The second dual of a tensor product of C*-algebras, II

By

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1. Introductinn

Let C be a C*-algebra, and let π_C be the universal represensation of C in the universal representation Hilbert space H_C. The second dual C^{**} of C may be identified with the closure of $\pi_C(C)$ in weak operator topology [1: p. 236]. For C*-algebras A and B we denote by $A \underset{\alpha}{\otimes} B$ the C*-tensor product of A and B, $A^{**} \otimes B^{**}$ the W*-tensor product of A^{**} and B^{**}. Since there exists the canonical *-isomorphism $\pi_A \otimes \pi_B$ from $A \underset{\alpha}{\otimes} B$ into $A^{**} \underset{\alpha}{\otimes} B^{**}$, $A \underset{\alpha}{\otimes} B$ may be identified with the weak dense subalgebra $\pi_A \otimes \pi_B (A \underset{\alpha}{\otimes} B)$ of $A^{**} \otimes B^{**}$. In this paper we shall study positive linear functionals of $A \underset{\alpha}{\otimes} B$ which has the normal extension to $A^{**} \otimes B^{**}$.

In §2, we shall show a characterization of pure states having the normal extension to $A^{**} \otimes B^{**}$.

In §3, we shall show that $(A \underset{\alpha}{\otimes} B)^{**}$ is *-isomorphic to $A^{**} \otimes B^{**}$ when either A or B is a dual C*-algebra, and the *-isomorphism $\pi_A \otimes \pi_B$ has no normal extension to $(A \underset{\alpha}{\otimes} B)^{**}$ when A and B are UHF algebras [2: Definition 1. 1].

2. Theorem

THEOREM. *Let A and B be C*-algebras and π be an irreducible representation of $A \underset{\alpha}{\otimes} B$ on a Hilbert space $H\pi$. Then the following two assertions are equivalent.*

(*a*) *π is equivalent with a representation $\pi_1 \otimes \pi_2$ where π_1 and π_2 are representations of A and B, respectively.*

(*b*) *A positive linear functional f of $A \underset{\alpha}{\otimes} B$ has the normal extension to $A^{**} \otimes B^{**}$, where f is given by the formula*

$$f(x) = (\pi(x)\xi, \xi), \quad x \in A \underset{\alpha}{\otimes} B, \quad \xi \in H_\pi.$$

PROOF. It is obvious that (a) implies (b).

If (b) holds, f can be expressed such that

$$f(x) = (x\xi, \xi), \quad x \in A \underset{\alpha}{\otimes} B, \quad \xi \in H_A \otimes H_B.$$

Now, ξ can be written such that

$$\xi=\sum_{i=1}^{\infty}\xi_i \otimes \eta_i$$

where $\{\xi_i\}$, $\{\eta_i\}$ are orthogonal families in H_A and H_B.

If S is a family of operators acting on a Hilbert space H and K is a set of vectors in H, the $[SK]$ denotes the closed subspace of H generated by vectors of the form Ta with T in S and a in K. Let P_A and P_B be projections on $[\pi_A(A)\xi_i]_{i=1,2,\dots}$ and $[\pi_B(B)\eta_i]_{i=1,2,\dots}$

If P is a projection in $\pi(A)'$ such that $P_A \gneqq P$. Then there exists a vector ξ_i such that $P\xi_i \neq \xi_i$. We have $P\otimes P_B\xi \neq \xi$.

Now, we get

$$f(x)=(x(P_A\otimes P_B-P\otimes P_B)\xi, \xi)+(xP\otimes P_B\xi, \xi)$$

for $x\in A\underset{\alpha}{\otimes}B$. This is a contradiction. Therefore the restriction $\pi_{A|P_A}$ of π_A to $[\pi_A(A)\xi_i]_{i=1,2,\dots}$ is an irreducible representation of A.

Similarly $\pi_{B|P_B}$ is an irreducible representation of B.

Since we have $[A\underset{\alpha}{\otimes}B\xi]\subset P_A\otimes P_B$, $[A\otimes B\xi]=P_A\otimes P_B$.

Consequently the representation: $x\to x|_{[A\otimes B\xi]}$ of $A\underset{\alpha}{\otimes}B$ is equivalent with $\pi_{A|P_B}\otimes \pi_{B|P_B}$. This completes the proof.

3. Examples

Example 1. *If either A or B is a dual C*-algebra, then $(A\underset{\alpha}{\otimes}B)^{**}$ is *-isomorphic to $A^{**}\otimes B^{**}$.*

Proof. We assume A is a dual C*-algebra.

First, we shall consider in case A is an elementary C*-algebra which has a *-isomorphism ι to the C*-algebra of all compact operators on a Hilbert space H.

Let f be a positive linear functional of $A\underset{\alpha}{\otimes}B$. For a representation π_f defined by f in a Hilbert space H_f, we have representations π_1 and π_2 of A and B in H_f such that

$$\pi_f(a\otimes b)=\pi_1(a)\pi_2(b)=\pi_2(b)\pi_1(a),$$

for $a\in A$, $b\in B$. Because of the property of the algebra of all compact operators, π_1 is equivalent with a representation $\iota\otimes I$ in a suitable Hilbert space $H\otimes K$. Then there exists a representation ρ of B in the Hilbert space $H\otimes K$. Then there exists a representation ρ of B in the Hilbert space K such that π_2 is equivalent with $I\otimes\rho$ in $H\otimes K$. Hence π_f is equivalent with $\iota\otimes\rho$, and so f has the normal extension to $A^{**}\otimes B^{**}$. By [3: Corollary] $(A\underset{\alpha}{\otimes}B)^{**}$ is *-isomorphic to $A^{**}\otimes B^{**}$.

Next, we shall consider in case A is a dual C*-algebra, that is, it is the C*-direct sum of A_i, where A_i is an elementary C*-algebra.

Since $A_i\underset{\alpha}{\otimes}B$ is a closed two-sided ideal in $A\underset{\alpha}{\otimes}B$, there exists a central projection p_i of

$(A \underset{\alpha}{\otimes} B)^{**}$ such that $(A \underset{\alpha}{\otimes} B)^{**} p_i = \overline{A_i \underset{\alpha}{\otimes} B}$, where $\overline{A_i \underset{\alpha}{\otimes} B}$ denotes the weak closure of $A_i \otimes B$ in $(A \underset{\alpha}{\otimes} B)^{**}$. Then $(A_i \underset{\alpha}{\otimes} B)^{**}$ is *-isomorphic to $\overline{A_i \underset{\alpha}{\otimes} B}$. We also have a central projection z_i of A^{**} such that $A_i^{**} = A^{**} z_i$. Since $(A \underset{\alpha}{\otimes} B)^{**} = \sum_i (A \underset{\alpha}{\otimes} B)^{**} p_i$, and $A^{**} \otimes B^{**} = \sum_i (A^{**} z_i \otimes B^{**})$, $(A \otimes B)^{**}$ is *-isomoprphic to $A^{**} \otimes B^{**}$.

EXAMPLE 2. *Let A and B be UHF algebras. The *-isomorpism* $\pi_A \otimes \pi_B$ *from* $A \underset{\alpha}{\otimes} B$ *into* $A^{**} \otimes B^{**}$ *has no normal extension to* $(A \underset{\alpha}{\otimes} B)^{**}$.

PROOE. By [4: Theorem 4] and Theorem, there exists a pure state of $A \underset{\alpha}{\otimes} B$ which has no normal extension to $A^{**} \otimes B^{**}$. By [3: Corollary] $\pi_A \otimes \pi_B$ has no normal extension to $(A \underset{\alpha}{\otimes} B)^{**}$.

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References

1. J. DIXMIER: Les C*-algēbres et leurs reprśentations Gauthier-Villars, Paris, 1969.
2. J. GLIMM: *On a certain class of operator algebras.* Trans. Amer. Math. Soc., 95 (1960), 318–340.
3. T. HURUYA: *The second dual of a tensor product of C*-algebras.* Sci. Rep. Niigata Univ., Ser. A, 9 (1972), 35–38.
4. A. WULFSOHN: *Produit tensoriel de C*-algèbres.* Bull. Sci. Math., 87 (1963), 13–27.