Holomorphically projective curvature tensors in certain almost Kählerian spaces

By

Hideo MIZUSAWA and Satoshi KOTŌ

(Received April 20, 1960)

Introduction

Recently one of the authors has defined an almost Kählerian space which is a generatization of a Kählerian space and called it an $*O$-almost Kählerian space or briefly an $*O$-space [5]. An $*O$-space is characterized by the fact that the covariant derivative of the structure tensor fields $\nabla_j F_i^{\cdot h}$ is pure with respect to j and i, where ∇_j denotes the covariant derivative with respect to the Riemannian connection.

On the other hand, in an almost complex space with a φ-connection, in a Kählerian space or in a K-space, a holomorphically projective transformation and a holomorphically projective curvature tensor have been studied in [8], [2], [3], [4], and [10]. In this paper, we shall define the notion of the holomorphically projective transformation, and the holomorphically projective curvature tensor in an $*O$-space.

In the next place, we shall consider an $*O$-space of constant holomorphic sectional curvature and an $*O$-space satisfying the axiom of holomophic planes.

When the holmorphically projective curvature tensor vanishes, we shall prove that the space is of constant holomorphic sectional curvature and satisfies the axiom of holomorphic planes. In the last section, we shall show that a K-space with a vanishing holmorphically projectitive curvature is necessarilly a Kählerian space.

§1. *O-almost Kählerian spaces and K-spaces

A $2n$-dimensional differentiable space, with a tensor field $F_j^{\cdot i}$ and a positive definite Riemannian metric tensor field g_{ji} satisfying

$$F_j^{\cdot r} F_r^{\cdot i} = -\delta_j^i. \tag{1.1}$$

$$g_{ji} = F_j^{\cdot b} F_i^{\cdot a} g_{ba}. \tag{1.2}$$

is called an almost Hermitian space.

An almost Hermitian space is called an $*O$-almost Kählerian or a K-space, if a tensor $F_{ji} = F_j^{\cdot r} g_{ri}$ satisfies.

(1.3) $$\nabla_j F_{ih} + F_j^{\cdot b} F_i^{\cdot a} \nabla_b F_{ah} = 0$$

or

(1.4) $$\nabla_j F_{ih} + \nabla_i F_{jh} = 0,$$

respectively. Transvecting (1.3) and (1.4) with g^{jh}, we see that an *O-space and a K-space both satisfy

(1.5) $$\nabla_r F_i^{\cdot r} = 0.$$

Let T_{jih}, T_{kjih} be tensors in an almost Hermitian space and we define the following operation

(1.6) $$O_{ji} T_{jih} = \frac{1}{2}(T_{jih} - F_j^{\cdot b} F_i^{\cdot a} T_{bah}),$$

$$*O_{ji} T_{jih} = \frac{1}{2}(T_{jih} + F_j{}^b F_j{}^a T_{bah}).$$

For the tensor T_{ji} we denote $*O_{ji}T_{ji} = *OT_{ji}$ briefly.

(1.7) $$*O_{kj}*O_{ih}T_{kjih} = \frac{1}{4}(T_{kjih} + F_k^{\cdot b} F_j^{\cdot a} T_{baih} + F_i^{\cdot b} F_h^{\cdot a} T_{kjba} + F_k^{\cdot b} F_j^{\cdot a} F_i^{\cdot d} F_h^{\cdot c} T_{badc}).$$

We see

(1.8) $$O_{ji}O_{ji} = O_{ji},\quad *O_{ji}*O_{ji} = *O_{ji},\quad *O_{ji}O_{ji} = O_{ji}*O_{ji} = 0,$$
$$*O_{kj}*O_{ih} = *O_{ih}*O_{kj},\quad O_{kj}O_{ih} = O_{ih}O_{kj}.$$

A tensor is called pure (hybrid) in two indices if the tensor vanishes by transvection of $*O(O)$ on these indices.

From this definition, the condition (1.3) can be written in the form;

(1.9) $$*O_{ji} \nabla_j F_{ih} = 0.$$

In an almost Hermitian space, using (1.2), we have

(1.10) $$*O_{ji} \nabla_h F_{ji} = 0.$$

Since an *O-space and K-space are an almost Hermitian space, we shall operate $*O_{ih}$ to (1.4) and using (1.4) and (1.10), we have

$$*O_{ji} \nabla_j F_{ih} = 0.$$

Hence a K-space is necessarily an *O-space.

Let $K_{kji}{}^h$ be the curvature tensor, i.e.

(1.11) $$K_{kji}{}^h = \partial_k \left\{ {h \atop ji} \right\} - \partial_j \left\{ {h \atop ki} \right\} + \left\{ {h \atop kr} \right\} \left\{ {r \atop ji} \right\} - \left\{ {h \atop jr} \right\} \left\{ {r \atop ki} \right\}$$

where $\partial_k = \partial / \partial x^k$, and denote

(1.12) $$K_{kjih} = K_{kji}{}^r g_{rh},\quad K_{ji} = K_{rji}{}^r,\quad \tilde{K}_{ji} = F_j^{\cdot r} K_{ir},\quad K = g^{ji} K_{ji}$$

$$H_{ji}=\frac{1}{2}F^{ab}K_{baji},\quad \tilde{H}_{ji}=F_{j}^{\cdot r}H_{ir},\quad H=F^{ji}H_{ji}.$$

From which we see

$$F^{kh}K_{kjih}=H_{ji}. \tag{1.13}$$

Now, if we assume that $K_{ji}=\tilde{H}_{ji}$, then by the symmetry of K_{ji} we have $\tilde{H}_{ji}=\tilde{H}_{ij}$, which means that $OH_{ji}=0$ by definition (1.12). The relation $K_{ji}=\tilde{H}_{ji}$ is equivalent to $\tilde{K}_{ji}=H_{ji}$, from which we have $OK_{ji}=0$ by the anti-symmetry of H_{ji}. Transvecting $K_{ji}=\tilde{H}_{ji}$ with g^{ji}, we get $K=H$.

We notice that a semi-Kählerian space of type II and an almost Kählerian space with an almost analytic Nijenhuis tensor satisfy the relation $K_{ji}=\tilde{H}_{ji}$. S. Koto. [6], [7].

§2. Holomorphically projective transformations and Holomorphycally projective curvature tensors

We introduce the curves satisfying the differential equations

$$\frac{d^2x^h}{dt^2}+\left\{{h \atop ji}\right\}\frac{dx^j}{dt}\frac{dx^i}{dt}=\alpha(t)\frac{dx^h}{dt}+\beta(t)F_{i}^{\cdot h}\frac{dx^i}{dt}. \tag{2.1}$$

Such a curve is called a holomorphically flat curve or a complex geodesic [4].

If in an *O-space there are two connections $\left\{{h \atop ji}\right\}$ and $'\left\{{h \atop ji}\right\}$, and if the two connections have all holomorphically flat curves in common, then

$$'\left\{{h \atop ji}\right\}=\left\{{h \atop ji}\right\}+\delta_j^h\rho_i+\delta_i^h\rho_j+F_{j}^{\cdot h}\sigma_i+F_{i}^{\cdot h}\sigma_j \tag{2.2}$$

holds for certain vectors fields ρ_i and σ_i.

Under the restriction (1.5) on both of the connections, we have

$$\sigma_i=-\tilde{\rho}_i$$

where
$$\tilde{\rho}_i=F_{i}^{\cdot r}\rho_r.$$

Accordingly (2.2) becomes

$$'\left\{{h \atop ji}\right\}=\left\{{h \atop ji}\right\}+\delta_j^h\rho_i+\delta_i^h\rho_j-F_{j}^{\cdot h}\tilde{\rho}_i-F_{i}^{\cdot h}\tilde{\rho}_j. \tag{2.3}$$

This transformation is called a holomorphically projective transformation (*H. P.* transformation) in an almost Hermitian space with the relation (1.5).

After some calculations, from (1.11) and (2.3), we obtain

(2.4)
$$'K_{kji}{}^{h}=K_{kji}{}^{h}+\delta_i{}^h(P_{kj}-P_{jk})+(\delta_j{}^hP_{ki}-\delta_k{}^hP_{ji})-(P_{kl}F_j^{\cdot h}-P_{jl}F_k^{\cdot h})F_i^{\cdot l}$$
$$-(P_{kl}F_j^{\cdot l}-P_{jl}F_k^{\cdot l})F_i^{\cdot h}+[F_i^{\cdot h}(\nabla_jF_k^{\cdot l}-\nabla_kF_j^{\cdot l})+F_i^{\cdot l}(\nabla_jF_k^{\cdot h}-\nabla_kF_j^{\cdot h})$$
$$+F_k^{\cdot h}\nabla_jF_i^{\cdot l}-F_j^{\cdot h}\nabla_kF_i^{\cdot l}+F_k^{\cdot l}\nabla_jF_i^{\cdot h}-F_j^{\cdot l}\nabla_kF_i^{\cdot h}]\rho_l,$$

where we have put

(2.5)
$$P_{ji}=\nabla_j\rho_i-\rho_j\rho_i+\tilde{\rho}_j\tilde{\rho}_i.$$

By contraction over i and h in (2.4), we have

(2.6)
$$P_{ji}=P_{ij}.$$

By contraction over k and h in (2.4), and using (1.3), we have

(2.7)
$$A_{ji}=-2(nP_{ji}+F_j^{\cdot b}F_i^{\cdot a}P_{ba})+F_r^{\cdot l}(\nabla_jF_i^{\cdot r}+\nabla_iF_j^{\cdot r})\rho_l$$

where we have put $A_{ji}='K_{ji}-K_{ji}$.
Operating $*O_{ji}$ to (2.7) and using (1.3), we have

(2.8)
$$P_{ji}+F_j^{\cdot b}F_i^{\cdot a}P_{ba}=-\frac{1}{n+1}*OA_{ji},$$

From which

(2.9)
$$F_j^{\cdot l}P_{li}-F_i^{\cdot l}P_{lj}=-\frac{1}{2(n+1)}(F_j^{\cdot l}A_{li}-F_i^{\cdot l}A_{lj}).$$

Next, transvecting (2.4) with $F_h^{\cdot k}$, we get

(2.10)
$$B_{ji}=-2F_j^{\cdot l}P_{li}+2nF_i^{\cdot l}P_{lj}-[\nabla_iF_j^{\cdot l}-(2n+1)\nabla_jF_i^{\cdot l}]\rho_l$$

where we have put $B_{ji}='H_{ji}-H_{ji}$.

Since B_{ji} is skew symmetric with respect to j and i, we get

(2.11)
$$B_{ji}=(n+1)[-(F_j^{\cdot l}P_{li}-F_i^{\cdot l}P_{lj})+(\nabla_jF_i^{\cdot l}-\nabla_iF_j^{\cdot l})\rho_l],$$

(2.12)
$$0=(n-1)(F_j^{\cdot l}P_{li}+F_i^{\cdot l}P_{lj})+n(\nabla_jF_i^{\cdot l}+\nabla_iF_j^{\cdot l})\rho_l.$$

Operating $*O_{ji}$ to (2.11) and comparing with (2.9), we get

$$*O_{ji}(F_j{}^lA_{li})=*OB_{ji}.$$

Transvecting (2.12) with $F_j^{\cdot r}$ and using (1.3), we have

(2.13)
$$F_r^{\cdot l}(\nabla_jF_i^{\cdot r}+\nabla_iF_j^{\cdot r})\rho_l=\frac{n-1}{n}(-P_{ji}+F_j^{\cdot b}F_i^{\cdot a}P_{ba}).$$

From (2.7), (2.8) and (2.13), we have

(2.14) $$P_{ji}=\frac{1}{2(n^2-1)}(*OA_{ji}-nA_{ji}).$$

Substituting (2.9) into (2.11), we get

(2.15) $$(\nabla_j F_i^{\cdot l}-\nabla_i F_j^{\cdot l})\rho_l=\frac{1}{2(n+1)}(-F_j^{\cdot l}A_{li}+F_i^{\cdot l}A_{lj}+2B_{ji}).$$

Substituting (2.14) into (2.12), we get

(2.16) $$(\nabla_j F_i^{\cdot l}+\nabla_i F_j^{\cdot l})\rho_l=\frac{1}{2(n+1)}(F_j^{\cdot l}A_{li}+F_i^{\cdot l}A_{lj}).$$

From (2.15) and (2.16), we obtain

(2.17) $$(\nabla_j F_i^{\cdot l})\rho_l=\frac{1}{2(n+1)}(F_i^{\cdot l}A_{lj}+B_{ji}).$$

We substitute (2.14) and (2.17) into (2.4), and operate $*O_{kj}*O_{ih}$ to this equation. Then by virtue of (1.10) we see that the tensor

(2.18) $$\begin{aligned}P_{kjih}\equiv *O_{kj}*O_{ih}[K_{kjih}-\frac{1}{2(n^2-1)}\{&g_{jh}(*OK_{ki}-nK_{ki})-g_{kh}(*OK_{ji}-nK_{ji})\\ &-F_{jh}F_i^{\cdot l}(*OK_{kl}-K_{kl})+F_{kh}F_i^{\cdot l}(*OK_{jl}-K_{jl})\\ &-(n-1)(F_{jh}H_{ki}-F_{kh}H_{ji})-2(n-1)F_{ih}*OH_{kj}\}]\end{aligned}$$

is invariant under the *H. P.* transformation. We call it the *H. P.* curvature tensor in an $*O$-space.

Taking account of (1.7), it is written down as follows:

(2.19) $$P_{kjih}\equiv *O_{kj}*O_{ih}K_{kjih}+\frac{1}{4(n+1)}(g_{jh}L_{ki}-g_{kh}L_{ji}+F_{jh}\tilde{L}_{ki}-F_{kh}\tilde{L}_{ji}+2F_{ih}*OH_{kj}),$$

where we have put

(2.20) $$L_{ji}\equiv *OK_{ji}+*O\tilde{H}_{ji},\quad \tilde{L}_{ji}\equiv F_j^{\cdot l}L_{il}=*OH_{ji}+*O\tilde{K}_{ji}.$$

THEOREM 2.1 *In an $*O$-space the H. P. curvature tensor with the form* (2.19) *is invariant under the H. P. transformation* (2.3).

We notice that, if the space is Kählerian, the following relations are known:

(2.21) $$*O_{kj}*O_{ih}K_{kjih}=K_{kjih},\quad *OK_{ji}=K_{ji},\quad *OH_{ji}=H_{ji},\quad \tilde{K}_{ji}=H_{ji}.$$

From which in a Kählerian space, we find

(2.22) $$P_{kjih}=K_{kjih}+\frac{1}{2(n+1)}(g_{jh}K_{ki}-g_{kh}K_{ji}+F_{jh}H_{ki}-F_{kh}H_{ji}+2F_{ih}H_{kj}).$$

§3. *O-spaces with a vanishing H. P. curvature tensor

In an *O-space, if the *H. P.* curvature tensor identically vanishes, then by virtue of (1.7) and (2.19), we obtain

$$*O_{kj}*O_{ih}K_{kjih}=-\frac{1}{4(n+1)}(g_{jh}L_{ki}-g_{kh}L_{ji}+F_{jh}\tilde{L}_{ki}-F_{kh}\tilde{L}_{ji}+2F_{ih}*OH_{kj}), \tag{3.1}$$

$$*O_{kj}*O_{ih}K_{kjih}=\frac{1}{4}(K_{kjih}+F_{k}^{\cdot b}F_{j}^{\cdot a}K_{baih}+F_{i}^{\cdot b}F_{h}^{\cdot a}K_{kjba}+F_{k}^{\cdot b}F_{j}^{\cdot a}F_{i}^{\cdot d}F_{h}^{\cdot c}K_{badc}). \tag{3.2}$$

Transvecting (3.1) and (3.2) with g^{ji}, we have

$$2(n-1)*OK_{ji}+2(n+1)*O\tilde{H}_{ji}=g_{ji}(K+H). \tag{3.3}$$

Transvecting (3.1) with g^{kh}, we have

$$*OK_{ji}=*O\tilde{H}_{ji}. \tag{3.4}$$

From (3.3) and (3.4), we have

$$*OK_{ji}=*O\tilde{H}_{ji}=\frac{1}{4n}(K+H)g_{ji}. \tag{3.5}$$

From which, we have

$$*O\tilde{K}_{ji}=-*OH_{ji}=\frac{1}{4n}(K+H)F_{ji}. \tag{3.6}$$

Substituing (3.5) and (3.6) into (3.1), we obtain

$$*O_{kj}*O_{ih}K_{kjih}=\frac{k}{4}(g_{ji}g_{kh}-g_{ki}g_{jh}+F_{ji}F_{kh}-F_{ki}F_{jh}-2F_{kj}F_{ih}), \tag{3.7}$$

where

$$k=\frac{1}{2n(n+1)}(K+H).$$

THEOREM 3.1. *In an *O-space, if a H. P. curvature tensor vanishes, then the curvature tensor of the space has the form* (3.7).

Notice that if the space is Kählerian, (3.7) be reduced to

$$K_{kjih}=\frac{k}{4}(g_{ji}g_{kh}-g_{ki}g_{jh}+F_{ji}F_{kh}-F_{ki}F_{jh}-2F_{kj}F_{ih}), \tag{3.8}$$

$$k=\frac{K}{n(n+1)}.$$

§4. Almost Hermitian spaces of constant holomorphic sectional curvatures

We consider in an almost Hermitian space a holomorphic sectional curvature with respect to a vector u^h

$$k=\frac{-K_{mjlh}F_q^{\cdot m}u^qF_p^{\cdot l}u^pu^ju^h}{g_{kj}u^ku^jg_{ih}u^iu^h}. \tag{4.1}$$

If $k=$ constant with respect to any vector at any point of the space, then the space is called a space of constant holomorphic sectional curvature.

In this case

$$K_{mjlh}F_q^{\cdot m}F_p^{\cdot l}u^qu^ju^pu^h=-kg_{qj}g_{ph}u^qu^ju^pu^h \tag{4.2}$$

should be satisfied for any vector u^h, from which we get

$$4!\,F_{(q}^{\cdot m}F_p^{\cdot l}K_{|m|j|l|h)}=-8k(g_{qp}g_{hj}+g_{qh}g_{jp}+g_{qj}g_{ph}). \tag{4.3}$$

Transvecting (4.3) with $F_k^{\cdot q}F_i^{\cdot h}$, we have

$$2[(K_{kjih}+K_{ijkh})-F_k^{\cdot q}F_h^{\cdot l}(K_{ljiq}+K_{ijlq})-F_i^{\cdot p}F_h^{\cdot m}(K_{kjmp}+K_{mjkp})$$

$$-F_j^{\cdot m}F_i^{\cdot l}(K_{mhkp}+K_{khmp})-F_k^{\cdot q}F_j^{\cdot l}(K_{ihlq}+K_{lhiq})+F_k^{\cdot q}F_i^{\cdot p}F_j^{\cdot m}F_h^{\cdot l}(K_{mqlp}+K_{lqmp})$$

$$=-8kF_k^{\cdot q}F_i^{\cdot p}(g_{qp}g_{hj}+g_{qh}g_{jp}+g_{qj}g_{ph}).$$

This equation is written as follows

$$\begin{aligned}&{}^*O_{kj}{}^*O_{ih}K_{kjih}+{}^*O_{ij}{}^*O_{kh}K_{ijkh}-F_k^{\cdot q}F_h^{\cdot l}{}^*O_{lj}{}^*O_{iq}K_{ljiq}\\&=-k(g_{ki}g_{hj}+F_{kh}F_{ij}+F_{kj}F_{ih}).\end{aligned} \tag{4.4}$$

Taking the alternating part with respect to k and j, we obtain

$$\begin{aligned}&2{}^*O_{kj}{}^*O_{ih}K_{kjih}+2{}^*O_{kj}({}^*O_{ki}{}^*O_{jh}K_{kijh}-{}^*O_{ji}{}^*O_{kh}K_{jikh})\\&=k(g_{kh}g_{ji}-g_{jh}g_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}).\end{aligned} \tag{4.5}$$

THEOREM 4.1. *If an almost Hermitian space has a constant holomorphic sectional curvature at every point, then the curvature tensor of the space satisfies* (4.5).

Now we shall prove that if an $^*O$-space satisfies the relation $K_{ji}=\tilde{H}_{ji}$ then the k in (4.5) is an absolute constant, in §1 we have see that if $K_{ji}=\tilde{H}_{ji}$ holds then $OK_{ji}=0$, $OH_{ji}=0$ and $K=H$ are valid. Taking account of these relations, we shall apply the Bianchi identity to (4.4).

From (1.7) the first term of the left hand side of (4.4) is following:

$$4{}^*O_{kj}{}^*O_{ih}K_{kjih}=K_{kjih}+F_k^{\cdot l}F_j^{\cdot m}K_{lmih}+F_i^{\cdot t}F_h^{\cdot s}K_{kits}+F_k^{\cdot l}F_j^{\cdot m}F_i^{\cdot t}F_h^{\cdot s}K_{lmts}. \tag{4.6}$$

Applying the Bianchi identity to the first term of the right hand side of (4.6), we have

$$\nabla_p K_{kjih} + \nabla_k K_{jpih} + \nabla_j K_{pkih} = 0.$$

Transvecting with $g^{ph}g^{ji}$, we get

$$2\nabla^p K_{kp} - \nabla_k K = 0.$$

Next, as to the second term of (4.6), we get

$$\begin{aligned} g^{ph}g^{ji}[\nabla_p(F_k^{\cdot l} F_j^{\cdot m} K_{lmih}) + \nabla_k(F_j^{\cdot l} F_p^{\cdot m} K_{lmih}) + \nabla_j(F_p^{\cdot l} F_k^{\cdot m} K_{lmih})] \\ = 2\nabla^p \tilde{H}_{kp} - \nabla_k H \\ = 2\nabla^p K_{kp} - \nabla_k K \\ = 0. \end{aligned}$$

By this way, we find that the third term of (4.6) vanishes and the fourth term becomes

$$2\nabla^p(F_p^{\cdot s} F_k^{\cdot m} K_{sm}) - \nabla_k K = 0$$

by virtue of $OK_{pk}=0$.

Thus the first term of the left hand side of (4.4) vanishes. Similarly the second and third term of the left hand side of (4.4) are reduced to $-4\nabla_k K$ and $4\nabla_k K$ respectively. Therefore the left hand side of (4.4) is zero by this way. As to the right hand side of (4.4) we obtain $4(1-n^2)\nabla_l k$ in the same way. Hence we have $\nabla_l k=0$.

Notice that in a Kählerian space [1], [11], formula (4.5) be reduced to

$$K_{kjih} = \frac{k}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + F_{kh}F_{ji} - F_{jh}F_{ki} - 2F_{kj}F_{ih}). \tag{4.7}$$

In an *O-space by means of Theorem 3.1 and 4.1 we can easily have the following

THEOREM 4.2. *In an *O-space satisfying* $K_{ji}=\tilde{H}_{ji}$, *if a H. P. curvature tensor vanishes, then the space is of constant holomorphic curvature.*

§5. Almost Hermitian spaces satisfying the axiom of holomorphic planes

In an almost Hermitian space, there is given a holomorphic plane element determined by two vectors u^h and $F_i{}^h u^i$ at a point. When we can always draw a 2-dimensional totally geodesic surface passing through this point and being tangent to the given holomorphic plane element, we say that the space satisfies the axiom of holomorphic planes.

If we represent such a surface by the parametric equation

(5.1) $$x^h = x^h(y^a) \qquad a.\ b.\ c.\ d. = 1,2,$$

then the fact the surface is totally geodesic is represented by the equation

(5.2) $$\partial_c B_b^h + B_c^j B_b^i \{{}^{\,h}_{ji}\} - B_a^h {}' \{{}^{\,a}_{cb}\} = 0$$

where $B_b^h = \frac{\partial x^h}{\partial y^b}$ and $'\{{}^{\,a}_{cb}\}$ is the Christoffel symbol formed with the $g_{cb} = B_c^j B_b^i g_{ji}$ of the surface.

The integrability condition of (5.2) are

(5.3) $$B_d^k B_c^j B_b^i K_{kji}{}^h = B_a^h {}' K_{dcb}{}^a.$$

If we put

$$B_1^h = u^h, \quad B_2^h = F_i^{\cdot h} u^i$$

equation (5.3) must be satisfied by any unit vector u^h. Thus we have

(5.4) $$\begin{cases} F_s^{\cdot m} u^s u^j u^i K_{mji}{}^h = \alpha u^h + \beta F_p^{\cdot h} u^p, \\ F_s^{\cdot m} u^s u^j F_q^{\cdot l} u^q K_{mjl}{}^h = \lambda u^h + \mu F_p^{\cdot h} u^p. \end{cases}$$

From the first equation of (5.4), we obtain

(5.5) $$(F_s^{\cdot m} K_{mji}{}^h - \alpha g_{sj} \delta_i^h - \beta g_{sj} F_i^{\cdot h}) u^s u^j u^i = 0,$$

from which

(5.6) $$F_{(s}^{\cdot m} K_{|m|ji)h} = \alpha g_{(sj} g_{i)h} + \beta g_{(sj} F_{i)h}.$$

Contracting by g_{ji}, we get $\alpha = 0$.

Transvecting this with $F_k^{\cdot s}$ and taking the alternating part with respect to k and j, we obtain

(5.7) $$\begin{aligned} &{}^*O_{kj} {}^*O_{ih} K_{kjih} + {}^*O_{kj} ({}^*O_{ki} {}^*O_{jh} K_{kijh} - {}^*O_{ji} {}^*O_{kh} K_{jikh}) \\ &= 2\beta (g_{kh} g_{ji} - g_{jh} g_{ki} + F_{kh} F_{ji} - F_{jh} F_{ki} - 2F_{kj} F_{ih}) \end{aligned}$$

which shows that the space is of constant holomorphic curvature. Thus we have

THEOREM 5.1. *If an almost Hermitian space satisfying* $K_{ji} = \widetilde{H}_{ji}$ *admits the axiom of holomorphic planes, then the space is of constant holomorphic sectional curvature.*

In an *O-space, by means of Theorem 3.1, we can easily have the following

THEOREM 5.2. *In an *O-space, if a H. P. curvature tensor vanishes, then the space admits the axiom of holomorphic planes.*

Notice that in a Kählerian space, formula (5.7) be reduced to

(5.8) $$K_{kjih}=\frac{\beta}{4}(g_{kh}g_{ji}-g_{jh}g_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}).$$

§6. K-spaces with a vanishing H. P. curvature

In a K-space, if we operate ∇_k to (1.4) and taking alternating part with respect to k and j, we have

(6.1) $$-K_{kji}{}^{r}F_{rh}-K_{kjh}{}^{r}F_{ir}+\nabla_k\nabla_iF_{jh}-\nabla_j\nabla_iF_{kh}=0.$$

Transvecting (6.1) with g^{ji} and using (1.5) we have

(6.2) $$\tilde{K}_{kh}+H_{hk}=\nabla^r\nabla_rF_{hk}$$

from which

$$\tilde{K}_{ji}+\tilde{K}_{ij}=0$$

Hence we have

(6.3) $$OK_{ji}=0$$

Transvecting (6.1) with F^{kj}, we get directly

(6.4) $$OH_{ji}=0$$

Using (6.2) aud (6.3), we have [9]

(6.5) $$O\tilde{K}_{ji}=0$$

(6.6) $$O\tilde{H}_{ji}=0$$

Lemma (6.1) [5] *A necessary and sufficient condition that a K-space be Kählerian is*

(6.7) $$K_{ji}=\tilde{H}_{ji}$$

holds good.

Proof. In a Kählerian space $\nabla_hF_{ji}=0$, is valid, hence from (6.2) it follows that

$$\tilde{K}_{ji}+H_{ij}=0.$$

This is equivalent to (6.7). Conversely we assume a K-space satisfies (6.7), then using (1.5), we get

$$0=\nabla_r[\nabla_j(F^{ji}F_i^{\cdot r})]=(\nabla_rF_{ji})(\nabla_jF_i^{\cdot r})+F_{ji}(\nabla_r\nabla_jF_i^{\cdot r}).$$

By virtue of (1.4), we have

(6.8) $$F^{ji}(\nabla^r\nabla_rF_{ji})+(\nabla_rF_{ji})(\nabla^rF^{ji})=0.$$

From (6.2) and (6.8), we obtain

$$(\nabla_r F_{ji})(\nabla^r F_{ji}) = F^{ji}(\tilde{K}_{ij} + H_{ji}).$$

Thus we have

$$\nabla_r F_{ji} = 0.$$

This means that the space is Kählerian. q. e. d.

Using (6.3), (6.4) and (2.18) we obtain

THEOREM 6.1. *In a K-space the H. P. curvature tensor has the form*

$$(6.9)\quad P_{kjih} = {}^*O_{kj}{}^*O_{jh}\left[K_{kjih} + \frac{1}{2(n+1)}(g_{jh}K_{ki} - g_{kh}K_{ji} + F_{jh}H_{ki} - F_{kh}H_{ji} + 2F_{ih}H_{kj})\right].$$

In a K-space, if a $H.\ P.$ curvature vanishes, then using (6.1), (6.4) and (3.7) we have

$$K_{ji} = \tilde{H}_{ji}.$$

By virtue of Lemma (6.1), we obtain

THEOREM 6.2 *A K-space with a vanishing H. P. curvature is necessarilly Kählerian.*

Bibliography

[1] Bochner, S.: *Curvature in Hermitian manifolds.* Bull. Amer. Math. 53(1947), 179–195.

[2] Ishihara, S.: *Holomorphically projective changes and their groups in an almost complex maniforld.* Tôhoku Math. J. 9(1957), 273–297.

[3] Ishihara, S.: *On holomorphic planes.* Ann. di. Mate. Pure ed App. Serie IV - Tomo XLVII (1959), 197–241.

[4] Ishihara. S. and Tachibana, S.: *On infinitesimal holomorphically projective transformations in Kählerian manifolds.* Tôhoku Math. J. 12(1960), 77–101.

[5] Kotō, S.: *Some theorems on almost Kählerian spaces,* J. Math. Soc. Japan 12(1960), 422–433.

[6] Kotō, S.: *Curvatures on Hermitian spaces.* to appear.

[7] Kotō, S.: *Almost Kählerian spaces with an almost analytic Nijenhuis tensor.* to appear.

[8] Tashiro, Y.: *On a holomorphically projective correspondence in an almost complex space,* Math. J. Okayama Univ. 6(1957), 147–152.

[9] Tachibana, S.: *On almost analytic victors in certain almost Hermitian manifolds.* Tohoku Math. J. 11(1959), 352–363.

[10] Tachidana, S.: *On infinitesimal holomorphically projective transformations in certain almost-Hermitian spaces.* Nat. Sci. Rep. Ochanomizu Univ. 10(1959), 45–51.

[11] Yano, K. and Mogi, I.: *On real representations of Kählerian manifolds.* Ann. Math. 61(1955), 169–189.

Facu'ty of Science and Faculty of Education,
Niigata University