On the asymptotic distribution of eigenvalues of non-symmetric operators associated with strongly elliptic sesquilinear forms

By

Michiaki WATANABE

(Received October 29, 1973)

1. Introduction

The asymptotic behaviour of eigenvalues of elliptic operators has been investigated by a number of authors. Combining an asymptotic expansion theorem for resolvent kernels, established by S. Agmon and Y. Kannai [3], and Malliavin's tauberian theorem together with the formula of Å. Pleijel [8], S. Agmon [2] deduced asymptotic formulas with remainder estimates for eigenvalues of operators whose coefficients are infinitely differentiable. Recently K. Maruo and H. Tanabe [7] have devised a method of estimating resolvent kernels of a class of operators with wider domains and improved the results of S. Agmon to obtain estimates for remainders in the asymptotic formulas under less smoothness assumptions on the coefficients of operators. And these results were extended to the case of some non-symmetric operators and the remainder estimates were strengthened further by K. Maruo [6]. We refer to the above results for our discussion.

The above authors have, however, always confined themselves to the case where operators dealt with are self-adjoint in some sense or other. From the viewpoint of pure theory at least we consider it desirable to extend the above results to more general non-symmetric cases. The purpose of this paper is to deal with operators which are not necessarily symmetric but satisfy the condition (3) stated below and to deduce asymptotic formulas for their eigenvalues slightly different from those obtained by the above authors.

Let Ω be a bounded domain in the real space R^n. We denote by $H_m(\Omega)$ for an integer $m\geqq 0$ the subclass of functions $u\in L^2(\Omega)$ with *distribution derivatives* $D^\alpha u\in L^2(\Omega)$ for all $|\alpha|\leqq m$, where and in the following $\alpha=(\alpha_1,\dots, \alpha_n)$ is a multiindex of length $|\alpha|=|\alpha_1|+\cdots+|\alpha_n|$ and $D^\alpha=D_1{}^{\alpha_1}\dots D_n{}^{\alpha_n}$, $D_k=(\partial/\partial x_k)$, $k=1,\dots, n$.

In $H_m(\Omega)$ we introduce as usual the inner product and the norm:

$$(u, v)_m=(u, v)_{m,\Omega}=\Big(\int_\Omega \sum_{|\alpha|\leqq m} D^\alpha u\,\overline{D^\alpha v}\,dx\Big)^{1/2};\ \|u\|_m=\|u\|_{m,\Omega}=((u, u)_m)^{1/2}.$$

$\mathring{H}_m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H_m(\Omega)$.

Let B be an integro-differential sesquilinear form of order m

$$B[u, v]=\int_\Omega \sum_{|\alpha|,|\beta|\leqq m} a_{\alpha\beta}(x)\, D^\alpha u\, \overline{D^\beta v}\, dx$$

and satisfy

(1) $\quad \mathrm{Re}\, B[u, u]\geqq\delta\|u\|_m$ for any $u\in v$

for some positive constant δ independent of u, where v is a closed subspace of $H_m(\Omega)$ containing $\mathring{H}_m(\Omega)$. We assume that

$$\text{for } |\alpha|, |\beta|\leqq m \quad a_{\alpha\beta} \text{ belongs to } L^\infty(\Omega)$$

which implies that

(2) $\quad |B[u, v]|\leqq K\|u\|_m\|v\|_m$ for any $u, v\in H_m(\Omega)$,

where K is some positive constant independent of u, v.

Let A be the operator associated with B: an element u of v belongs to the domain $D(A)$ of A and $Au=f\in L^2(\Omega)$ *if* $B[u, v]=(f, v)_0$ is valid for any $v\in V$. It is easy to see that the spectrum $\sigma(A)$ of A is contained in a sector

$$\Gamma=\{\lambda: |\arg\lambda|\leqq\theta\},\ 0\leqq\theta\leqq\cos^{-1}(\delta/K)$$

of the complex plane. In this paper we investigate the asymptotic distribution of eigenvalues of this A, or more precisely, the behaviour as $t\rightarrow\infty$ of the number $N(t)$ of eigenvalues whose real parts do not exceed t.

The assumptions (1), (2) themselves imply that

$$B[u, u] \text{ belongs to } \Gamma \text{ for any } u\in V. \tag{1. 1}$$

With the aid of this fact we obtain an estimate for the resolvent kernel of A in a similar way to that in the case of self-adjoint operators.

In order to study the behaviour of $N(t)$, however, we need some modification since eigenvalues of A distribute in the sector Γ. Motivated by this problem we have tried to generalize the tauberian theorem by P. Malliavin [5] and deduced the asymptotic behaviour of a positive measure μ defined on a sector

$$\Gamma_0=\{\lambda: |\arg\lambda|\leqq\theta\},\quad 0\leqq\theta<\pi/4$$

in the complex plane from the behaviour of the transform

$$\int_{\Gamma_0}\frac{\mu(d\lambda)}{\lambda-z},\quad z\notin\Gamma_0$$

along a curve. Lemma 5.1 is a simple consequence of our recent result [9] and plays the most important role in the present paper. To apply this lemma we must assume in addition to (1), (2) that B satisfies

(3) $\qquad B[u, u]\in\Gamma_0$ for any $u\in v$.

From (1.1) it is easy to see that this condition holds good when δ and K in (1), (2) satisfy $\sqrt{2}\,K<\delta\leqq K$. It is also evident that (3) holds for $\theta=0$ if B is symmetric.

To clearify our intention we restrict ourselves in this paper to the case where

(4) $\qquad$ for $|\alpha|=|\beta|=m$, $a_{\alpha\beta}$ is uniformly Hoelder continuous of order h.

Making some other assumptions on Ω itself and following the method of the above authors, we obtain an estimate of $N(t)$:

as $t\to\infty$

$$\left|N(t)-\mathrm{Re}\,C_0\left\{\sin\left(\frac{n}{2m}(\pi-\varphi)\right)\Big/\left(\frac{n\pi}{2m}\right)-C_1\sin\theta\right\}\sec^{n/2m}2\theta\; t^{n/2m}\right|$$

$$\leqq \mathrm{Re}\,C_0\,C_2\sin\theta\sec^{n/2m}2\theta\; t^{n/2m}+O(t^{n/2m-(h-\sigma(h+2))/4m})\sqrt{\sec 2\theta-1}+O(t^{(n-\sigma)/2m})$$

for $0<\sigma<h/(h+2)$, where $\varphi=\tan^{-1}\sqrt{\sec 2\theta-1}$ and C_0, C_1 and C_2 are some constants given in Theorem I below.

This estimate for $N(t)$ may be vague unless θ is small enough to satisfy

$$(C_1+C_2)\sin\theta<\sin\left(\frac{n}{2m}(\pi-\varphi)\right)\Big/\left(\frac{\pi n}{2m}\right)$$

but still an extention of some part of the above authors' results where θ was always assumed to be zero or sufficiently small.

2. Main result

Throughout this paper we assume that Ω is a bounded domain in R^n which possesses the *restricted cone property* (p. 11 of S. Agmon [1]) and such that $\dim\Omega<2m$. Writing $\delta(x)=\min(1, \mathrm{dist}(x, \partial\Omega))$ for $x\in\Omega$, we assume that

(5) $\qquad \int_\Omega \delta(x)^{-p}\,dx<\infty$ for some positive number $p<1$

which will be specified later.

THEOREM I. *Under the assumptions* (1)−(5) *the following asymptotic formulas for* $N(t)$ *hold as* $t\to\infty$:

$$\left|N(t)-\mathrm{Re}\,C_0\left\{\sin\left(\frac{n}{2m}(\pi-\varphi)\right)\Big/\left(\frac{\pi n}{2m}\right)-C_1\sin\theta\right\}\sec^{n/2m}2\theta\, t^{n/2m}\right|$$

$$\leqq \mathrm{Re}\,C_0\,C_2\sin\theta\sec^{n/2m}2\theta\; t^{n/2m}+O(t^{n/2m-(h-\sigma(h+2))/4m})\sqrt{\sec 2\theta-1}+O(t^{(n-\sigma)/2m})$$

for $0<\sigma<h/(h+2)$;

$$\left|N(t)-\mathrm{Re}\, C_0\left\{\sin\left(\frac{n}{2m}(\pi-\varphi)\right)\Big/\left(\frac{\pi n}{2m}\right)-C_1\sin\theta\right\}\sec^{n/2m}2\theta\ t^{n/2m}\right|$$

$$\leqq \mathrm{Re}\, C_0\, C_2 \sin\theta\ \sec^{n/2m} 2\theta\ t^{n/2m}+O(t^{(n-\sigma)/2m})$$

for $0<\sigma\leqq h/(h+3)$, *where* $\varphi=\tan^{-1}\sqrt{\sec 2\theta-1}$,

$$\begin{cases} C_0=(2\pi)^{-n}\int_\Omega dx\int_{R^n}\Big(\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}+1\Big)^{-1}d\xi, \\ C_1=\sqrt{2}\cos((1-n/2m)(\pi-\varphi))/\pi, \\ C_2=\sqrt{2}\sin((1-n/2m)(\pi-\varphi))/\cos(\theta+\pi/4). \end{cases}$$

REMARK 1. If A is self-adjoint, then the above formulas hold true for $\theta=0$ and are written as

$$N(t)=C_0\sin\left(\frac{\pi n}{2m}\right)\Big/\left(\frac{\pi n}{2m}\right)t^{n/2m}+O(t^{(n-\sigma)/2m})$$

for $0<\sigma<h/(h+2)$ and $0<\sigma\leqq h/(h+3)$ respectively. The latter formula coincides with that obtained by K. Maruo and H. Tanabe [7] or by R. Beals [4]. The former has been given by K. Maruo [6] under the assumption that $\theta>0$ is sufficiently small.

REMARK 2. In case when δ and K in (1), (2) satisfy $\sqrt{2}\ K<\delta\leqq K$, the condition (3) always holds with $\theta=\cos^{-1}(\delta/K)$ as is easily verified. Therefore we have the above formulas with θ replaced by $\cos^{-1}(\delta/K)$.

REMARK 3. The second formula can be written as

$$\mathrm{Re}\, C_0\left\{\sin\left(\frac{n}{2m}(\pi-\varphi)\right)\Big/\left(\frac{\pi n}{2m}\right)-(C_1+C_2)\sin\theta\right\}\sec^{n/2m}2\theta\ t^{n/2m}+O(t^{(n-\sigma)/2m})$$

$$\leqq N(t)\leqq \mathrm{Re}\, C_0\left\{\sin\left(\frac{n}{2m}(\pi-\varphi)\right)\Big/\left(\frac{\pi n}{2m}\right)-(C_1-C_2)\sin\theta\right\}\sec^{n/2m}2\theta\, t^{n/2m}+O(t^{(n-\sigma)/2m})$$

as $t\to\infty$. In view of this we find that $N(t)$ is estimated not only from above but from below and the estimate is strengthened to some sxtent as long as θ satisfies in addition to the condition $0\leqq\theta<\pi/4$

$$(C_1+C_2)\sin\theta < \sin\left(\frac{n}{2m}(\pi-\varphi)\right)\Big/\left(\frac{\pi n}{2m}\right).$$

A similar thing holds for the first formula.

3. Preliminary results on A

In this section we study fundamental properties of the operator A associated with B under the assumptions (1) and (2). Identifying $L^2(\Omega)$ with its antidual we consider as usual $V\subset L^2(\Omega)\subset V^*$ algebraically and topologically and extend A to a mapping from V to the antidual V^* of V since it is not necessarily assumed that $D(A)\subset H_{2m}(\Omega)$. This extended operator denoted again by A is defined by

$$B[u, v]=\langle Au, v\rangle_V \text{ for any } v\in V$$

where $\langle f, v\rangle_V$ astands for the duality between V^* and V. The resolvent of A is a bounded linear operator on V^* to V or on $L^2(\Omega)$ to V and its norm is denoted by $\|(A-\lambda)^{-1}\|_{V^*\to V}$ or $\|(A-\lambda)^{-1}\|_{L^2\to V}$.

LEMMA 3.1. *Eigenvalues* $\{\lambda_j\}_{j=0}^{\infty}$ *of* A *which have finite multiplicity distribute in the sector*

$$\Gamma=\{\lambda:\ |\arg\lambda|\leqq\theta\},\quad 0\leqq\theta\leqq\cos^{-1}(\delta/K)$$

and can have only ∞ *as a limit point.*

PROOF. We have only to show that $\sigma(A)$ is contained in Γ and to recall Rellich's theorem. Noting (1.1) and putting $d(\lambda, \Gamma)=\text{dist}(\lambda, \Gamma)$, we have for $\lambda\notin\Gamma$ and $u\in D(A)$

$$|((A-\lambda)u, u)|=|B[u, u]-\lambda\|u\|^2|\geqq d(\lambda, \Gamma)\|u\|^2,$$

that is, $$\|(A-\lambda)u\|\geqq d(\lambda, \Gamma)\|u\|.$$

Here and later on we use (u, v), $\|u\|$ to denote $(u, v)_0$, $\|u\|_0$ respectively. On the other hand for the adjoint operator A^* associated with the sesquilinear form $B^*[u, v]=\overline{B}[v, u]$ a similar estimate holds:

$$\|(A^*-\overline{\lambda})u\|\geqq d(\lambda, \Gamma)\|u\| \quad \text{for any } \lambda\notin\Gamma \quad \text{and} \quad u\in D(A^*),$$

which implies that the null space of $(A-\lambda)^*$ consists only of 0 and hence that the resolvent set $\rho(A)$ of A contains the complement of Γ.

We next have for $\lambda\notin\Gamma$ and $u\in V$

$$\|(A-\lambda)u\|_{V^*}\|u\|_m\geqq|\langle(A-\lambda)u, u\rangle_V|=|B[u, u]-\lambda\|u\|^2|,$$

where $|B[u, u]-\lambda\|u\|^2|$ dominates

$$|B[u, u]|\sin(|\arg\lambda|-\theta)=|B[u, u]|d(\lambda, \Gamma)/|\lambda| \text{ if } \theta<|\text{ang}\,\lambda|\leqq\theta+\pi/2$$

and

$$|B[u, u]|\geqq|B[u, u]|d(\lambda, \Gamma)/|\lambda| \quad \text{if} \quad \theta+\pi/2<|\arg\lambda|\leqq\pi.$$

Hence by virtue of (1) we obtain

$$\|(A-\lambda)u\|_{V^*}\geqq\delta\|u\|_m d(\lambda, \Gamma)/|\lambda| \quad \text{for} \quad \lambda\notin\Gamma \quad \text{and} \quad u\in V$$

and similarly

$$\|(A-\lambda)^*u\|_{V^*}\geqq\delta\|u\|_m\, d(\lambda, \Gamma)/|\lambda| \text{ for } \lambda\notin\Gamma \text{ and } u\in V.$$

Consequently $\rho(A)$ contains the complement of Γ in this case too. Q. E. D.

LEMMA 3. 2. *If* $\lambda\notin\Gamma$, *then*

i) $\|(A-\lambda)^{-1}f\|_m+|\lambda|^{1/2}\|(A-\lambda)^{-1}f\|\leqq C|\lambda|^{1/2}\|f\|/d(\lambda,\Gamma)$ for $f\in L^2(\Omega)$,

ii) $\|(A-\lambda)^{-1}f\|_m+|\lambda|^{1/2}\|(A-\lambda)^{-1}f\|\leqq C|\lambda|\,\|f\|_{V^*}/d(\lambda,\Gamma)$ for $f\in V^*$,

where C is a positive constant.

PROOF. Putting $u=(A-\lambda)^{-1}f$ for $f\in L^2(\Omega)$, we have

$$\|f\|\,\|u\|\geqq|((A-\lambda)u,u)|=|B[u,u]-\lambda\|u\|^2|.$$

As was seen in the proof of the preceeding lemma, it generally holds that

$$|B[u,u]-\lambda\|u\|^2\geqq\begin{cases} d(\lambda,\Gamma\|u\|^2| \\ B[u,u]d(\lambda,\Gamma)/|\lambda| \end{cases} \quad \text{for any } \lambda\notin\Gamma \text{ and } u\in V, \tag{3. 1}$$

from which

$$C|B[u,u]-\lambda\|u\|^2|\geqq(\|u\|_m+|\lambda|^{1/2}\|u\|)^2d(\lambda,\Gamma)/|\lambda| \tag{3. 2}$$

follows. Thus we conclude

$$\|u\|_m+|\lambda|^{1/2}\|u\|\leqq C\|f\|\,|\lambda|^{1/2}/d(\lambda,\Gamma).$$

Similarly we have for $u=(A-\lambda)^{-1}f, f\in V^*$

$$\|f\|_{V^*}\|u\|_m\geqq|<(A-\lambda)u,u>_V|=|B[u,u]-\lambda\|u\|^2|.$$

Combining this with (3. 2) we get

$$\|u\|_m+|\lambda|^{1/2}\|u\|\leqq C\|f\|_{V^*}\|\lambda|/d(\lambda,\Gamma),$$

which completes the proof.

LEMMA 3. 3. *There exists a constant C such that for any integers* $0\leqq k\leqq m$

i) $\|(A-\lambda)^{-1}f\|_k\leqq C|\lambda|^{k/2m}\|f\|/d(\lambda,\Gamma)$ *for* $f\in L^2(\Omega)$,

ii) $\|(A-\lambda)^{-1}f\|_k\leqq C|\lambda|^{1/2+k/2m}\|f\|_{V^*}/d(\lambda,\Gamma)$ *for* $f\subset V^*$.

PROOF. This is a simple consequence of the preceeding lemma and the following lemma.

LEMMA 3. 4. *For any* $u\in H_m(\Omega)$, *integer* $0\leqq k\leqq m$ *and positive number M*

$$\|u\|_k\leqq CM^{-(m-k)/2m}(\|u\|_m+M^{1/2}\|u\|),$$

where C is a positive constant independent of u, k and M.

PROOF. Under our assumption on Ω the following interpolation inequality holds

$$\|u\|_k\leqq C\|u\|_m^{k/m}\|u\|^{1-k/m} \text{ for any } u\in H_m(\Omega).$$

From this inequality and Young's inequality it follows that

$$\|u\|_k = C(M^{-(m-k)/2m}\|u\|_m)^{k/m}(M^{k/2m}\|u\|)^{1-k/m}$$
$$\leqq CM^{-(m-k)/2m}(\|u\|_m + M^{1/2}\|u\|).$$

REMARK. This lemma is also true in the case where $\Omega = R^n$. In fact, from

$$\int_{R^n}(1+|\xi|)^{2k}|\hat{u}(\xi)|^2\,d\xi \leqq \left(\int_{R^n}(1+|\xi|)^{2m}|\hat{u}(\xi)|^2 d\xi\right)^{k/m}\left(\int_{R^n}|\hat{u}(\xi)|^2 d\xi\right)^{1-k/m}$$

it follows that

$$\|u\|_{k,R^n} \leqq C\|u\|_{m,R^n}^{k/m}\ \|u\|_{0,R^n}^{1-k/m} \quad \text{forany } u\in H_m(R^n) \text{ and } 0\leqq k\leqq m.$$

The following lemma has been proved by K. Maruo and H. Tanabe [7] and plays essential roles also in the present paper. With the aid of this lemma we find that the resolvent kernel $K_\lambda(x, y)$ of A is continuous in $\bar{\Omega}\times\bar{\Omega}$ and estimated as

$$|K_\lambda(x, y)| \leqq C|\lambda|^{n/2m}/d(\lambda, \Gamma) \quad \text{for any } x, y\in\Omega \text{ and } \lambda\notin\Gamma. \tag{3. 3}$$

LEMMA 3. 5. *A bounded operator S on* V^* to V *has a kernel* M:

$$(Sf)(x) = \int_\Omega M(x, y)f(y)dy \quad \text{for} \quad f\in L^2(\Omega).$$

$M(x, y)$ *is continuous in* $\bar{\Omega}\times\bar{\Omega}$ *and there exists a constant C such that for any* $x, y\in\Omega$

$$|M(x, y)| \leqq C\|S\|_{V^*\to V}^{n^2/4m^2}\ \|S\|_{V^*\to L^2}^{n/2m-n^2/4m^2}\ \|S\|_{L^2\to V}^{n/2m-n^2/4m^2}\ \|S\|_{L^2\to L^2}^{(1-n/2m)^2}.$$

Finally we shall deal with the properties of

$$A'(x, \xi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}$$

which we need for later discussion. Under the assumptions (1) and (2) we have

LEMMA 3. 6. *For almost all* $x\in\Omega$ *and for all real vector* $\xi\in R^n$

i) $\mathrm{Re}\, A'(x, \xi) \geqq \delta \sum_{|\alpha|\leqq m} \xi^{2\alpha}$,

ii) $A'(x, \xi)$ *belongs* to Γ.

PROOF. The statement i) is well known. See Theorem 7. 12 of S. Agmon [1] for example. The proof of ii). Because of (1. 1) for any $v\in C_0^\infty(\Omega)$ $B[v, v]$ belongs to Γ and hence satisfies by (3. 1)

$$|B[v, v]-\lambda\|v\|^2| \geqq d(\lambda,\Gamma)\|v\|^2 \quad \text{for any } v\in C_0^\infty(\Omega) \text{ and } \lambda\notin\Gamma.$$

Put $v_t(x) = e^{it\cdot x}v(x)$ for an arbitrary $v\in C_0^\infty(\Omega)$ and $t>0$. Then v_t belongs to $C_0^\infty(\Omega)$ and satisfies the inequality just above. Hence letting λ be such that $|\lambda| = t^{2m}$, we have

$$\int_\Omega |v(x)|^2 dx \leqq \left|\int_\Omega A'(x, \xi)t^{2m}|v(x)|^2 dx - \int_\Omega \lambda|v(x)|^2 dx + O(t^{2m-1})\right| \Big/ d(\lambda, \Gamma)$$

$$\leqq \left\{\int_\Omega |A'(x, \xi) - e^{i\arg\lambda}|\ |v(x)|^2 dx + O(t^{-1})\right\} \Big/ C_\lambda,$$

where

$$C_\lambda=\begin{cases}\sin(|\arg\lambda|-\theta) & \text{if} \quad \theta<|\arg\lambda|\leqq\theta+\pi/2,\\ 1 & \text{if} \quad \theta+\pi/2<|\arg\lambda|\leqq\pi.\end{cases}$$

Letting $t\rightarrow\infty$ we have for any $v\in C_0^\infty(\Omega)$

$$\int_\Omega\left\{|A'(x,\xi)-e^{i\arg\lambda}|-C_\lambda\right\}|v(x)|^2\,dx\geqq 0,$$

which holds also for any $v\in L^2(\Omega)$ since $a_{\alpha\beta}\in L^\infty(\Omega)$.

Thus, for almost all $x\in\Omega$ it holds that

$$|A'(x,\xi)-e^{i\arg\lambda}|\geqq C_\lambda,$$

where C_λ is positive as long as $\lambda\notin\Gamma$. Hence $A'(x,\xi)$ must belong to Γ. *Q. E. D.*

4. Estimation of resolvent kernel of A

In this section we estimate $K_\lambda(x, x)-C_0(x)(-\lambda)^{n/2m-1}$, $C_0(x)=(2\pi)^{-n}\int_{R^n}(A'(x,\xi)+1)^{-1}d\xi$ under the assumptions (1), (2) and (4).

To this end we first estimate the difference between $K_\lambda(x, y)$ and the resolvent kernel $K_\lambda^0(x, y)$ of the operator A_0 associated with B under the Dirichlet boundary condition. Replacing V, V^* by $\mathring{H}_m(\Omega)$, the antidual $H_{-m}(\Omega)$ of $\mathring{H}_m(\Omega)$ respectively in the definition of A, we can define A_0 by a quite similar method:

$$B[u, v]=\langle A_0 u, v\rangle_m \quad \text{for any} \quad u, v\in\mathring{H}_m(\Omega),$$

where $\langle f, v\rangle_m$ stands for the pairing between $H_{-m}(\Omega)$ and $\mathring{H}_m(\Omega)$. As is easily seen the resolvent of A_0 is a bounded linear operator on $H_{-m}(\Omega)$ to $\mathring{H}_m(\Omega)$ or on $L^2(\Omega)$ to $\mathring{H}_m(\Omega)$ and the spectrum $\sigma(A_0)$ is contained in Γ. Obviously for this A_0 the analogues of Lemmas 3. 2 and 3. 3 hold true with the definition of unchanged.

LEMMA 4. 1. *For any positive p there exists a constant C_p depending only on p but not on x and λ such that*

$$|K_\lambda(x, x)-K_\lambda^0(x, x)|\leqq C_p\frac{|\lambda|^{n/2m}}{d(\lambda,\Gamma)}\left(\frac{|\lambda|^{1-1/2m}}{\delta(x)\,d(\lambda,\Gamma)}\right)^p$$

for any $x\in\Omega$ and $\lambda\notin\Gamma$ with $|\lambda|\geqq 1$.

PROOF. We denote by Λ a class of functions infinitely differentiable in R^n whose supports are contained in the set $\{x\in R^n\colon |x|<1\}$ and which equal to 1 at the origin. We put $\xi_r(x)=\xi((x-x_0)/r)$, $r=\delta(x_0)$ for $\xi\in\Lambda$ and an arbitrarily fixed $x_0\in\Omega$ and define the operator $S_{\lambda r}$ by

$$S_{\lambda r}f=\xi_r\{(A-\lambda)^{-1}f-(A_0-\lambda)^{-1}(rf)\} \quad \text{for any } f\in V^*,$$

where rf is the restriction of $f\in V^*$ to $\mathring{H}_m(\Omega)$. Evidently $S_{\lambda r}$ is a bounded operator on

V^* to $\overset{\circ}{H}_m(\Omega)$ and hence to V.

Putting $u=(A-\lambda)^{-1}f-(A_0-\lambda)^{-1}(rf)$, $v=\xi_r u$, we find

$$B[v, v]-\lambda(v, v)=B[v, v]-B[u, \xi_r v].$$

Clearly

$$|D^\gamma \xi_r(x)| \leqq C r^{-|\gamma|} \tag{4. 1}$$

with a constant C independent of x, x_0. With the aid of these things and the results given in the preceeding section, we prove:

For any non-negative integer j there exists a constant C_j depending only on j but not on x_0, λ such that

i) $$\|v\|_m+|\lambda|^{1/2}\|v\| \leqq C_j \frac{|\lambda|^{1/2}}{d(\lambda, \Gamma)}\left(\frac{|\lambda|^{1-1/2m}}{rd(\lambda, \Gamma)}\right)^j \|f\|, \quad f\in L^2(\Omega),$$

ii) $$\|v\|_m+|\lambda|^{1/2}\|v\| \leqq C_j \frac{|\lambda|}{d(\lambda, \Gamma)}\left(\frac{|\lambda|^{1-1/2m}}{rd(\lambda, \Gamma)}\right)^j \|f\|_{V^*}, \quad f\in V^*$$

for $\lambda \notin \Gamma$ with $r^{-1}|\lambda|^{-1/2m} \leqq 1$.

For $j=0$ these inequalities hold good because of Lemma 3. 2 and the fact $\|rf\|_{-m} \leqq \|f\|_{V^*}$.

We pick another function $\eta \in A$ such that $\eta(x)=1$ for any $x\in \mathrm{supp}\,\xi$ and denote $\eta_r(x)=\eta((x-x_0)/r)$. From (4. 1) it follows that

$$|B[v, v]-\lambda(v, v)|=|B[v, v]-B[u, \xi_r v]|$$

$$\leqq \left|\int_\Omega \sum_{|\alpha|, |\beta| \leqq m} a_{\alpha\beta}(x) \sum_{\alpha>\gamma} \binom{\alpha}{\gamma} D^{\alpha-\gamma}\xi_r D^\gamma(\eta_r u) \overline{D^\beta v}\, dx\right|$$

$$+\left|\int_\Omega \sum_{|\alpha|, |\beta| \leqq m} a_{\alpha\beta}(x) \sum_{\beta>\gamma} \binom{\beta}{\gamma} D^\alpha(\eta_r u) D^{\beta-\gamma}\xi_r \overline{D^\gamma v}\, dx\right|$$

$$\leqq C\sum_{k=0}^{m-1} r^{k-m}\|\eta_r u\|_k \|v\|_m + C\sum_{k=0}^{m-1} r^{k-m}\|\eta_r u\|_m \|v\|_k .$$

Using Lemma 3. 4 and nothing $r^{-1}|\lambda|^{-1/2m} \leqq 1$, we get

$$|B[v, v]-\lambda(v, v)|$$

$$\leqq C r^{-1}|\lambda|^{-1/2m}(\|\eta_r u\|_m+|\lambda|^{1/2}\|\eta_r u\|)(\|v\|_m+|\lambda|^{1/2}\|v\|)$$

and hence by (3. 2)

$$\|v\|_m+|\lambda|^{1/2}\|v\| \leqq C\frac{|\lambda^{1-1/2m}}{rd(\lambda, \Gamma)}(\|\eta_r u\|_m+|\lambda|^{1/2}\|\eta_r u\|).$$

Thus assuming that i) and ii) hold for some k with η in place of ξ, we have them for $k+1$.

Applying Lemma 3.5 to the kernel $\xi_r(x)(K_\lambda(x, y)-K_\lambda^0(x, y))$ of $S_{\lambda r}$ we have for λ with $r^{-1}|\lambda|^{-1/2m}\leqq 1$

$$|K_\lambda(x_0, x_0)-K_\lambda^0(x_0, x_0)|\leqq C_j \frac{|\lambda|^{n/2m}}{d(\lambda, \Gamma)}\left(\frac{|\lambda|^{1-1/2m}}{rd(\lambda, \Gamma)}\right)^j. \tag{4. 2}$$

On the other hand from (3. 3) it follows that

$$|K_\lambda(x_0, x_0)-K_\lambda^0(x_0, x_0)|\leqq C|\lambda|^{n/2m}/d(\lambda, \Gamma).$$

Since $d(\lambda, \Gamma)\leqq|\lambda|$ it follows from this inequality that (4. 2) holds in a trivial way also when $r^{-1}|\lambda|^{-1/2m}\geqq 1$, $|\lambda|\geqq 1$.

Thus we have proved (4. 2) for all integers $\geqq 0$. That (4. 2) also holds for non-integral values of j is obvious. Q. E. D.

We next estimate $K_\lambda^0(x, y)$. Let us consider the sesquilinear form:

$$B'[u, v]=\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_0)\int_{R^n} D^\alpha u\, \overline{D^\beta v}\, dx,$$

where $x_0\in R^n$ is an arbitrarily fixed point.

LEMMA 4. 2. *B' satisfies*

i) $\operatorname{Re} B'[u, u]\geqq\delta'\|u\|^2_{m, R^n}-C_3\|u\|^2_{0, R^n}$,

ii) $B'[u, u]\in\Gamma$

for any $u\in H_m(R^n)$, where $\delta'>0$ and $C_3\geqq 0$ are constants independent of u, x_0.

PROOF. We use Lemma 3.6 which is true for any $x_0\in\Omega$ and $\xi\in R^n$ in the present case. We have only to show that the lemma holds for any $v\in C_0^m(R^n)$.

i) is well known. The proof of ii). We have for any $v\in C_0^m(R^n)$

$$|\operatorname{Im} B'[v, v]|=|\operatorname{Im}\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_0)\int_{R^n} D^\alpha v\, \overline{D^\beta v}\, dx|$$

$$\leqq\int_{R^n}|\operatorname{Im} A'(x_0, \xi)|\,|\hat{v}(\xi)|^2 d\xi\leqq\int_{R^n}\tan\theta \operatorname{Re} A'(x_0, \xi)|\hat{v}(\xi)|^2 d\xi$$

$$=\tan\theta \operatorname{Re}\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_0)\int_{R^n} D^\alpha v\, D^\beta v\, dx = \tan\theta \operatorname{Re} B'[v, v],$$

where $\hat{v}(\xi)$ stands for the Fourier transform:

$$\hat{v}(\xi)=(2\pi)^{-n/2}\int_{R^n} v(x)\, e^{-i\xi\cdot x}\, dx.$$

That is,

$$|\arg B'[v, v]|\leqq\theta$$

for any $v\in C_0^m(R^n)$. Q. E. D.

For a function $u\in\mathring{H}_m(\Omega)$ let $\tilde{u}=u$ in Ω and $\tilde{u}=0$ in $R^n-\Omega$. Then $\tilde{u}$ belongs to

$H_m(R^n)$ and $\mathring{H}_m(\Omega)$ may be considered as a closed subspace of $H_m(R^n)$ by this correspondance.

Let A_1 be the operator associated with the sesquilinear form:

$$B^1[u, v]=B'[u, v]+C_3(u, v)_{0, R^n}$$

restricted to $\mathring{H}_m(\Omega)\times\mathring{H}_m(\Omega)$. By definition we have

$$B^1[u, v]=\langle A_1u, v\rangle_m \quad \text{for} \quad u, v\in\mathring{H}_m(\Omega).$$

Because of Lemma 4. 2 $B^1[u, u]=B'[u, u]+C_3\|u\|^2$ belongs to Γ for any $u\in\mathring{H}_m(\Omega)$ and hence the analogues of Lemmas 3. 2 and 3. 3 hold also for A_1 with Γ unchanged.

We are now in a position to estimate the difference between $K_\lambda^0(x, y)$ and the resolvent kernel $K_\lambda^1(x, y)$ of A_1.

LEMMA 4. 3. *For any positive integer j there is a constant C_j depending only on j but not on x, λ and ε such that*

$$|K_\lambda^0(x, x)-K_\lambda^1(x, x)|$$

$$\leqq C_j(|\lambda|^{-1/2m}+\varepsilon^h)|\lambda|^{n/2m-1}|\lambda|^2/d(\lambda, \Gamma)^2+C_j\frac{|\lambda|^{n/2m}}{d(\lambda, \Gamma)}\left(\frac{|\lambda|^{1-1/2m}}{\varepsilon d(\lambda, \Gamma)}\right)^j$$

for $\lambda\notin\Gamma$ with $\varepsilon^{-1}|\lambda|^{1-1/2m}/d(\lambda, \Gamma)\leqq 1$.

PROOF. Putting $\xi_\varepsilon(x)=\xi((x-x_0)/\xi)$ for $\xi\in\Lambda$, $x_0\in\Omega$ and an arbitrary $\xi>0$, we define an operator $S_{\lambda\varepsilon}$ as follows:

$$S_{\lambda\varepsilon}f=\xi_\varepsilon\{(A_0-\lambda)^{-1}f-(A_1-\lambda)^{-1}f\}$$

for any $f\in H_{-m}(\Omega)$. Evidently $S_{\lambda\varepsilon}$ is a bounded operator on $H_{-m}(\Omega)$ to $\mathring{H}_m(\Omega)$ whether the support of ξ_ε is contained in Ω or not. Writing $v=\xi_\varepsilon u$, $u=(A_0-\lambda)^{-1}f-(A_1-\lambda)^{-1}f$ we find

$$B[v, v]-\lambda(v, v)$$

$$=(B^1-B)[(A_1-\lambda)^{-1}f, \xi_\varepsilon v]+B[v, v]-B[u, \xi_\varepsilon v].$$

We shall first prove that for any positive integer there exists a constant C_j such that for $\lambda\notin\Gamma$ with $\varepsilon^{-1}|\lambda|^{1-1/2m}/d(\lambda, \Gamma)\leqq 1$

i) $$\|v\|_m+|\lambda|^{1/2}\|v\|\leqq C_j(|\lambda|^{-1/2m}+\varepsilon^h)|\lambda|^{3/2}/d(\lambda, \Gamma)^2 +\frac{|\lambda|^{1/2}}{d(\lambda, \Gamma)}\left(\frac{|\lambda|^{1-1/2m}}{\varepsilon d(\lambda, \Gamma)}\right)^j\|f\|,\ f\in L_2(\Omega),$$

ii) $$\|v\|_m+|\lambda|^{1/2}\|v\|\leqq C_j(|\lambda|^{-1/2m}+\varepsilon^h)|\lambda|^2/d(\lambda, \Gamma)^2 +\frac{|\lambda|}{d(\lambda, \Gamma)}\left(\frac{|\lambda|^{1-1/2m}}{\varepsilon d(\lambda, \Gamma)}\right)^j\|f\|_{-m},\ f\in H_{-m}(\Omega).$$

From Lemma 3. 4 and

$$(B^1-B)[(A_1-\lambda)^{-1}f,\ \xi_\varepsilon v]$$

$$=\int_\Omega \sum_{|\alpha|=|\beta|=m}(a_{\alpha\beta}(x_0)-a_{\alpha\beta}(x))D^\alpha((A_1-\lambda)^{-1}f)\overline{D^\beta(\xi_\varepsilon v)}dx$$

$$-\int_\Omega \sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}(x)D^\alpha((A_1-\lambda)^{-1}f)\overline{D^\beta(\xi_\varepsilon v)}dx+C_3\int_\Omega (A_1-\lambda)^{-1}f\xi_\varepsilon\bar{v}\,dx$$

it follows that

$$|(B^1-B)[(A_1-\lambda)^{-1}f,\ \xi_\varepsilon v]|$$

$$\leqq C\varepsilon^h\|(A_1-\lambda)^{-1}f\|_m\|\xi_\varepsilon v\|_m+C\sum_{k=0}^{1}\|(A_1-\lambda)^{-1}f\|_{m-k}\|\xi_\varepsilon v\|_{m-1+k}$$

$$+C(\varepsilon^h+|\lambda|^{-1/2m})(\|(A_1-\lambda)^{-1}f\|_m+|\lambda|^{1/2}\|(A_1-\lambda)^{-1}f\|)(\|\xi_\varepsilon v\|_m+|\lambda|^{1/2}\|\xi_\varepsilon v\|).$$

Hence nothing that

$$\|\xi_\varepsilon v\|_m+|\lambda|^{1/2}\|\xi_\varepsilon v\|\leqq C(\|v\|_m+|\lambda|^{1/2}\|v\|)$$

holds because of $\varepsilon^{-1}|\lambda|^{1-1/2m}/d(\lambda,\ \Gamma)\leqq 1$, we get

$$|(B^1-B)[(A_1-\lambda)^{-1}f,\ \xi_\varepsilon v]|$$

$$\leqq C(\varepsilon^h+|\lambda|^{-1/2m})(\|(A_1-\lambda)^{-1}f\|_m+|\lambda|^{1/2}\|(A_1-\lambda)^{-1}f\|)(\|v\|_m+|\lambda|^{1/2}\|v\|).$$

On the other hand remembering the proof of Lemma 4. 1 we have

$$|B[v,\ v]-B[u,\ \xi_\varepsilon v]|\leqq C\varepsilon^{-1}|\lambda|^{-1/2m}(\|u\|_m+|\lambda|^{1/2}\|u\|)(\|v\|_m+|\lambda|^{1/2}\|v\|)$$

and hence

$$\|v\|_m+|\lambda|^{1/2}\|v\|\leqq C\{(|\lambda|^{-1/2m}+\varepsilon^h)(\|(A_1-\lambda)^{-1}f\|_m+|\lambda|^{1/2}\|(A_1-\lambda)^{-1}f\|)$$
$$+\varepsilon^{-1}|\lambda|^{-1/2m}(\|u\|_m+|\lambda|^{1/2}\|u\|)\}|\lambda|/d(\lambda,\ \Gamma).$$

Thus i) and ii) are true for $j=1$ by Lemma 3. 2.

Let $\eta\in\Lambda$ be such that $\eta(x)=1$ for any $x\in\operatorname{supp}\xi$ and put $\eta_\varepsilon(x)=((x-x_0)/\varepsilon)$. Then by a similar method to that in the proof of Lemma 4. 1 we get

$$\|v\|_m+|\lambda|^{1/2}\|v\|\leqq C\{(|\lambda|^{-1/2m}+\varepsilon^h)(\|(A_1-\lambda)^{-1}f\|_m+|\lambda|^{1/2}\|(A_1-\lambda)^{-1}f\|)$$
$$+\varepsilon^{-1}|\lambda|^{-1/2m}(\|\eta_\varepsilon u\|_m+|\lambda|^{1/2}\|\eta_\varepsilon u\|)\}|\lambda|/d(\lambda,\ \Gamma).$$

Thus if $\eta_\varepsilon u$ satisfies i) or ii) for $j=k$, then v satisfies i) or ii) for $j=k+1$.

Using Lemma 3. 5 for the kernel $\xi_\varepsilon(x)(K_\lambda{}^0(x,\ y)-K_\lambda{}^1(x,\ y))$ of $S_{\lambda\varepsilon}$, we coclude the proof of the lemma.

Finally we estimate $K_\lambda{}^1(x,\ y)$. Let A_2 be the operator associated with B^1 on $H_m(R^n)\times H_m(R^n)$:

$$B^1[u,\ v]=\langle A_2u,\ v\rangle_{m,R^n},\quad u,\ v\in H_m(R^n),$$

wher $\langle f,\ v\rangle_{m,R^n}$ denotes the duality between $H_{-m}(R^n)$ and $H_m(R^n)$.

By Lemma 4. 2 $B^1[u, u]$ does belong to Γ for any $u \in H_m(R^n)$. Therefore by a similar argument to that in Section 3 we find that $\sigma(A_2)$ is surely included in Γ and the resolvent of A_2 is a bounded operator on $H_{-m}(R^n)$ to $H_m(R^n)$ or on $L^2(R^n)$ to $H_m(R^n)$. Moreover, by the regularity theorem on *weak solutions* $(A_2-\lambda)^{-1}L^2(R^n)$ is contained in $H_{2m}(R^n)$ and $(A_2-\lambda)^{-1}$ is a bounded linear operator on $L^2(R^n)$ to $H_{2m}(R^n)$. As is well known $(A_2-\lambda)^{-1}$ has a continuous kernel $K_\lambda^2(x, y)$ given by

$$K^2_\lambda(x, y)=(2\pi)^{-n}\int_{R^n}\frac{e^{i(x-y)\cdot\xi}}{A'(x_0, \xi)+C_3-\lambda}d\xi.$$

We define the operator $S'_{\lambda r}$, $r=\delta(x_0)$ as follows

$$S'_{\lambda r}f=\xi_r\{((A_1-\lambda)^{-1}(rf))^{\sim}-(A_2-\lambda)^{-1}f\}$$

for any $f \in H_{-m}(R^n)$, where rf is the restriction of f to $\mathring{H}_m(\Omega)$. Evidently $S'_{\lambda r}$ is a bounded linear operator on $H_{-m}(R^n)$ to $H_m(R^n)$. Putting $v=\xi_r u$, $u=((A^1-\lambda)^{-1}(rf))^{\sim}-(A_2-\lambda)^{-1}f$ we find

$$B^1[v, v]-\lambda(v, v)_{0, R^n}=B^1[v, v]-B^1[u, \xi_r v].$$

By a quite similar method to that in the proof of Lemma 4. 1, we get

Lemma 4. 4 *For any positive p there exists a constant C_p depending only on p but not on x and λ such that*

$$|K_\lambda^1(x, x)-C_0(x)(C_3-\lambda)^{n/2m-1}|\leqq C_p\frac{|\lambda|^{n/2m}}{d(\lambda, \Gamma)}\left(\frac{|\lambda|^{1-1/2m}}{\delta(x)d(\lambda, \Gamma)}\right)^p$$

for $\lambda \notin \Gamma$ with $|\lambda|\geqq 1$.

Here we denote $(-\lambda)^{n/2m-1}$ the analytic branch of power in the complex plane cut along the positive axis which is positive on the negative axis.

Thus we have proved the following

Theorem II. *Under the assumptions* (1), (2) and (4) *there exist, for any positive number p, ε and positive integer j, constants C_p and C_j independent of x, ε and λ such that*

$$\begin{aligned}&|K_\lambda(x, x)-C_0(x)(C_3-\lambda)^{n/2m-1}|\\&\leqq C_j(|\lambda|^{-1/2m}+\dot{\varepsilon}h)\frac{|\lambda|^{n/2m+1}}{d(\lambda, \Gamma)^2}+C_j\frac{|\lambda|^{n/2m}}{d(\lambda, \Gamma)}\left(\frac{|\lambda|^{1-1/2m}}{\varepsilon d(\lambda, \Gamma)}\right)^j\\&+C_p\frac{|\lambda|^{n/2m}}{d(\lambda, \Gamma)}\left(\frac{|\lambda|^{1-1/2m}}{\delta(x)d(\lambda, \Gamma)}\right)^p\end{aligned}$$

for any $x \in \Omega$ and $\lambda \notin \Gamma$ with $|\lambda|\geqq 1$, $\varepsilon^{-1}|\lambda|^{1-1/2m}/d(\lambda, \Gamma)\leqq 1$.

This theorem informs us the behaviour as $|\lambda|\to\infty$ with $d(\lambda, \Gamma)\geqq|\lambda|^{1-1/2m+2\varepsilon}$ of the resolvent kernel $K_\lambda(x, y)$ of A:

$$K_\lambda(x, x)=C_0(x)(C_3-\lambda)^{n/2m+1}+O(|\lambda|^{n/2m+1+h-h/2m+\varepsilon h}/d(\lambda, \Gamma)^{h+2})$$

$$+O(|\lambda|^{n/2m+1-1/2m}/d(\lambda, \Gamma)^2)+O(|\lambda|^{n/2m+p-p/2m}/d(\lambda, \Gamma)^{p+1}). \qquad (4.3)$$

This can be verified without difficulty by replacement of ε with $|\lambda|^{1-1/2m+\varepsilon}/d(\lambda, \Gamma)$ for a large j in the formula of the theorem.

5. Asymptotic formulas for eigenvalues of A

In the last section we shall show under the assumptions (1)—(5) how Theorem II, when combined with the following modified tauberian theorem yields the asymptotic formulas for eigenvalues of A. As was mentioned in the introduction, the most important tool in our paper is the following lemma, which is a simple consequence of our recent paper [9]:

LEMMA 5. 1. *Let* $\{\lambda_j\}_{j=0}^{\infty}$ *be a sequence in the sector of the complex plane*

$$\Gamma_0=\{\lambda: |\arg \lambda|\leqq\theta\},\ 0\leqq\theta<\pi/4$$

and have only ∞ *as a limit point. Suppose that*

$$\sum_{j=0}\frac{1}{\lambda_j-\lambda}=a(-\lambda)^{-\alpha}+O(|\lambda|^{-\beta}(|\lambda|^{\gamma}/d(\lambda, \Gamma_0))^p)$$

as $|\lambda|\to\infty$ *with* $d(\lambda, \Gamma_0)\geqq|\lambda|^{\gamma}$, *where* $0<\alpha<\beta<1$, $0<\gamma<1$, $p>0$ *and* $\mathrm{Re}\, a\geqq 0$. *Then as* $t\to\infty$

$$\left|\sum_{Re\lambda_j\leqq t}1-\mathrm{Re}\, a\left\{\frac{\sin((1-\alpha)(\pi-\varphi))}{(1-\alpha)\pi}-\frac{\sqrt{2}}{\pi}\cos(\alpha(\pi-\varphi))\sin\theta\right\}\sec^{(1-\alpha)/2}2\theta\, t^{1-\alpha}\right|$$
$$\leqq \mathrm{Re}\, a\sqrt{2}(\cos(\theta+\pi/4))^{-1}\sin(\alpha(\pi-\varphi))\sin\theta\sec^{(1-\alpha)/2}2\theta\, t^{1-\alpha}$$
$$+O(t^{1-\beta})\sqrt{\sec 2\theta-1}+O(t^{\gamma-\alpha}),\quad \varphi=\tan^{-1}\sqrt{\sec 2\theta-1}.$$

In order to verify this lemma we have only to note that

$$\sum_{j=0}^{\infty}\frac{1}{\lambda_j-Z}=\int_{\Gamma_0}\frac{\mu(d\lambda)}{\lambda-Z},\qquad z\notin\Gamma_0$$

for a positive measure μ defined on Γ_0 by

$$\mu(E)=\sum_{\lambda_j\in E}1\quad\text{for } E\subset\Gamma_0.$$

As for the eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ of A we have in addition to Lemma 3. 1

LEMMA 5. 2.

i) $\displaystyle\int_{\Omega}K_\lambda(x, x)dx=\sum_{j=0}^{\infty}\frac{1}{\lambda_j-\lambda},\qquad \lambda\notin\Gamma_0,$

ii) $\displaystyle\sum_{j=0}^{\infty}\frac{1}{\lambda_j-\lambda}=C_0(-\lambda)^{n/2m-1}+O(|\lambda|^{(n-h)/2m+1+h+\varepsilon h}/d(\lambda, \Gamma_0)^{h+2})+$

$$+O(|\lambda|^{(n-1)/2m+1}/d(\lambda, \Gamma_0)^2)+O(|\lambda|^{(n-p)/2m+p}/d(\lambda, \Gamma_0)^{p+1})$$

as $|\lambda|\to\infty$ *with* $d(\lambda, \Gamma_0)\geqq^{1-1/2m+2\varepsilon}$, *where* $C_0=\int_\Omega C_0(x)dx$.

PROOF. The first equality i) is proved in p. 228 of S. Agmon [1] for example. On the other hand a simple computation shows

$$(C_3-\lambda)^{n/2m-1}=(-\lambda)^{n/2m-1}+O(|\lambda|^{n/2m}/d(\lambda, \Gamma_0)^2)$$

as $|\lambda|\to\infty$ with $\lambda\notin\Gamma_0$. Hence using the assumption of (5) and integrating (4. 3) on Ω we obtain ii). Q. E. D.

In view of this lemma we find that

$$\sum_{j=0}^{\infty}\frac{1}{\lambda_j-\lambda}=C_0(-\lambda)^{n/2m-1}+O(|\lambda|^{-\beta_1}(|\lambda|^{\gamma}/d(\lambda, \Gamma_0))^{h+2})+$$

$$+O(|\lambda|^{-\beta_2}(|\lambda|^{\gamma}/d(\lambda, \Gamma_0))^2)+O(|\lambda|^{-\beta_3}(|\lambda|^{\gamma}/d(\lambda\ \Gamma_0))^{p+1})$$

as $|\lambda|\to\infty$ with $d(\lambda, \Gamma_0)\geqq|\lambda^{1-1/2m+2\varepsilon}$, where

$$\begin{cases} 1-\beta_1=n/2m-(h-\sigma(h+2))/2m-\varepsilon(h+4), \\ 1-\beta_2=n/2m-(1-2\sigma)/2m-4\varepsilon, \\ 1-\beta_3=n/2m-(p-\sigma(p+1))/2m-2\varepsilon(p+1), \\ \quad \gamma=1-\sigma/2m+2\varepsilon,\ 0<\sigma<1. \end{cases}$$

It is to be noted that Re C_0 is positive since Re $(A'(x, \xi)+1)^{-1}$ is positive for all $x\in\Omega$ and $\xi\in R^n$.

We are now ready to apply Lemma 5. 1.

THEOREM III. *Under the assumptions* (1)—(5) $N(t)=\sum_{Re\lambda_j\leqq t}1$ *satisfies as* $t\to\infty$

$$\left|N(t)-\mathrm{Re}\,C_0\left\{\sin\left(\frac{n}{2m}(\pi-\varphi)\right)\Big/\left(\frac{\pi n}{2m}\right)-C^1\sin\theta\right\}\sec^{n/2m}2\theta\, t^{n/2m}\right|$$

$$\leqq\mathrm{Re}\,C_0\,C_2\sin\theta\sec^{n/2m}2\theta\, t^{n/2m}+O(t^{(n-\sigma)/2m})$$

$$+O(t^{n/2m-(h-\sigma(h+2))/2m}+t^{n/2m-(p-\sigma(p+1))/2m})\sqrt{\sec 2\theta-1}, \qquad \varphi=\tan^{-1}\sqrt{\sec 2\theta-1}$$

for $0<\sigma<h/(h+2)$, $0<\sigma<p/(p+1)$, *where* C_1, C_2 *are constants given in* Theorem I.

Assuming (5) for $h/2\leqq p<1$, we obtain the first formula of Theorem I. Similarly assuming (5) for $2h/3\leqq p<1$ we obtain the second.

NIIGATA UNIVERSITY

References

[1] S. AGMON: Lectures on Elliptic Boundary Value Problem, Van Nostrand Mathematical Studies, Princeton, 1965.

[2] S. AGMON: *Asymptotic formulas with remainder estimates for eigenvalues of elliptic operators.* Arch. Rational Mech. Anal. 28 (1968), 165–183.

[3] S. AAMON and Y. KANNAI: *On the asymptotic behevior of spectral functions and resolvent kernels of elliptic operators.* Israel J. Math. 5 (1967), 1–30.

[4] R. BEALS: *Asymptotic behavior of the Green's function and spectral function of an elliptic operator.* J. Func. Anal. 5 (1970), 484–503.

[5] P. MALLIAVIN: *Un théorème taubérien avec reste pour la transformée de Stieltjes.* C. R. Acad. Sci. Paris 225 (1962), 2351–2352.

[6] K. MARUO: *Asymptotic distribution of eigenvalues of non-symmetric operators associated with strongly elliptic sesquilinear forms.* Osaka J. Math. 9 (1972), 547–560.

[7] K. MARUO and H. TANABE: *On the asymptotic distribution of eigenvalues of operators associated with strongly elliptic sesquilinear forms.* Osaka J. Math. 8 (1971), 323–345.

[8] Å. PLEIJEL. *On a theorem by P. Malliavin.* Israel J. Math. 1 (1963), 166–168.

[9] M. WATANABE: *A generalization of Malliavin's tauberian theorem.* Sci. Rep. Niigata Univ. Ser. A, No. 11 (1974) 61–68.