A characterization of double centralizer algebras of Banach algebras

By

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1. Introduction

Let $M(A)$ be the algebra of double centralizers of a Banach algebra A. Let A^* and A^{**} be the conjugate and the second conjugate spaces of A, respectively. Let π be the canonical mapping of A into A^{**} and $Q(A)$ is the idealizer of $\pi(A)$ in A^{**}. The purpose of this paper is to generalize a fact that $Q(A)$ is isometrically $*$-isomorphic onto $M(A)$ when A is a C*-algebra [8]. Suppose that A is a Banach algebra without order. Then there is a canonical map Φ [see §3] which is a norm-decreasing homomorphism of $Q(A)$ into $M(A)$. Also A has a weak bounded approximate identity if and only if Φ is onto. Moreover, we shall investigate a condition for $Q(A)$ to be isometrically isomorphic onto $M(A)$. Finally, we shall construct two interesting examples.

2. Notations and preliminaries

Let A be a Banach algebra. The two *Arens products* $*_1$ and $*_2$ are defined in stages according to the following rules [1, 4]. Let $x, y \in A$, $f \in A^*$, and $F, G \in A^{**}$. Then we have, by definition,

$$(f *_1 x)(y) = f(xy),\ (G *_1 f)(x) = G(f *_1 x),\ (F *_1 G)(f) = F(G *_1 f).$$

Then, $f *_1 x$, $G *_1 f \in A^*$ and $F *_1 G \in A^{**}$ and A^{**} is a Banach algebra with the Arens product $*_1$. A^{**} with the Arens product $*_1$ is denoted by $(A^{**}, *_1)$. Similarly, we define

$$(x *_2 f)(y) = f(yx),\ (f *_2 F)(x) = F(x *_2 f),\ (F *_2 G)(f) = G(f *_2 F).$$

Then, $x *_2 f$, $f *_2 F \in A^*$ and $F *_2 G \in A^{**}$, and A^{**} is a Banach algebra with the Arens product $*_2$. A^{**} with the Arens products $*_2$ is denoted by $(A^{**}, *_2)$. Furthermore a Banach algebra A is said to be Arens regular if the two Arens products coincide on A^{**}.

An ordered pair (T_1, T_2) of operators in A is said to be a *double centralizer on* A provided that $x(T_1 y) = (T_2 x)y$ for all $x, y \in A$. The set of all double centralizers of A will be denoted by $M(A)$. We say that a Banach algebra A has a *weak approximate identity* if there exists a net $\{e_\alpha\}_{\alpha\in\Lambda}$ in A such that $\lim f(e_\alpha x-x)=\lim f(xe_\alpha-x)=0$ for every $x\in A$ and $f\in A^*$. It is said to be bounded if there is some number M such that $\|e_\alpha\|\leqq M$ for all $\alpha\in\Lambda$. We put $P=\{x\in A: xA=(0) \text{ or } Ax=(0)\}$.

We say that A is without order when $P=(0)$. This is the case, if A either is semi-simple or has a weak approximate identity. Throughout this paper, we use the standard notations and terminologies from [7].

LEMMA 1. *Let A be a Banach algebra without order and let $(T_1, T_2)\in M(A)$. Then*

(i) *T_1 and T_2 are continuous linear operators in A,*

(ii) *$T_1(xy)=(T_1x)y$ for all $x, y\in A$,*

(iii) *$T_2(xy)=x(T_2y)$ for all $x, y\in A$,*

(iv) *if $(S_1, S_2)\in M(A)$, $(T_1S_1, S_2T_2\in M(A)$.*

PROOF. The proof of these statments is almost the same as that of [2, proposition 2.5 and Lemma 2.9].

DEFINITION. Let A be a Banach algebra without order and (T_1, T_2), $(S_1, S_2)\in M(A)$, and let α be a complex number,

(i) $(T_1, T_2)+(S_1, S_2)=(T_1+S_1, T_2+S_2)$,

(ii) $\alpha(T_1, T_2)=(\alpha T_1, \alpha T_2)$,

(iii) $(T_1, T_2)(S_1, S_2)=(T_1S_1, S_2T_2)$,

(iv) $\|(T_1, T_2)\|=\max(\|T_1\|, \|T_2\|)$.

Then $M(A)$ is seen to be a Banach algebra under above operations and norm.

Furthermore, we define a map $\mu: A\longrightarrow M(A)$ by the formula $\mu(x)=(L_x, R_x)$ where $L_x(y)=xy$ and $R_x(y)=yx$ for all $x, y\in A$. Then μ is an isomorphism from A into $M(A)$ and $\mu(A)$ is a 2-sided ideal of $M(A)$.

LEMMA 2. *Let A be a Banach algebra with a weak approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ such that $\|e_\alpha\|\leq 1$. Then we have*

$$\|x\|=\sup_{\|y\|\leqq 1}\|yx\|=\sup_{\|y\|\leqq 1}\|xy\| \quad \text{for all} \quad x\in A.$$

PROOF. Let $\{e_\alpha\}$ be a weak approximate identity such that $\|e_\alpha\|\leqq 1$ and $x\in A$. Then we have

$$f(x)=\lim f(xe_\alpha)=\lim f(e_\alpha x) \quad \text{for all} \quad f\in A^*,$$

and so $\|x\|=\sup_\alpha\|xe_\alpha\|=\sup_\alpha\|e_\alpha x\|$. This shows the lemma.

LEMMA 3. *Let A be as in Lemma 2 and $(T_1, T_2)\in M(A)$. Then we have $\|T_1\|=\|T_2\|$.*

PROOF. By Lemma 2, the proof of this statement is almost the same as that of [2, Lemma 2.6].

If A is a Banach *-algebra without order, then $M(A)$ can be made into a Banach *-algebra, by defining an involution by $(T_1, T_2)^*=(T_2^*, T_1^*)$, where $T_i^*(x)=(T_i(x^*))^*$ for all $x\in A$ and for $i=1, 2$. Then μ is seen to be a *-isomorphism from A into $M(A)$.

3. The main theorems

Let A be a Banach algebra. To simplify, we shall identify A with $\pi(A)$. Let $Q(A)$ be the idealizer of A in $(A^{**}, *_1)$; that is,

$$Q(A)=\{F\in A^{**}: x*_1F \text{ and } F*_1x\in A \text{ for all } x\in A\}.$$

Then $Q(A)$ is a closed subalgebra of $(A^{**}, *_1)$. Now put

$$L_F(x)=F*_1x,\ R_F(x)=x*_1F \text{ for all } x\in A \text{ and } F\in Q(A).$$

We have $(L_F, R_F)\in M(A)$. We define a map $\Phi: Q(A)\longrightarrow M(A)$

by the formula $\Phi(F)=(L_F, R_F)$. Clearly Φ is the extension of μ to $Q(A)$. Now put $K=\{F\in A^{**}: A^{**}*_1F=(0)\}$.

THEOREM 1. *Let A be a Banach algebra without order. Then the map Φ is a norm-decreasing homomorphism of $Q(A)$ into $M(A)$ with kernel $K\cap Q(A)$.*

PROOF. It is clear that Φ is a norm-decreasing homomorphism. Thus we shall show that $\ker\Phi=K\cap Q(A)$. If $F\in\ker\Phi$, we have $R_F(x)=x*_1F=0$ for all $x\in A$.

By Goldstine's theorem,

$$A^{**}*_1F=(0).$$

That is, $F\in K\cap Q(A)$.

Conversely if $F\in K\cap Q(A)$, we have $R_F(x)=x*_1F=0$ for all $x\in A$, and so $xL_F(y)=R_F(x)y=0$ for all $x, y\in A$.

Since A is without order, $L_F(y)=0$, and so $F\in\ker\Phi$. This completes the proof.

REMARK 1. If A is a commutative Banach algebra without order, then $K\cap Q(A)=K$.

LEMMA 4. *Let A be a Banach such that $K\cap Q(A)=(0)$. Then the two Arens products coincide on $Q(A)$. Furthermore, A has a weak bounded approximate identity if and only if $Q(A)$ has an identity.*

PROOF. As was noted in [1], $F*_1G$ is w*-continuous in F for fixed $G\in A^{**}$. For any $F, G\in Q(A)$ and $x\in A$, we have, by [4. Lemma 1. 5],

$$x*_1(F*_1G)=(x*_1F)*_1G=(x*_2F)*_2G=x*_2(F*_2G)=x*_1(F*_2G).$$

Hence, by Goldstine's theorem,

$$H*_1(F*_1G)=H*_1(F*_2G) \text{ for all } H\in A^{**}.$$

By our assumption, $F*_1G=F*_2G$, so that the two Arens products coincide on $Q(A)$.

Suppose now that A has a weak bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$.

Since there is some number M such that $\|e_\alpha\|\leq M$, the w*-compactness of the ball of radius M in A^{**}, implies the existence of a subnet $\{e_\beta\}_{\beta\in\Lambda'}$ such that w*-lim $e_\beta=I\in A^{**}$. By [3, Lemma 3. 8] I is a right identity for $(A^{**}, *_1)$ and a left identity for $(A^{**}, *_2)$. By [4, Lemma 1. 5], $I\in Q(A)$. Since the two Arens products conincide on $Q(A)$, I is the identity of $Q(A)$. Conversely suppose that $Q(A)$ has an identity I. By Goldstine's theorem, there is a net $\{e_\alpha\}_{\alpha\in\Lambda}$, with $\|e_\alpha\|\leqq\|I\|$, $\alpha\in\Lambda$, and w*-lim $e_\alpha=I$. It is easy to show that $\{e_\alpha\}$ is a weak bounded approximate identity of A. This completes the proof.

REMARK 2. The element I in the preceding proof is not necessarily an identity of $(A^{**}, *_1)$. However if A is Arens reguler, I is an identity of $(A^{**}, *_1)$

THEOREM 2. *Let A be a Banach algebra without order. Then A has a weak bounded approximate identity if and only if Φ is onto. Furthermore if $K\cap Q(A)=0$ and A has a weak approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ such that $\|e_\alpha\|\leqq 1$, Φ is an isometric isomorphism.*

PROOF. Suppose that A has a weak bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$. Let $T=(T_1, T_2)\in M(A)$. Since $\{T_1 e_\alpha\}$ is bounded, it has w*-limit points in A^{**} by Alaoglu's theorem. Thus there is a subnet $\{T_1e_\beta\}_{\beta\in\Lambda'}$ such that w*-lim $T_1e_\beta=F\in A^{**}$. Since $(T_1e_\beta)x=T_1(e_\beta x)$ and $f\circ T_1\in A^*$ for any $f\in A^*$, we have

$$\begin{aligned}(F*_1x)(f)&=\lim(T_1 e_\beta *_1 x)(f)=\lim f(T_1(e_\beta x))=\lim(f\circ T_1)(e_\beta x)\\&=f(T_1x)=(T_1x)(f).\end{aligned}$$

Consequently $F*_1x=T_1x$. Since $x(T_1y)=(T_2x)(y)$ for all $x, y\in A$, it follows that $x*_1F=T_2x$. Therefore there is an element $F\in Q(A)$ such that $\Phi(F)=T$. Hence Φ is onto. Conversely suppose that Φ is onto. Since $M(A)$ has an identity (E, E) where $Ex=x$ for all $x\in A$, there is an element $F\in Q(A)$ such that $\Phi(F)=(E, E)$. By Goldstine's theorem, there is a net $\{e_\alpha\}$, with $\|e_\alpha\|\leqq\|F\|$, $\alpha\in\Lambda$, and w*-lim $e_\alpha=F$. It is not hard to show that $\{e_\alpha\}$ is a weak boundet approximate identity of A. The first statement is thus proved.

Suppose that $K\cap Q(A)=(0)$ and A has a weak approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ such that $\|e_\alpha\|\leqq 1$ for all $\alpha\in\Lambda$. Now choose I as in the proof of Lemma 4. Since I is the identity of $Q(A)$, we have

$$\text{w*-lim } e_\beta *_1 F=I*_1F=F \quad \text{for all} \quad F\in Q(A).$$

This implies that $\|F\|\leqq\sup_\beta\|e_\beta *_1F\|$ and therefore

$$\|\Phi(F)\|=\|R_F\|=\sup_{\|x\|\leqq 1}\|x*_1 F\|\geqq\sup_\beta\|e_\beta *_1F\|\geqq\|F\|.$$

Since Φ is a norm-decreasing map, we have $\|\Phi(F)\|\leqq\|F\|$, and so $\|\Phi(F)\|=\|F\|$. Hence Φ is an isometry. This completes the proof.

By Remark 2 and Therorem 2, we have the following;

COROLEARY 1. *Let A be an Arens regular Banach algebra with a weak bound approximate identity. Then $Q(A)$ is isomorphic onto $M(A)$.*

COROLLARY 2. *Let A be a Banach algebra with a weak apporximate identity $\{e_\alpha\}$ such that* w-lim $f*_1e_\alpha=f$ for all $f\in A^{**}$.

Then $Q(A)$ is isomorphic onto $M(A)$.

PROOF. Choose I as the proof in Lemma 4. Then I is an identity of $(A^{**}, *_1)$ by our assumption. This completes the proof.

In the remainder of this section, we shall study the case of a Banach *-algebra. Let A be a Banach *-algebra with a continuous involution $x\longrightarrow x^*$. Mapping $f\longrightarrow f^*$ and $F\longrightarrow F^*$ are then defined on A^* and A^{**}, respectively, by

$$f^*(x)=\overline{f(x^*)} \quad (x\in A),$$

and

$$F^*(f)=\overline{F(f^*)} \quad (F\in A^{**}).$$

Itis clear that the correspondence $F\longrightarrow F^*$ maps A^{**} onto A^{**} such that

$$(\alpha F+\beta G)^*=\bar{\alpha}F^*+\bar{\beta}G^*,\ F^{**}=F$$

for F, $G\in A^{**}$ and for complex numbers α, β.

However it is not in general true that $(F*_1G)^*=G^**_1F^*$.

LEMMA 5. *Let A be a Banach *-algebra, with a continuous involution. If $K\cap Q(A)=(0)$, then $Q(A)$ is a Banach *-algebra.*

PROOF. It is straightfoward to verify that

$$(F*_1G)^*=G^**_2F^* \quad \text{for} \quad F,\ G\in A^{**}.$$

By Lemma 4, the two Arens products coincide on $Q(A)$ and so

$$(F*_1G)^*=G^**_1F^* \quad \text{for all} \quad F,\ G\in A^{**}.$$

The mapping $F\longrightarrow F^*$ is therefore an involution on $Q(A)$. This completes the proof.

THEOREM 3. *Let A be a Banach *-algebra with a continuous involution and with a weak bounded approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$. If $K\cap Q(A)=(0)$, then Φ is a *-isomorphism of $Q(A)$ onto $M(A)$. If, in addition, $\|e_\alpha\|\leqq 1(\alpha\in\Lambda)$, Φ is an isometric *-isomophism.*

PROOF. By Theorem 2, it is sufficient to show that Φ is a *-preserving mapping. Let $F\in Q(A)$. We have

$$\Phi(F)^*\equiv(L_F,\ R_F)^*=((R_F)^*,\ (L_F)^*)=(L_{F^*},\ R_{F^*})=\Phi(F^*).$$

Hence Φ is a *-isomorphism. This completes the proof.

4. Examples

EXAMPLE 1. *There is a semi-simple commutative Banach *-algebra A such that*

(i) *A has an approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ such that $\|e_\alpha\|=1$. $(\alpha\in\Lambda)$.*

(ii) $K\cap Q(A)=K\neq(0)$.

CONSTRUCTION. Let G be a locally compact abelian group which is not discrete and let $L(G)$ be the group algebra of G. Then $L(G)$ is a semi-simple commutative Banach *-algebra with an approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$ such that $\|e_\alpha\|=1$ $(\alpha\in\Lambda)$. By Remark 1, $K\cap Q(A)=K$. By the proof of [3, Theorem 3. 12], $K\neq(0)$. So Φ is not an isomorphism

EXAMPLE 2. *There is a semi-simple commutative Banach algebra A such that*

(i) *A has no weak approximate identity,*

(ii) $A^* *_1 A = A^*$ *and so* $K=(0)$,

(iii) $Q(A)=A$.

Construction. Let D denote the closed unit disc in the complex plane $\{z: |z| \leqq 1\}$, and let Γ denote the unit circle $\{z: |z|=1\}$.

We denote by B the collection of functions which are continuous on D and analytic in the interior of D. Now put $A=zB$. This Banach algebra A has the required properties (i), (ii) and (iii).

(i) Suppose that A has a weak approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$. Defining $f(x)=x'(0)$, where x' is the derivative of $x\in A$, we have $f\in A^*$ clearly. Therefore $\lim f(xe_\alpha)=f(x)=x'(0)$. Since $f(xe_\alpha)=(xe_\alpha)'(0)=0$, we have $x'(0)=0$. This is a contradiction.

Hence A has no weak approximate identity.

Let $(T_1, T_2)\in M(A)$. Since A is commutative, $T_1=T_2$. So we may consider $M(A)$ such as

$$M(A)=\{T: (Tx)y=x(Ty) \quad \text{for all} \quad x, y\in A\}.$$

Defining $T_y(x)=yx(x\in A)$ for each $y\in B$, we have

$$M(A)=\{T_y: y\in B\}.$$

Indeed, it is clear that $\{T_y: y\in B\}\subset M(A)$.

For any $T\in M(A)$,

$$(Tx)z=x(Tz)=(Tz)x \quad \text{for all} \quad x\in A,$$

then putting $y=Tz/z\in B$, we have $Tx=(Tz/z)x=T_y(x)$, and so

$$M(A)=\{T_y: y\in B\}.$$

(ii) Let $C(\Gamma)$ be the space of call ontinuous functions on Γ and let $M(\Gamma)$ be the space of Radon measures on Γ. Then $C(\Gamma)^*=M(\Gamma)$. Since A is the closed subalgebra of $C(\Gamma)$, we have, by Theorem of F. and M. Riesz [See 5],

$$A^*=M(\Gamma)/H^1,$$

where $H^1=\{\mu\in M(\Gamma): \int_{-\pi}^{\pi} e^{in\theta}\, d\mu(\theta)=0,\ n=1, 2,\dots\}$.

Let $\sim$ be the canonical map of $M(\Gamma)$ onto $M(\Gamma)/H^1$. Now putting $\nu(\cdot)=\mu(e^{i\theta}\cdot)$ for each $\mu\in M(\Gamma)$, we see that $\nu\in M(\Gamma)$.

For all $x\in A$, we have

$$(\tilde{\nu} *_1 e^{i\theta})(x)=\tilde{\nu}(e^{i\theta} x)=\nu(e^{i\theta} x)=\mu(e^{-i\theta} e^{i\theta} x)$$

$$=\mu(x)=\tilde{\mu}(x).$$

Thus $\tilde{\nu} *_1 e^{i\theta}=\tilde{\mu}$ and so $A^* *_1 e^{i\theta}=A^*$.

Therefore $A^* *_1 A=A^*$. Note that $K=(0)$ if and only if the linear span of $\{f *_1 x:$

$f \in A^*$, $x \in A\}$ is strongly dense in A^*. Thus $K=(0)$.

(iii) Since A has no weak approximate identity and $K=(0)$, Φ is not onto and one-to-one by Theorem 2. Hence we have $Q(A)=A$. This completes the construction.

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