Boundary representations of a tensor product of $C^*$-algebras

By

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1. Introduction

In [1] Arveson gives a non-commutative generalization of Choquet boundary and Silov boundary. We shall study those of a tensor product of $C^*$-algebras.

If $E$ is a subspace of a $C^*$-algebra and $M_n$ is the algebra of $n \times n$ complex matrices, then the algebraic tensor product $E \otimes M_n$ is the set of $n \times n$ matrices with entries in $E$. If $\varphi: E \rightarrow F$ is a linear map from one linear space into another, then, for each positive integer $n$, define $\varphi_n: E \otimes M_n \rightarrow F \otimes M_n$ by applying element by element to each matrix over $E$, i.e. $\varphi_n(T_{ij}) = (\varphi(T_{ij}))$. $\varphi$ is called completely positive (resp., completely isometric) if each $\varphi_n$ is positive (resp., isometric).

Following Arveson [1], let $B$ be a $C^*$-algebra with unit and $A$ a subspace of $B$ which contains unit and generates $B$ as a $C^*$-algebra.

An irreducible representation $\pi$ of $B$ is called a boundary representation for $A$ if the restriction $\pi|A$ has a unique completely positive linear extension to $B$.

A closed two-sided ideal $J$ in $B$ is called a boundary ideal for $A$ if the canonical quotient map $q_J: B \rightarrow B/J$ is completely isometric on $A$.

A boundary ideal is called the Silov boundary for $A$ if it contains every other boundary ideal.

$A$ is called an admissible subspace of $B$ if the intersection of the kernels of the boundary representations for $A$ is a boundary ideal for $A$.

Throughout this paper, we use the following notations. Let $B_1$ and $B_2$ be $C^*$-algebras, and let for each $i =1, 2$, $e_i$ be unit in $B_i$, $A_i$ a subspace of $B_i$ which contains $e_i$ and generates $B_i$ as a $C^*$-algebra.

2. Boundary representations

Let $A_1 \otimes A_2$ be the algebraic tensor product, and $B_1 \otimes_{a} B_2$ the $C^*$-tensor product [3]. Then $A_1 \otimes A_2$ generates $B_1 \otimes_{a} B_2$ as a $C^*$-algebra.
**Theorem 1.** Let \( \pi_1 \) (resp., \( \pi_2 \)) be a boundary representation of \( B_1 \) (resp., \( B_2 \)) for \( A_1 \) (resp., \( A_2 \)). Then \( \pi_1 \otimes \pi_2 \) is a boundary representation of \( B_1 \otimes B_2 \) for \( A_1 \otimes A_2 \).

**Proof.** Let \( \varphi \) be a completely positive extension to \( B_1 \otimes B_2 \) of the restriction \( \pi_1 \otimes \pi_2 |_{A_1 \otimes A_2} \). Then there is a representation \( \pi \) of \( B_1 \otimes B_2 \) on a Hilbert space \( H \) such that

\[
\varphi(x) = H_1 \otimes H_2 \pi(x) H_1 \otimes H_2, \quad x \in B_1 \otimes B_2,
\]

where \( H_1 \) and \( H_2 \) are representation spaces of \( \pi_1 \) and \( \pi_2 \).

Let \( L(H_1) \) and \( L(H_2) \) be the \( C^* \)-algebras of all bounded operators on \( H_1 \) and \( H_2 \). We define the bounded linear map \( L_{\xi, \eta} \) of \( L(H_1) \otimes L(H_2) \) to \( L(H_1) \) by

\[
L_{\xi, \eta}(x \otimes y) = (\eta | \xi) x, \quad x \in L(H_1), \quad y \in L(H_2), \quad \xi, \eta \in H_2.
\]

Then \( L_{\xi, \eta} \) is a completely positive map. By [1; Theorem 1.2.9] it has a completely positive extension to \( L(H_1 \otimes H_2) \), and is also denoted by \( L_{\xi, \eta} \).

Then the map: \( a \rightarrow L_{\xi, \eta} \varphi(a \otimes e_2) \) is completely positive and we have

\[
L_{\xi, \eta} \varphi(a \otimes e_2) = (\xi | \xi) \pi_1(a), \quad a \in A_1.
\]

Since \( \pi_1 \) is a boundary representation of \( B_1 \) for \( A_1 \), we have

\[
L_{\xi, \eta} \varphi(a \otimes e_2) = (\xi | \xi) \pi_1(a), \quad a \in B_1.
\]

Since \( L_{\xi, \eta} \) is a linear combination of maps of the form \( L_{\xi, \xi} \), we have

\[
L_{\xi, \eta} \varphi(a \otimes e_2) = (\eta | \xi) \pi_1(a), \quad a \in B_1.
\]

Hence we have

\[
\varphi(a \otimes e_2) = \pi_1(a) \otimes I_{H_2}, \quad a \in B_1.
\]

Consequently, by [1; p. 174], \( H_1 \otimes H_2 \) is a invariant subspace for \( \pi(B_1 \otimes B_2) \).

Similarly, we have \( \varphi(e_1 \otimes b) = I_{H_1} \otimes \pi_2(b), \quad b \in B_2 \) and \( H_1 \otimes H_2 \) is a invariant subspace for \( \pi(e_1 \otimes B_2) \).

Hence we have

\[
\varphi(a \otimes b) = H_1 \otimes H_2 \pi(a \otimes b) H_1 \otimes H_2
\]

\[
= H_1 \otimes H_2 \pi(a \otimes e_2) \pi(e_1 \otimes b) H_1 \otimes H_2
\]

\[
= \pi_1(a) \otimes \pi_2(b), \quad a \in B_1, \quad b \in B_2.
\]

Consequently, we have \( \varphi = \pi_1 \otimes \pi_2 \). This completes the proof.

In [2] Hopenwasser proved the following result.

Let \( B \) be a \( C^* \)-algebra with unit \( e_b \). Let \( S \) be a linear subspace of \( B \otimes M_n \) which generates \( B \otimes M_n \) and which contains the set of matrix units \( e_{ij} \otimes e_{ij}, i, j = 1, \ldots, n \). Let \( T \) be the set of operators in \( B \) which appear as a matrix entry in some element of \( S \). Then an irreducible representation \( \pi \) of \( B \) is a unique completely positive extension of \( \pi |_T \) to \( B \) if and only if \( \pi \otimes I_n \).
is a boundary representation for $S$.

We shall give the proof of the "if" part in a slightly different way.

**Proof.** Let $\pi$ be a boundary representation for $T$, acting on the Hilbert space $H$, and let $\varphi$ be a completely positive extension to $B \otimes M_n$.

Then, by [1: p. 146], we have a representation $\pi_b$ of $B \otimes M_n$ such that

$$\varphi(x \otimes y) = H \otimes H_n \pi_b(x \otimes y) H \otimes H_n, \quad x \in B, \text{ and } y \in M_n,$$

where $H_n$ is $n$-dimensional Hilbert space.

Since $e_b \otimes e_{ij} \in S$,

$$\varphi(e_b \otimes e_{ij}) = P \pi_b(e_b \otimes e_{ij}) P = I_H \otimes e_{ij},$$

where $P$ is the projection on $H \otimes H_n$.

Hence the map: $x \mapsto \varphi(e_b \otimes x)$ is a representation of $M_n$, and so $P$ is invariant for $\pi_b(e_b \otimes M_n)$.

Now, we have

$$\varphi(x \otimes e_{ij}) = P \pi_b(x \otimes e_n) \pi_b(e_b \otimes e_{ij}) P$$

$$= P \pi_b(x \otimes e_n) P I_H \otimes e_{ij},$$

where $e_n$ is unit of $M_n$.

On the other hand, we have

$$\varphi(x \otimes e_{ij}) = P \pi_b(e_b \otimes e_{ij}) \pi_b(x \otimes e_n)$$

$$= I_H \otimes e_{ij} P \pi_b(x \otimes e_n) P.$$

Hence, we have $P \pi_b(x \otimes e_n) P \in (I_H \otimes L(H_n))^\prime$, and so there is a positive linear map $\rho$ such that

$$P \pi_b(x \otimes e_n) P = \rho(x) \otimes I_{H_n}.$$

Since we have for each $s \in S$, $\varphi \otimes I_n(s) = \pi \otimes I_n(s)$, we have $\rho = \pi$ on $T$.

On the other hand, the map: $x \mapsto \varphi(x \otimes e_n)$ is completely positive, and $\pi$ is a boundary representation for $T$ we have $\pi = \rho$ on $B$.

Then $P$ is invariant for $\pi_b(B \otimes e_n)$.

Consequently, we have $\varphi = \rho \otimes I_n = \pi \otimes I_n$. This completes the proof.

3. Boundary ideals

We assume throughout this section, for each $i = 1, 2$, $B_i$ acts on a Hilbert space $H_i$.

**Theorem 2.** Let $J_i$ be a boundary ideal for $A_i$ of $B_i$. Then $\ker(q_{J_1} \otimes q_{J_2})$ is a boundary ideal of $B_1 \otimes B_2$ for $A_1 \otimes A_2$.

**Proof.** The map $q_{J_1}(a) \rightarrow a$ is completely isometric on $q_{J_1}(A_1)$ by [1: Theorem 1.
2. 9], this map has a completely positive linear extension to $B_1/J_1$. There are a representation $\pi_1$ of $B_1/J_1$ and a linear isometric map $V_1$ from $H_1$ into a representation space $H_{\pi_1}$ of $\pi_1$ such that
\[
a = V_1^* \pi_1(q_{f_1}(a)) V_1, \quad a \in A_1.
\]

Similarly, there are a representation $\pi_2$ of $B_2/J_2$ and a linear isometric map $V_2$ from $H_2$ into a representation space $H_{\pi_2}$ of $\pi_2$ such that
\[
b = V_2^* \pi_2(q_{f_2}(b)) V_2, \quad b \in A_2.
\]

We have for $a \in A_1$ and $b \in A_2$
\[
a \otimes b = (V_1 \otimes V_2)^* \pi_1 \otimes \pi_2(q_{J\iota}(a) \otimes q_{f_2}(b)) V_1 \otimes V_2.
\]

Hence the map: $q_{ker}(q_{f_1} \otimes q_{J\iota})(x) \rightarrow x$ is completely contractive.

Consequently, ker$(q_{K1} \otimes q_{K2})$ is a boundary ideal.

**THEOREM 3.** Let $A_1$ (resp., $A_2$) be an admissible subspace of $B_1$ (resp., $B_2$), and $K_1$ (resp., $K_2$) be the intersection of all kernels of boundary representations of $B_1$ (resp., $B_2$) for $A_1$ (resp., $A_2$). Then $A_1 \otimes A_2$ is an admissible subspace of $B_1 \otimes B_2$, and ker$(q_{K1} \otimes q_{K2})$ is the Silov boundary for $A_1 \otimes A_2$.

**PROOF.** Let $B_i$ denote the set of boundary representations of $B_i$ for $A_i$, and let $\rho_i = \sum_{\pi_{ij} \in B_i} \otimes \pi_{ij}$ be the direct sum of boundary representations of $B_i$. Let $J$ be the intersection of the kernels of representations of the form $\pi_{1m} \otimes \pi_{2n}$ where $\pi_{1m}$ and $\pi_{2n}$ are boundary representations of $B_1$ and $B_2$. Since $q_{K1} \otimes q_{K2}(B_1 \otimes B_2)$ is *-isomorphic to $\rho_1 \otimes \rho_2(B_1 \otimes B_2)$, we have
\[
ker(q_{K1} \otimes q_{K2}) = J.
\]

Let $K$ be the intersection of all kernels of boundary representations of $B_1 \otimes B_2$ for $A_1 \otimes A_2$.

By Theorem 1, $\pi_{1m} \otimes \pi_{2n}$ is a boundary representation, then we have
\[
J \supset K.
\]

On the other hand, by Theorem 2, ker$(q_{K1} \otimes q_{K2})$ is a boundary ideal. Therefore, $K$ is a boundary ideal, and so $A_1 \otimes A_2$ is admissible. Then $K$ is the Silov boundary ideal [1: Theorem 2. 2. 3], hence we have
\[
K \supset ker(q_{K1} \otimes q_{K2}).
\]

Consequently, we have
\[
K = ker(q_{K1} \otimes q_{K2}).
\]

This completes the proof.

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References