = \psi(x)\ f_1(x) \varphi^1{}_-(x) = f_0(x) \varphi^1{}_-(x)$ on A^*.

Let $C_0 = \mathrm{Im}\ \varphi^0{}_+ \cup \mathrm{Im}\ \bar{\varphi}^0{}_-$ and $C_1 = \mathrm{Im} \varphi^1{}_- \cup \mathrm{Im}\ \bar{\varphi}^1{}_+$. Then we have $GC_0 = GC_1 = M$.

Define a map ψ: $C_0 \longrightarrow C_1$ by $\psi(\varphi^0{}_-x)) = \varphi^1{}_-(x)$ and $\psi(\varphi^0{}_+(x)) = \overline{\varphi}^1{}_+(x)$. We have $\psi(f_0(x)\varphi^0{}_-(x)) = \psi(\varphi^0{}_+(x)) = \overline{\varphi}^1{}_+(x) = f_0(x)\varphi_{1-}(x) = f_0(x)\psi(\varphi^0{}_-(x))$.

It follows from the following Lemma and Lemma 7 that any action of $U(2)$ on S^6 with the property of Theorem in Introduction is continuously equivalent to the action φ_1.

LEMMA 9. *Let G be a compact connected Lie group and let X_1 and X_2 be Hausdorff spaces on which G acts as a topological transformation group. Let $C_i \subset X_i$ $(i=1, 2)$ be closed subsets such that $GC_i = X_i$ and let ψ: $C_1 \longrightarrow C_2$ be a map such that for every $g \in G$ and $x \subset C_1$ such that $gx \in C_1$ we have $\psi(gx) = g(\psi(x))$. Then ψ can be extended uniquely to an equivariant map from X_1 to X_2.*

See [2]. Thus we have proved the Theorem B for the case $G = U(2)$.

Next we shall consider the case 3. We may assume that $G = Sp(2) \times S$. Then we see that $K = Sp(1) \times Sp(1) \times S$, $L = \left\{\left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, b\right] \subset Sp(2) \times S\right\}$ and $H = \left\{\left[\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a\right] \in Sp(2) \times S\right\}$ $\approx S$. It is easy to see that $N_H{}^0 = H$ when $S = Sp(1)$. By the same arguments as case 1 and 2, we can show that there are cross sections φ_+ and φ_- from $M^*{}_+$ and $M^*{}_-$ to M respectively.

We have a continuous function $f: (-1, 1) \longrightarrow G$ such that $f(x)\varphi_+(x) = \varphi_-(x)$ and hence $f(x) \in N_H/H$, which is assumed to be in $N_{H^\circ}/H = \{1\}$. Then we can take the constant map 1: $[-1, 1] \longrightarrow G$ as the comparison function of cross sections (φ_+, φ_-), which shows that two actions of G are continuously equivalent each other. When $S = T^1$, we see that $N_{H^\circ}/H = T^2$. Since $N_H \subset N_L$, it is not difficult to see that Theorem B holds in this case. Thus we have completed the proof of the Theorem B.

References

[1] BREDON, G. E.: Introduction to Compact transformation groups, Academic Press 1972

[2] BREDON, G. E.: *Transformation groups on spheres with two types of orbits*, Topology vol. 3 (1965) 103–113

[3] BREDON, G. E.: *On homogeneous cohomology spheres*, Ann. of Math. 73 (1961) 556–565

[4] STEENROD, N. E.: The Topology of Fibre Bundles, Princeton Univ. Press 1951

[5] WANG, H. C.: *Compact transformation groups of S^n with an $(n-1)$-dimensional orbit*, Amer. J. Math. 82 (1960) 698–748

[6] HSIANG, W. C. and HSIANG, W. Y.: *Differentiable actions of compact connected classical groups, II*. Ann. of Math. 92 (1970) 189–223