On a Galton-Watson process with state-dependent immigration

By

Masamichi SATO

(Received October 31, 1974)

1. Introduction

Consider a Galton-Watson process in which immigration is allowed in a generation if the number of the previous generation is smaller than or equal to i_0, where i_0 is a fixed positive integer.

Denote the size of the n-th generation by X_n, and we set up this process as follows;

1) $A(x)=\sum_{j=0}^{\infty} a_j x^j$, $(|x|\leqq 1)$, is the probability generating function of the offspring distribution;

2) $B_k(x)=\sum_{j=0}^{\infty} b_{kj} x^j$, $(|x|\leqq 1, 0\leqq k\leqq i_0)$, is the probablity generating function of the immigrants distribution, where b_{kj} is the probability that j immigrants enter in a generation when the number of the previous generation is equal to k;

3) transition probability p_{ij} is given by

$$
\begin{aligned}
p_{ij} \equiv P\{X_{n+1}=j\,|\,X_n=i\} &= a_j^{(i^*)}, \qquad i\geqq i_0+1, \\
&= \sum_{k=0}^{j} b_{ik}\cdot a_{j-k}^{(i^*)}, \qquad 0\leqq i\leqq i_0,
\end{aligned} \tag{1}
$$

where $a_j^{(i^*)}$ is the j-th term in the i-th fold convolution of the sequence $\{a_j\}$.
The state-dependent immigration have been studied by Pakes (1971) [9], Foster (1971) [4] and Nakagawa-Sato (1974) [8]. The setting up of Pakes or Foster is the case that $B_0(x)=B(x)$ and $B_k(x)=1 (k=1, 2, \cdots, i_0)$, and Nakagawa-Sato's is the case that $B_k(x)=B(x)$ for all k, in our process.

From now on it will be assumed that

1. $0<a_0, a_0+a_1, b_{k0}<1, \qquad (k=0, 1, \cdots\cdots, i_0);$

2. $\alpha=A'(1-)<\infty$ and $\beta_k=B_k'(1-)\leqq\infty, \qquad (k=0, 1, \cdots\cdots, i_0).$

2. Preliminary Considerations

It is clear that $\{X_n\}$ is a Morkov chain with nonnegative integers as state space. This Morkov chain will be said to be subcritical, critical or supercritical according as $\alpha<1$, $=1$ or >1, respectively. A sufficient condition for irreducibility (and aperiodicity) is that $a_1>0$.

Let $p_{ij}^{(n)}(n,\ i,\ j=0,\ 1,\ 2\cdots)$ be the n-th step transition probability from state i to j, $p_{ij}^{(1)}=p_{ij}$ and $p_{ij}^{(0)}=\delta_{ij}$, the Kronecker delta, and finally let

$$P_i^{(n)}(x)=\sum_{j=0}^{\infty}p_{ij}^{(n)}x^j, \qquad (|x|\leqq 1).$$

Clearly

$$P_i^{(n+1)}(x)=P_i^{(n)}(A(x))-\sum_{k=0}^{i_0}(1-B_k(x))(A(x))^k\cdot p_{ik}^{(n)},$$

(2)

$$P_i^{(0)}(x)=x^i, \qquad (n,i=0,1,2,\cdots).$$

By iterating this equation, we obtain

(3) $$P_i^{(n)}(x)=[A_n(x)]^i-\sum_{k=0}^{i_0}\left[\sum_{m=0}^{n-1}(1-B_k(A_m(x)))(A_{m+1}(x))^k p_{ik}^{(n-m-1)}\right]$$

where $A_{n+1}(x)=A(A_n(x))$ and $A_0(x)=x$.

Letting $x=0$ in equation (3) and taking generating function, we get

$$\sum_{n=0}^{\infty}p_{i0}^{(n)}y^n=\sum_{n=0}^{\infty}(A_n(0))^iy^n-\sum_{k=0}^{i_0}\left[y\cdot\sum_{m=0}^{\infty}p_{ik}^{(m)}y^m\cdot\sum_{n=0}^{\infty}(1-B_k(A_n(0)))(A_{n+1}(0))^ky^n\right],$$

whence we may rewrite the above equation as

(4) $$P_{i0}(y)=C_i(y)-\sum_{k=0}^{i_0}y\cdot P_{ik}(y)\cdot D_k(y), \qquad (|y|<1),$$

where

$$P_{ik}(y)=\sum_{n=0}^{\infty}p_{ik}^{(n)}\cdot y^n, \qquad C_i(y)=\sum_{n=0}^{\infty}(A_n(0))^iy^n,$$

and

$$D_k(y)=\sum_{n=0}^{\infty}(1-B_k(A_n(0)))(A_{n+1}(0))^ky^n, \qquad (|y|<1).$$

Denote by q the extinction probability of the branching process defined by $A(x)$, so that q is the least positive solution of $x=A(x)$ and $q=1$ if $\alpha\leqq 1$ and $0<q<1$ if $\alpha<1$.

LEMMA 1. *When* $\alpha\leqq 1$ *then* $D_k(1-)\equiv\sum_{n=0}^{\infty}(1-B_k(A_n(0)))(A_{n+1}(0))^k$ *converges if and only if*

(5) $$\int_0^1 \frac{1-B_k(u)}{A(u)-u}\,du<\infty.$$

And when $\alpha>1$ *then* $D_k(1-)$ *diverges for any* k $(k=0, 1, \cdots, i_0)$.

For the proof see Nakagawa-Sato (1974) [8].

3. Classification of $\{X_n\}$

In this section we consider the classification of the Morkov chain, $\{X_n\}$, when it is irreducible (and aperiodic). In the case where $B_k(x)=B(x)$ for all k, the classification of the Morkov chain $\{X_n\}$ is considered by Nakagawa-Sato (1974) [8].

The following theorem is an extension of Nakagawa-Sato's result.

THEOREM 1. *Let the Morkov chain* $\{X_n\}$ *be irreducible (and appriodic).*

(i) *The necessary and sufficient conditions for the chain to be positive recurrent (i.e. for the stationary distribution to exist) are that*

$$\alpha\leqq 1 \textit{ and } \int_0^1 \frac{1-B_k(u)}{A(u)-u}\,du<\infty, \qquad \textit{for all } k(k=0, 1, \cdots, i_0).$$

(ii) *When* $\alpha=1$, $\sigma^2=A''(1-)<\infty$ *and* $\beta_k<\infty$ *for all* $k(k=0, 1, \cdots, i_0)$, *then* $\{X_n\}$ *is null-recurrent.*

(iii) *When* $\alpha>1$ *then* $\{X_n\}$ is transient.

Proof.

(i) Since the chain is assumed to be irreducible and aperiodic, we need to consider one state only, and a convenient one is the empty state. Letting $x=i=0$ in the equation (3), we have

$$p_{00}^{(n)}=1-\sum_{k=0}^{i_0}\left[\sum_{m=0}^{n-1}(1-B_k(A_m(0)))(A_{m+1}(0))^k p_{0k}^{(n-m-1)}\right].$$

It is clearly sufficient for positive recurrence to show that $p_{00}^{(n)}$ approaches to a positive limit as $n\longrightarrow\infty$, and it is sufficient to show that

(6) $$\lim_{n\to\infty} p_{00}^{(n)}=1-\sum_{k=0}^{i_0}\left[\lim_{n\to\infty}\sum_{m=0}^{n}(1-B_k(A_m(0)))(A_{m+1}(0))^k p_{0k}^{(n-m)}\right]$$

is positive.

Now, since $(1-B_k(A_m(0)))(A_{m+1}(0))^k>0$, we have

$$\lim_{n\to\infty}\frac{(1-B_k(A_n(0)))(A_{n+1}(0))^k}{\sum_{\nu=0}^{n}(1-B_k(A_\nu(0)))(A_{\nu+1}(0))^k}=0.$$

Hence, using a well-known fact (see, e.g. Chung (1967) [2], p. 22), we have

$$\text{(7)} \qquad \frac{\lim_{n\to\infty}\sum_{m=0}^{n}(1-B_k(A_m(0)))(A_{m+1}(0))^k p_{0k}^{(n-m)}}{\sum_{n=0}^{\infty}(1-B_k(A_n(0)))(A_{n+1}(0))^k} = \lim_{n\to\infty} p_{0k}^{(n)},$$

whenever the limit of $p_{0k}^{(n)}$ as $n\to\infty$ exists.

Equation (6) and (7) imply that whenever $\sum_{m=0}^{n}(1-B_k(A_m(0)))(A_{m+1}(0))$ diverges as $n\to\infty$ for some k ($k=0, 1, 2, \cdots, i_0$), it is impposible that the limit of $p_{0j}^{(n)}$ exists and is positive for any j ($j=0, 1, \cdots$).

Furthermore, from the lemma in the section 2 and (3), if $D_k(1-)\equiv\sum_{n=0}^{\infty}(1-B_k(A_n(0)))(A_{n+1}(0))^k$ converges for all $k(k=0, 1, \cdots, i_0)$, equivalently, if

$$\alpha\leqq 1 \text{ and } \int_0^1 \frac{1-B_k(u)}{A(u)-u}du<\infty \qquad \text{for all } k\ (k=0, 1, \cdots, i_0),$$

then the limit of $P_i^{(n)}(x)$ as $n\to\infty$ exists, and so the limit of $p_{ij}^{(n)}$ as $n\to\infty$ exists ($i, j=0, 1, \cdots$).

Now we suppose that $D_k(1-)$ converges as $n\to\infty$ and $\lim_{n\to\infty} p_{00}^{(n)}=0$. Then $\lim_{n\to\infty} p_{0j}^{(n)}=0$ for any j ($j=0, 1, 2, \cdots$).

Thus, equation (6) and (7) imply that

$$\lim_{n\to\infty} p_{00}^{(n)}=1,$$

which is contradicting the assumption that $\lim_{n\to\infty} p_{00}^{(n)}=0$.

Hence, whenver $D_k(1-)$ converges as $n\to\infty$ for all k ($k=0, 1, \cdots i_0$),

$$\lim_{n\to\infty} p_{00}^{(n)}>0.$$

This completes the proof of (i).

(ii) Note that when $\alpha=1$ and $\sigma^2<\infty$ then, as shown by Kesten, Ney and Spitzer (1966),

$$\text{(8)} \qquad \sum_{\nu=0}^{n}(1-A_\nu(0))\sim\frac{2}{\sigma^2}\log n, \qquad \text{as } n\to\infty.$$

Thus, for any k

$$\sum_{n=0}^{\infty}(1-B_k(A_n(0)))\sim\frac{2\beta_k}{\sigma^2}\log n, \qquad \text{as } n\to\infty,$$

and hence $D_k(1-)$ diverges and

(9) $$D_k(y) \sim \beta_k \sum_{n=0}^{\infty} (1-A_n(0))y^n, \qquad \text{as } y \longrightarrow 1-.$$

From lemma 1 and (i) of this theorem, we have only to show that $P_{00}(1-)=\infty$, because this yields that the Markov chain $\{X_n\}$ is null-recurrent.

Now we suppose that $P_{00}(1-)<\infty$, then $P_{0k}(1-)<\infty$ for any $k(k=0,1,\cdots)$. In this case, since

$$(1-y)P_{00}(y)=1-y(1-y)\sum_{k=0}^{i_0} P_{0k}(y)\cdot D_k(y),$$

we obtain

$$\lim_{y\to 1-}(1-y)\sum_{k=0}^{i_0} P_{k0}(y)\cdot D_k(y)=1,$$

equivalently,

$$\sum_{k=0}^{i_0} P_{0k}(y)\cdot D_k(y) \sim \frac{1}{1-y}, \qquad \text{as } y \longrightarrow 1-.$$

Thus, from equation (9) we have

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{i_0} P_{0k}(1-)\cdot\beta_k\right)(1-A_n(0))y^n \sim \frac{1}{1-y}, \qquad \text{as } y \longrightarrow 1-.$$

Hence, by the Tauberian theorem (see, e.g. Feller (1966) [3] p. 423),

$$\sum_{\nu=0}^{n} (1-A_\nu(0)) \sim \frac{n}{\sum_{k=0}^{i_0} \beta_k P_{0k}(1-)}, \qquad \text{as } n \longrightarrow \infty,$$

which is contradicting to equation (8).

Hence, we have $P_{00}(1-)=\infty$ and the proof of (ii) is complete.

(iii) In the supercritical case we have only to show that $P_{00}(1-)<\infty$. From lemma 1, $D_k(1-)=\infty$, and then we have

$$D_k(y) \sim [1-B_k(q)]q^k \frac{1}{1-y}, \qquad \text{as } y \longrightarrow 1-.$$

This implies that

$$P_{00}(y) \sim \frac{1}{1-y}\{1-y\sum_{k=0}^{i_0}(1-B_k(q))q^k P_{0k}(y)\}, \qquad \text{as } y \longrightarrow 1-.$$

Now we suppose that $P_{00}(1-)=\infty$, then $P_{0k}(1-)=\infty$ for any $k(k=0,1,\cdots)$. Therefore

$$\lim_{y\to 1-} \frac{1}{1-y}\{1-y\sum_{k=0}^{i_0}(1-B_k(q))\cdot q^k \cdot P_{0k}(y)\}=-\infty$$

whichis contradicting the assumption that $P_{00}(1-)=\infty$.

Hence we have $P_{00}(1-)<\infty$ and the proof of (iii) is complete. Thus the proof of the theorem is finished.

Corollary 1. *When* $\alpha\leqq 1$ *and* $\int_0^1 \frac{1-B_k(u)}{A(u)-u}du<\infty$ *for all* k $(k=0, 1, \cdots, i_0)$, *then the probability generating function,* $\Pi(x)$, *of the stationary distribution of the Markov chain* $\{X_n\}$ *satisfies the equation*

$$\Pi(x)=\Pi(A(x))-\sum_{k=0}^{i_0}(1-B_k(x))(A(x))^k\cdot\pi_k \tag{10}$$

where

$$\Pi(x)=\sum_{k=0}^{\infty}\pi_k x^k, \qquad (|x|\leqq 1).$$

Corollary 2. *In the supercritical case, even if* $i_0=\infty$, *the Morkov chain* $\{X_n\}$ *is transient.*

4. Continuously Subcritical Class

For the continuously subcritical (c. s. c.) class of offspring p.g.f.'s (Seneta (1969) [12]) we shall state an asymptotic relation that suggests an approximation to $\Pi(x)$ for α close to unity. The c.s.c. class of offspring distributions is defined by

(i) $A(x)=A(\alpha; x)$ is a p.g.f. for all $\alpha\in(1-\varepsilon, 1)$ for some $\varepsilon>0$ and $A(\alpha; x)\longrightarrow A(*; x)$ $(\alpha\uparrow 1; 0\leqq x\leqq 1)$, a proper p.g.f.,

(ii) $A''(*; 1-)=\tau^2>0$ and

(iii) $A'''(\alpha; 1-)<c\equiv\text{const.}$ for $\alpha\in(1-\varepsilon, 1)$.

Now we suppose

(iv) $B_k''(1-)<\infty$ for all $k=0, 1, \cdots, i_0$,

which implies that $\beta_k<\infty$, thus this assumption is sufficient condition for a stationary distribution to exist.

First, we have the following equations

$$\Pi'(1-)=\Big(\sum_{k=0}^{i_0}\pi_k\beta_k\Big)/(1-\alpha), \tag{11}$$

$$\Pi''(1-)=\frac{1}{1-\alpha^2}\sum_{k=0}^{i_0}\pi_k\Big[\frac{A''(1-)\beta_k}{1-\alpha}+B_k''(1-)+2k\alpha\beta_k\Big]. \tag{12}$$

For subsequenct use we now iotroduce the notation in terms of Laplace transforms:

$$\Pi(e^{-t})=\overline{\Pi}(t),\quad A(e^{-t})=\overline{A}(t), B_k(e^{-t})=\overline{B}_k(t) \qquad (k=0, 1, \cdots, i_0)$$

$$\overline{\Pi}_\delta(t)=\overline{\Pi}(t\delta)=\Pi(e^{-t(1-\alpha)})$$

where we have put $\delta=(1-\alpha)$. The results (11) and (12) enable us to assert that

$$\overline{\Pi}_\delta'(0+)=-\sum_{k=0}^{i_0}\pi_k\beta_k \tag{13}$$

$$\overline{\Pi}_\delta''(0+)=\frac{1-\alpha}{1+\alpha}\sum_{k=0}^{i_0}\pi_k\Big[\frac{A''(1-)\beta_k}{1-\alpha}+B_k''(1-)+2k\alpha\beta_k\Big]+(1-\alpha)\sum_{k=0}^{i_0}\pi_k\beta_k \tag{14}$$

It therefore follows that for all $t \geqq 0$, and for all δ sufficiently close to zero, under our conditions, we have the uniform bounds

$$(15)\qquad \begin{cases} -\sum_{k=0}^{i_0} \pi_k \beta_k \leqq \overline{\Pi}_\delta'(t) \leqq 0 \\ 0 \leqq \overline{\Pi}_\delta''(t) \leqq \chi \equiv \text{const.} \end{cases}$$

THEOREM 2. *Under the conditions (i)—(iv),*

$$\frac{1-\Pi[\exp\{-(1-\alpha)t\}]}{\sum_{k=0}^{i_0} \pi_k \beta_k} \longrightarrow \frac{2}{\tau^2} \log\left(1+\frac{1}{2}t\tau^2\right), \qquad \text{as } \alpha \longrightarrow 1-$$

where $0<t<\infty$.

Proof. We rewrite (10) as

$$(16)\qquad \overline{\Pi}_\delta(t)=\overline{\Pi}_\delta\left(-\frac{\log \overline{A}(\delta t)}{\delta}\right)-\sum_{k=0}^{i_0} \pi_k(1-\overline{B}_k(\delta t))(\overline{A}(\delta t))^k.$$

Now we arbitrarily chose $T>0$, which is finite, but fixed, then by the argument in Quine-Seneta (1969) [10] it follows from (15) that for $t \in [0, T]$ as $\delta \longrightarrow 0+$

$$(17)\qquad \overline{\Pi}_\delta\left(-\frac{\log \overline{A}(\delta t)}{\delta}\right)=\overline{\Pi}_\delta(t)-\left(1+\frac{1}{2}tu^2\right)\delta t \overline{\Pi}_\delta'(t)+\bar{o}(\delta)$$

where $\qquad u^2 \equiv A''(\alpha; 1-)+\alpha-\alpha^2,$

$$(18)\qquad \overline{B}_k(\delta t)=1-\beta_k \delta t+\bar{o}(\delta), \qquad (k=0, 1, \cdots, i_0), \text{ and}$$

$$(19)\qquad (\overline{A}(\delta t))^k=1-k\alpha\delta t+\bar{o}(\delta)$$

where the notation $\bar{o}(\delta^i)$ is used to signify $o(\delta^i)$ uniformly with respect to $t \in [0, T]$.

Hence from (17), (18) and (19), using (16), that

$$\overline{\Pi}_\delta(t)=\overline{\Pi}_\delta(t)-\left(1+\frac{1}{2}tu^2\right)\delta t \overline{\Pi}_\delta'(t)+\bar{o}(\delta)-\sum_{k=0}^{i_0} \pi_k(\beta_k \delta t+\bar{o}(\delta))(1-k\alpha\delta t+\bar{o}(\delta))$$

i.e.

$$\left(1+\frac{1}{2}tu^2\right)\overline{\Pi}_\delta'(t)=-\sum_{k=0}^{i_0} \pi_k \beta_k+\bar{o}(1),$$

for $t \in [0, T]$. By integrating this differential equation on $[0, T]$, we have

$$\overline{\Pi}_\delta(T)-1=-\sum_{k=0}^{i_0} \pi_k \beta_k \int_0^T \frac{1}{\left(1+\frac{1}{2}tu^2\right)} dt+\int_0^T \frac{\xi_t(\delta)}{\left(1+\frac{1}{2}tu^2\right)} dt$$

where we have put $\bar{o}(1)=\xi_t(\delta)$.

But

$$\left|\int_0^T \frac{\xi_t(\delta)}{\left(1+\frac{1}{2}tu^2\right)} dt\right| \leqq \varepsilon \int_0^T \frac{dt}{\left(1+\frac{1}{2}tu^2\right)}=\varepsilon \log\left(1+\frac{1}{2}u^2 t\right)^{\frac{2}{u^2}}$$

for arbitrary positive ε if δ is made sufficiently small.

Hence, since $u^2 \longrightarrow \tau^2$ as $\delta \longrightarrow 0$,

$$\lim_{\delta\to 0+} \frac{1-\overline{\Pi}_\delta(T)}{\sum_{k=0}^{i_0} \pi_k\beta_k} = \lim_{\delta\to 0+} \int_0^T \frac{1}{\left(1+\frac{1}{2}tu^2\right)}dt$$

$$= \frac{2}{\tau^2}\log\left(1+\frac{\tau^2}{2}T\right).$$

Now since $T>0$ is arbitrarily chosen, we have for each $t>0$

$$\frac{1-\overline{\Pi}_\delta(t)}{\sum_{k=0}^{i_0} \pi_k\beta_k} \longrightarrow \frac{2}{\tau^2}\log\left(1+\frac{2}{\tau^2}t\right), \qquad \text{as } \delta \longrightarrow 0+,$$

which completes the proof of the theorem.

5.A Limit Theorem for the Supercritical Process

We prove now a limit theorem for the supercritical process, which is an extension of Pakes' result.

Throughout this section we assume that $1<\alpha<\infty$.

LEMMA 2.

$$\sum_{k=0}^{i_0}\left[(1-B_k(q))q^k\sum_{n=0}^{\infty} p_{ik}^{(n)}\right]=q^i. \tag{20}$$

Proof. Note that in the supercritical case $P_{ik}(1-)<\infty$ for any i, $k(i, k=0, 1, \cdots)$. We have seen that if $\alpha>1$ then

$$D_k(y)\sim(1-B_k(q))q^k\frac{1}{1-y} \quad \text{and} \quad C_i(y)\sim\frac{q^i}{1-y}, \qquad \text{as } y\longrightarrow 1-.$$

Hence, from equation (4) we have

$$\sum_{k=0}^{i_0}\left[\left(\sum_{n=0}^{\infty} p_{ik}^{(n)}y^n\right)(1-B_k(q))q^k\frac{1}{1-y}\right]\sim\frac{q^i}{1-y}, \qquad \text{as } y\longrightarrow 1-.$$

This yields that

$$\sum_{k=0}^{i_0}\left[(1-B_k(q))q^k\sum_{n=0}^{\infty} p_{ik}^{(n)}\right]=q^i,$$

and the proof of the lemma is complete.

THEOREM 3. *Threr exists a sequence of positive constants* $\{c_n\}$ $(n=1, 2, \cdots)$ *with* $c_n\longrightarrow\infty$ *and* $c_n^{-1}\cdot c_{n+1}\longrightarrow\alpha$ *as* $n\longrightarrow\infty$, *such that* X_n/c_n *converges almost surely to a random variable* X, *with* $P(X=0)=0$, *whose Laplace transform* $\Phi(\theta)$ *is given by*

$$\Phi(\theta)=(\Psi(\theta))^i-\sum_{k=0}^{i_0}\left[\sum_{n=0}^{\infty}\left\{1-B_k\left(\Psi\left(\frac{\theta}{\alpha^{n+1}}\right)\right)\right\}\left(\Psi\left(\frac{\theta}{\alpha^n}\right)\right)^k p_{ik}^{(n)}\right], \quad (0\leqq\theta<\infty) \tag{21}$$

where $\Psi(\theta)$ satisfies the equation

$$\Psi(\alpha\theta)=A(\Psi(\theta)).$$

Let θ_0 be any fixed mumber in $(0, -\log q)$. Then, c_n can be taken as $(h_n(\theta_0))^{-1}$ where $h_n(\theta)$ is the inverse function of $k_n(\theta)=-\log A_n(e^{-\theta})$.

Proof. Let $\{c_n\}$ be as given in the statement of the theorem. Then, as shown by Seneta (1968) [11], $\{c_n\}$ has the properties given in the theorem.

Now let $\Psi(\theta)=\lim_{n\to\infty} A_n(e^{-\frac{\theta}{c_n}})$, $0\leqq\theta<\infty$, then $\Psi(\theta)$ satisfies

$$\Psi(\alpha\theta)=A(\Psi(\theta)),$$

and it follows that

$$A_{n-m}(e^{-\frac{\theta}{c_n}})\longrightarrow\Psi\left(\frac{\theta}{\alpha^{m+1}}\right), \qquad \text{as } n\longrightarrow\infty,$$

(see Seneta (1968)).

From equation (3), for $\theta\in[0, \infty)$

$$\text{(22)} \qquad P_i^{(n)}(e^{-\frac{\theta}{c_n}})=(A_n(e^{-\frac{\theta}{c_n}}))^i-\sum_{k=0}^{i_0}\left[\sum_{m=0}^{n-1}\{1-B_k(A_{n-m-1}(e^{-\frac{\theta}{c_n}}))\}(A_{n-m}(e^{-\frac{\theta}{c_n}}))^k p_{ik}^{(m)}\right].$$

Since $\sum p_{ik}^{(n)}<\infty$, so letting $n\longrightarrow\infty$ in equation (22), then we have equation (21), which shows that X_n/c_n converges in law to a random variable X.

Furthermore, using the equation (20), we obtain

$$\lim_{\theta\to\infty}\Phi(\theta)=0,$$

equivalenty,

$$P\{X=0\}=0.$$

It remains only to prove that X_n/c_n converges almost surely to a random variable X.

Denote by $B_{X_n}(x)$ the probability generating function of the immigrants entering in the $(n+1)$th generation, so that if $X_n\geqq i_0+1$, then $B_{X_n}(x)=1$ for $x\in[0, 1]$.

Now define the random variable $Y_n=\exp\{-h_n(\theta_0)X_n\}$ $(n=0, 1, \cdots)$. Then, it follows from the Markov property that

$$E(Y_{n+1}\mid Y_n,\cdots,Y_0)=E(Y_{n+1}\mid Y_n), \qquad (n=0, 1, \cdots).$$

From the difinition of $\{X_n\}$,

$$\begin{aligned} E(Y_{n+1}\mid Y_n) &= B_{X_n}\{\exp(-h_{n+1}(\theta_0))\}[A\{\exp(-h_{n+1}(\theta_0))\}]^{X_n} \\ &\leqq [A\{\exp(-h_{n+1}(\theta_0))\}]^{X_n} \\ &= [\exp\{-k(h_{n+1}(\theta_0))\}]^{X_n}=Y_n. \end{aligned}$$

Thus $\{Y_n\}$ is a bounded submartingale and so converges almost surely to some random variable.

This completes the proof of the theorem.

Acknowledgement

The author has much pleasure in thanking Professor Tetsuo Kaneko, Tetsuo Nakagawa and Hideyuki Fujihira for their encouragement and having given some advices.

NIIGATA UNIVERSITY

References

[1] ATHREYA, K. B. and NEY, P. E. (1972): Branching Processes. Springer Verlag Berlin.

[2] CHUNG, K. L. (1967): Markov Chains with Stationary Transition Probabilities. Springer Verlag Berlin.

[3] FELLER, W. (1966): An Introduction to Probability Theory and its Application. Vol. II. Wiley New York.

[4] FOSTER, J. H. (1971): *A limit theorem for a branching process with state-dependent immigration.* Ann. Math. Statist. 42, 1773–1776.

[5] FOSTER, J. H. and WILLIAMSON, J. A. (1971): *Limit theorems for the Galton-Watson process with time-dependent immigration.* Z. Wahr. verw Gev. 20, 227–235.

[6] HEATHCOTE, C. R. (1966): *Corrections and comments to the paper "A branching process allowing immigration."* J. R. Statist. Soc. B28, 213–217,

[7] KESTEN, H., NEY, P. E. and SPITZER, R. (1966): *The Galton-Watson process with mean one and finite variance.* Teor. Veroyat. Primen. 11, 579–611.

[8] NAKAGAWA, T. and SATO, M. (1974): *A Galton-Watson process with state dependent immigration.* Res. Rep. Nagaoka Tech. Coll. 9, 177–182.

[9] PAKES, A. G. (1971): *A branching process with a state-dependent immigration component.* Adv. Appl. Pord 3, 301–314.

[10] QUINE, M. P. and SENETA, E. (1969): *A limmitt theorem for the Galton-Watson process with immigration.* Aus. J. Statist. 11, 166–173.

[11] SENETA, E. (1968): *On recent theorems concerning the supercritical Galton-Watson process.* Ann. Math. Statist. 39, 2098–2102.

[12] SENETA, E. (1969): *Functional equations and the Galton-Waton process.* Adv. Prob. 1, 1–43.

[13] SENETA, E. (1970): *On the supercritical Galton-Watson process with immigration.* Math. Biosciences. 7, 9–14.