On almost-analytic vectors in almost-Hermitian spaces

By

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(Received December 20, 1959)

§1. Introduction

Let M be an almost-complex space of class C^∞ and denote by $\varphi_a{}^b$ its almost-complex structure tensor satisfying $\varphi_a{}^b\varphi_b{}^c=-\delta_a^c$.

If $\varphi_a{}^b$ and a positive definite Riemannian metric tensor g_{ji} on M satisfies $g_{rs}\varphi_j{}^r\varphi_i{}^s=g_{ji}$, then M is called an almost-Hermitian space and if an almost-Hermitian space satisfies $\partial_{[j}\varphi_{ih]}=0$ where $\varphi_{ji}=g_{ri}\varphi_j{}^r$, then the space is called an almost-Kählerian space.

Almost-complex space, almost-Hermitian space and almost-Kählerian space are called complex space, Hermitian space and Kählerian space respectively if the Nijenhuis tensor $N_{ji}{}^h$ constructed from $\varphi_j{}^i$ vanishes [6].

In a 2n-dim. Hermitian space X_{2n}, we consider a contravariant vector v^i and a covariant vector u_i, and if v^i satisfies

$$(1.1)\qquad \partial_{\bar\lambda}v^\alpha=0,\ \partial_\lambda v^{\bar\alpha}=0,$$

where the Latin indices take the values 1, $\cdots$, n, $\bar 1$, $\cdots$, $\bar n$ and the Greek indices run over the range 1, $\cdots$, n, then v^i is called analytic. It is well known that in a Kählerian space, (1.1) is equivalent to

$$\nabla_{\bar\lambda}v^\alpha=0,\ \nabla_\lambda v^{\bar\alpha}=0 \text{ or } \varphi_i{}^l\nabla_l v^h-\varphi_l{}^h\nabla_i v^l=0$$

where ∇ denotes the operator of covariant derivative w. r. t. the Riemannian connection.

Similarly if u_i satisfies the following

$$(1.2)\qquad \partial_{\bar\lambda}u_\alpha=0,\ \partial_\lambda u_{\bar\alpha}=0$$

which is, in a Kählerian space, equivalent to

$$\nabla_{\bar\lambda}u_\alpha=0,\ \nabla_\lambda u_{\bar\alpha}=0 \text{ or } \varphi_j{}^l\nabla_i u_l-\varphi_i{}^l\nabla_l u_j=0,$$

then u_i is called analytic [6].

Now if we try to generalize the notion of analytic vector in a Hermitian space X_{2n} to an n-dim. almost-Hermitian space X_n, then we shall have the following definition [3], i.e. in an almost-Hermitian space, we say that a contravariant vector

v^i is almost-analytic if it satisfies

(1.3) $$\nabla_j v_k + \varphi_j{}^r \varphi_{lk} \nabla_r v^l - v^r (\nabla_r \varphi_j{}^l) \varphi_{lk} = 0$$

and a covariant vector u_i is almost-analytic if it satisfies

(1.4) $$\nabla_j u_k + \varphi_j{}^r \varphi_k{}^t \nabla_r u_t + \varphi_j{}^t (\nabla_t \varphi_k{}^s) u_s - \varphi_j{}^t (\nabla_k \varphi_t{}^s) u_s = 0$$

where the Latin indices run over 1, 2, $\cdots$ n.

S. Tachibana, in [3], gave a necessary and sufficient condition that a vector in an almost-Kählerian space be almost-analytic and recently S. Koto, in [1] gave a necessary and sufficient condition that a contravariant vector in his *O*-space be almost-analytic.

The main purpose of this paper is to try the same thing for vectors in an almost-Hermitian space.

§2. Curvature tensor

We considers the Riemannian connection $\left\{ {h \atop ji} \right\}$ in an n-dim. almost-Hermitian space X_n and let $R_{kji}{}^h$ be the curvature tensor field defined in the usual manner, that is,

(2.1) $$R_{kji}{}^h = \partial_k \left\{ {h \atop ji} \right\} - \partial_j \left\{ {h \atop ki} \right\} + \left\{ {h \atop kl} \right\} \left\{ {l \atop ji} \right\} - \left\{ {h \atop jl} \right\} \left\{ {l \atop ki} \right\}$$

and put $$R_{ji} = R_{rji}{}^r, \quad R_{kjih} = g_{rh} R_{kji}{}^r \text{ and}$$

$$R^*{}_{kj} = \frac{1}{2} \varphi^{ab} R_{abrj} \varphi_k{}^r.$$

From the definition (2.1), we have $R_{(kj)ih} = 0$, $R_{kj(ih)} = 0$.

The Ricci's identities are given by the following forms for tensor $\varphi_i{}^j$ and vector [2],

(2.2) $$\nabla_k \nabla_j \varphi_i{}^h - \nabla_j \nabla_k \varphi_i{}^h = R_{kjr}{}^h \varphi_i{}^r - R_{kji}{}^r \varphi_r{}^h$$

transvecting (2.2) with g^{ki}, we get

(2.3) $$\nabla^r \nabla_j \varphi_r{}^h = R_{ijr}{}^h \varphi^{ir} + R_j{}^r \varphi_r{}^h + \nabla_j \nabla_r \varphi^{rh}$$

where $$\varphi^{ir} = g^{li} \varphi_l{}^r.$$

On the other hand, by the Bianchi's identity, we have

(2.4) $$R_{ijr}{}^h \varphi^{ir} = \frac{1}{2} \varphi^{ir} (R_{ijr}{}^h - R_{rji}{}^h) = \frac{1}{2} \varphi^{ir} R_{irj}{}^h.$$

Hence, (2.3) turns to

(2.5) $$\nabla^r \nabla_j \varphi_{rh} = \frac{1}{2} \varphi^{ir} R_{irjh} + R_j{}^r \varphi_{rh} + \nabla_j \nabla^r \varphi_{rh}$$

and therefore

$$\varphi_l{}^i\nabla^j\nabla_r\varphi_{ji}=\frac{1}{2}\varphi^{ab}R_{abri}\varphi_l{}^i+R_{rl}+\varphi_l{}^i\nabla_r\nabla^j\varphi_{ji} \tag{2.6}$$

$$=R_{lr}-R^*{}_{lr}+\varphi_l{}^i\nabla_r\nabla^j\varphi_{ji}.$$

Moreover, by the Ricci's identity, we have

$$\varphi_l{}^i\varphi^{ab}\nabla_a\nabla_b v_i=\frac{1}{2}\varphi_l{}^i\varphi^{ab}(\nabla_a\nabla_b v_i-\nabla_b\nabla_a v_i) \tag{2.7}$$

$$=\frac{1}{2}\varphi_l{}^i\varphi^{ab}(-R_{abi}{}^s v_s)=-v^s R^*{}_{ls}.$$

§3. Somo properties of *O-space, almost-Kählerian space and K-space

Here an *O-space implies an almost-Hermitian space satisfying

$$*O_{ji}^{ab}\nabla_a\varphi_{bh}=0 \tag{3.1}$$

where $*O_{ji}^{ab}=\frac{1}{2}(\delta_j^a\delta_i^b+\varphi_j{}^a\varphi_i{}^b)$ [1].

In an *O-space, since $\varphi_i{}^l\nabla_l\varphi_{jh}=\varphi_j{}^l\nabla_i\varphi_{lh}$, $N_{jih}=g_{rh}N_{ji}{}^r$ can be written in the form

$$N_{jih}=\varphi_j{}^l(\nabla_l\varphi_{ih}-\nabla_i\varphi_{lh})-\varphi_i{}^l(\nabla_l\varphi_{jh}-\nabla_j\varphi_{lh}) \tag{3.2}$$
$$=2\varphi_j{}^l(\nabla_l\varphi_{ih}-\nabla_i\varphi_{lh})$$

and we get, easily, from (3.1),

$$\nabla_j\varphi_i{}^j=0. \tag{3.3}$$

Almost-Kählerian space, as we stated in the preceding paragraph, implies an almost-Hermitian space satisfying

$$F_{jih}\overset{\text{def.}}{=}\nabla_j\varphi_{ih}+\nabla_i\varphi_{hj}+\nabla_h\varphi_{ji}=0 \tag{3.4}$$

and K-space [4] implies an almost-Hermitian space satisfying

$$\nabla_j\varphi_{ik}+\nabla_i\varphi_{jk}=0. \tag{3.5}$$

Moreover, it can be verified that almost-Kählerian space and K-space are both *O-spaces. In fact, putting

$$P_{jih}=*O_{ji}^{ab}\nabla_a\varphi_{bh}=\nabla_j\varphi_{ih}+\varphi_j{}^a\varphi_i{}^b\nabla_a\varphi_{bh},$$

$$Q_{jih}=*O_{ji}^{ab}(\nabla_a\varphi_{bh}+\nabla_b\varphi_{ha}+\nabla_h\varphi_{ab})=\nabla_j\varphi_{ih}+\nabla_i\varphi_{hj}+\varphi_j{}^a\varphi_i{}^b(\nabla_a\varphi_{bh}+\nabla_b\varphi_{ha})$$

and writing out their squares, we get

$$\frac{1}{2} P_{jih}P^{jih}=\frac{1}{2}Q_{jih}Q^{jih}=(\nabla^j\varphi^{ih})\nabla_j\varphi_{ih}+\varphi_j{}^a\varphi_i{}^b(\nabla_a\varphi_{bh})\nabla^j\varphi^{ih} \tag{3.6}$$

which shows that $P_{jih}=0$ is equivalent to $Q_{jih}=0$. Thus, we see that if $\nabla_{[j}\varphi_{ih]}=0$, then we get $*O^{ab}_{ji}\nabla_a\varphi_{bh}=0$. On the other hand, from (3.5), it follows easily that $*O^{ab}_{ji}\nabla_a\varphi_{bh}=0$.

And therefore, in an almost-Kählerian space, we have, from (3.2),

$$N_{jih}=-2\varphi_j{}^l\nabla_h\varphi_{li}=-2\varphi_h{}^l\nabla_l\varphi_{ji} \tag{3.7}$$

and in a K-space, we have

$$N_{jih}=4\varphi_j{}^l\nabla_l\varphi_{ih}. \tag{3.8}$$

But, in a K-space, we have also following relations [4]:

$$(\nabla_j\varphi_{ab})\nabla_i\varphi^{ab}=R_{ji}-R^*_{ji},\quad R^*_{ji}=R^*_{ij},\quad N_{j(ih)}=0. \tag{3.9}$$

Hence, we get, from (3.8)

$$\begin{aligned} N_{jih}N^{ji}{}_k&=16\varphi_j{}^l(\nabla_l\varphi_{ih})\varphi^{js}\nabla_s\varphi^i{}_k=16(\nabla_h\varphi_{is})\nabla_k\varphi^{is}\\ &=16(R_{hk}-R^*_{hk}). \end{aligned} \tag{3.10}$$

§4. Contravariant almost-analytic vectors in an almost-Hermitian space

Let X_n be an almost-Hermitian space and v^i be a contravariant vector in X_n. When v^i is a contrvariant almost-analytic vector, we have, from (1.3),

$$-P_{jk}\overset{\text{def.}}{=}\nabla_j v_k+\varphi_j{}^r\varphi_{lk}\nabla_r v^l-v^r(\nabla_r\varphi_j{}^l)\varphi_{lk}=0 \tag{4.1}$$

and therefore we have, from $\frac{1}{2}\varphi_b{}^k(\nabla_t\varphi^{jb})P_{jk}=0$,

$$\frac{1}{2}\varphi_b{}^k(\nabla_t\varphi^{jb})\nabla_j v_k+\frac{1}{2}\varphi_j{}^r(\nabla_t\varphi^{jb})\nabla_r v_b-\frac{1}{2}v^r(\nabla_t\varphi^{jb})\nabla_r\varphi_{jb}=0 \quad \textit{i.e.}$$

$$\frac{1}{2}v^r(\nabla_r\varphi_{jb})\nabla_k\varphi^{jb}+\varphi_{rb}(\nabla_k\varphi^{jb})\nabla_j v^r=0 \tag{4.2}$$

and

$$\nabla^j P_{jk}=0. \tag{4.3}$$

And moreover, we get

$$\nabla^j P_{jk}+\frac{1}{2}v^r(\nabla_r\varphi_{jb})\nabla_k\varphi^{jb}+\varphi_{rb}(\nabla_k\varphi^{jb})\nabla_j v^r=0. \tag{4.4}$$

Hence (4.2) and (4.3), or (4.4) is a necessary condition for v^i to be almost-analytic.

To prove the converse, writing out the square of P_{jk}, we get

$$(4.5)\quad \frac{1}{2}P_{jk}P^{jk}$$
$$=(\nabla_j v_k)\nabla^j v^k+\varphi_j{}^r\varphi_{lk}(\nabla^j v^k)\nabla_r v^l-2v^r\varphi_{lk}(\nabla^j v^k)\nabla_r\varphi_j{}^l+\frac{1}{2}v^a v^r(\nabla_r\varphi_{jb})\nabla_a\varphi^{jb}.$$

Consequently, we have

$$(4.6)\quad \frac{1}{2}P_{jk}P^{jk}+\nabla^j(P_{jk}v^k)=\frac{1}{2}P_{jk}P^{jk}+(\nabla^j P_{jk})v^k+P_{jk}\nabla^j v^k$$
$$=v^k[\nabla^j P_{jk}+\frac{1}{2}v^r(\nabla_r\varphi_{jb})\nabla_k\varphi^{jb}+\varphi_{bj}(\nabla_k\varphi_a{}^j)\nabla^a v^b].$$

From (4.6), it follows that if the space X_n is compact, then by virtue of Green's theorem, we have

$$(4.7)\quad \int_{X_n}[v^k\{\nabla^j P_{jk}+\frac{1}{2}v^r(\nabla_r\varphi_{jb})\nabla_k\varphi^{jb}+\varphi_{bj}(\nabla_k\varphi_a{}^j)\nabla^a v^b\}-\frac{1}{2}P_{jk}P^{jk}]d\sigma=0$$

where $d\sigma$ means the volume element of the space X_n. (4.7) shows that, if the space is compact, then (4.2) and (4.3), or (4.4) is a sufficient condition for a vector v^i to be almost-analytic.

In this place, by using (2.6) and (2.7), $\nabla^j P_{jk}$ can also be written as

$$\begin{aligned}\nabla^j P_{jk}=&-\nabla^j\nabla_j v_k-\varphi_j{}^r\varphi_{lk}\nabla^j\nabla_r v^l+v^r(\nabla^j\nabla_r\varphi_j{}^l)\varphi_{lk}-\varphi_j{}^r(\nabla_r v^l)\nabla^j\varphi_{lk}\\&+\nabla^j v^r(\nabla_r\varphi_j{}^l)\varphi_{lk}+v^r(\nabla_r\varphi_j{}^l)\nabla^j\varphi_{lk}-\nabla^j\varphi_j{}^r(\nabla_r v^l)\varphi_{lk}\\=&-\nabla^j\nabla_j v_k-v^s R^*_{ks}+v^r(R^*_{kr}-R_{kr}-\varphi_k{}^i\nabla_r\nabla^j\varphi_{ji})-\varphi_j{}^r(\nabla_r v^l)\nabla^j\varphi_{lk}\\&+\nabla^j v^r(\nabla_r\varphi_j{}^l)\varphi_{lk}+v^r(\nabla_r\varphi_j{}^l)\nabla^j\varphi_{lk}-\nabla^j\varphi_j{}^r(\nabla_r v^l)\varphi_{lk}\\=&-\nabla^j\nabla_j v_k-v^r R_{rk}+\nabla^j v^r(\nabla_s\varphi_{rk}+\nabla_r\varphi_{ks})\varphi_j{}^s+v^r(\nabla_r\varphi_j{}^l)\nabla^j\varphi_{lk}\\&-v^r\varphi_k{}^i\nabla_r\nabla^j\varphi_{ji}-\nabla^j\varphi_j{}^r(\nabla_r v^l)\varphi_{lk}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\nabla^j P_{jk}+\frac{1}{2}v^r(\nabla_r\varphi_{jl})\nabla_k\varphi^{jl}+\varphi_{rl}(\nabla_k\varphi^{jl})\nabla_j v^r\\&=-\nabla^j\nabla_j v_k-v^r R_{rk}+\nabla^j v^r(\nabla_l\varphi_{rk}+\nabla_r\varphi_{kl})\varphi_j{}^l+\nabla^j v^r(\nabla_k\varphi_j{}^l)\varphi_{rl}+v^r(\nabla_r\varphi_j{}^l)\nabla^j\varphi_{lk}\\&\quad+\frac{1}{2}v^r(\nabla_r\varphi_{jl})\nabla_k\varphi^{jl}-v^r\varphi_k{}^i\nabla_r\nabla^j\varphi_{ji}-\nabla^j\varphi_j{}^r(\nabla_r v^l)\varphi_{lk}\\&=-\nabla^j\nabla_j v_k-v^r R_{rk}+\nabla^j v^r(\nabla_l\varphi_{rk}+\nabla_r\varphi_{kl}+\nabla_k\varphi_{lr})\varphi_j{}^l+\frac{1}{2}v^r(\nabla_r\varphi^{jl})\\&\times(\nabla_j\varphi_{lk}+\nabla_l\varphi_{kj}+\nabla_k\varphi_{jl})\\&=-\nabla^j\nabla_j v_k-v^r R_{rk}+(\varphi_r{}^j\nabla^r v^l+\frac{1}{2}v^r\nabla_r\varphi^{jl})F_{jlk}-v^r\varphi_k{}^i\nabla_r\nabla^j\varphi_{ji}-\nabla^j\varphi_j{}^r(\nabla_r v^l)\varphi_{lk}.\end{aligned}$$

Thus, we have the following

THEOREM 4.1. *In a compact almost-Hermitian space, a necessary and sufficient condition that a contravariant vector v^i be almost-analytic is that v^i satisfy*

(1) $$\nabla^j\nabla_j v_k + v^r R_{rk} - \nabla^j v^r(\nabla_s\varphi_{rk} + \nabla_r\varphi_{ks})\varphi_j{}^s - v^r(\nabla_r\varphi_j{}^l)\nabla^j\varphi_{lk}$$
$$+v^r\varphi_k{}^i\nabla_r\nabla^j\varphi_{ji} + \nabla^j\varphi_j{}^r(\nabla_r v^l)\varphi_{lk} = 0,$$
$$\frac{1}{2}v^r(\nabla_r\varphi_{jl})\nabla_k\varphi^{jl} + \varphi_{rl}(\nabla_k\varphi^{jl})\nabla_j v^r = 0$$

or

(2) $$\nabla^j\nabla_j v_k + v^r R_{rk} - (\varphi_r{}^j\nabla^r v^l + \frac{1}{2}v^r\nabla_r\varphi^{jl})F_{jlk} + v^r\varphi_k{}^i\nabla_r\nabla^j\varphi_{ji}$$
$$+\nabla^j\varphi_j{}^r(\nabla_r v^l)\varphi_{lk} = 0.$$

Next, we consider a contravariant almost-analytic v^i in an *O-space.

We have, from (1.3),

(4.8) $$v^r\nabla_r\varphi_j{}^i - \varphi_j{}^r\nabla_r v^i + \varphi_r{}^i\nabla_j v^r = 0$$

and then multiplying (4.8) by $\nabla^j\varphi_{ki}$, we get

(4.9) $$v^r(\nabla^j\varphi_{ki})\nabla_r\varphi_j{}^i - \varphi_j{}^r(\nabla^j\varphi_{ki})\nabla_r v^i + \varphi_r{}^i(\nabla^j\varphi_{ki})\nabla_j v^r = 0$$

but since, by (3.1), $\varphi_r{}^i(\nabla^j\varphi_{ki})\nabla_j v^r = -\varphi_i{}^j(\nabla^i\varphi_{kr})\nabla_j v^r$, (4.9) can also be written as

(4.10) $$v^r(\nabla^j\varphi_{ki})\nabla_r\varphi_j{}^i - 2\varphi_j{}^r(\nabla^j\varphi_{ki})\nabla_r v^i = 0$$

and similarly, multiplying (4.8) by $\nabla_i\varphi_k{}^j$, we get

(4.11) $$v^r(\nabla_i\varphi_k{}^j)\nabla_r\varphi_j{}^i - 2\varphi_j{}^r(\nabla_i\varphi_k{}^j)\nabla_r v^i = 0.$$

Forming (4.10) - (4.11), we find

$$v^r(\nabla_r\varphi_j{}^l)\nabla^j\varphi_{lk} + \nabla^j v^r(\nabla_s\varphi_{rk} + \nabla_r\varphi_{ks})\varphi_j{}^s = 0.$$

Hence, on taking account of (3.3), we have, from (1) in Theorem 4.1.,

(4.12) $$\nabla^j\nabla_j v_k + v^r R_{rk} = 0$$

and therefore, we get, from (2),

(4.13) $$(\varphi_r{}^j\nabla^r v^l + \frac{1}{2}v^r\nabla_r\varphi^{jl})F_{jlk} = 0 \quad i.e. \quad F_{jlk}\underset{v}{\pounds}\varphi^{jl} = 0$$

where $\underset{v}{\pounds}$ is the operator of Lie derivative.

Thus we have the following

THEOREM 4.2. *In a compact *O-space, a necessary and sufficient condition that a contravariant vector v^i be almost-analytic is that v^i satisfy*

$$\nabla^j\nabla_j v_k + v^r R_{rk} = 0, \quad F_{jlk}\underset{v}{\pounds}\varphi^{jl} = 0.$$

That is the result obtained by S. Koto [1].

If the space is an almost-Kählerian space, we have, from Theorem 4.2 or (2) in Theorem 4.1, the following result obtained by S. Tachibana [3]:

THEOREM 4.3. *In a compact almost-Kählerian space, a necessary aud sufficient condition that a contravariant vector v^i be almost-analytic is that it satisfy*

$$\nabla^j\nabla_j v_k+v^r R_{rk}=0.$$

Finally, if the space is a K-space, then

$$(\varphi_r{}^j\nabla^r v^l+\frac{1}{2}v^r\nabla_r\varphi^{jl})F_{jlk}=3[\varphi_r{}^j(\nabla_j\varphi_{lk})\nabla^r v^l+\frac{1}{2}v^r(\nabla_r\varphi^{jl})\nabla_k\varphi_{jl}]$$
$$=3[\frac{1}{4}N_{rlk}\nabla^r v^l+\frac{1}{2}v^r(R_{rk}-R^*{}_{rk})].$$

Thus we have, from Theorem 4.2, the following

THEOREM 4.4 *In a compact K-space, a necessary and sufficient condition that a contravariant vector v^i be almost-analytic is that v^i satisfy*

$$\nabla^j\nabla_j v_k+v^r R_{rk}=0,\quad N_{rlk}\nabla^r v^l+2v^r(R_{rk}-R^*{}_{rk})=0$$

This is the result obtained by S. Tachibana [4].

§5. Covariant almost-analytic vectors in an almost-Hermitian space

Again let X_n be an almost-Hermitian space and u_i be a covariant vector in X_n. If u_i is almost-analytic, then we have, from (1.4),

$$(5.1)\qquad -P_{jk}\overset{\text{def}}{=}\nabla_j u_k+\varphi_j{}^r\varphi_k{}^t\nabla_r u_t+\varphi_j{}^t(\nabla_t\varphi_k{}^s)u_s-\varphi_j{}^t(\nabla_k\varphi_t{}^s)u_s=0$$

or from $\varphi_j{}^r\varphi_k{}^t P_{rt}=0$,

$$(5.2)\qquad \nabla_j u_k+\varphi_j{}^r\varphi_k{}^t\nabla_r u_t-\varphi_k{}^t(\nabla_j\varphi_t{}^s)u_s+\varphi_k{}^t(\nabla_t\varphi_j{}^s)u_s=0$$

and then forming (5.1) - (5.2), we find

$$(5.3)\qquad u^s[\varphi_j{}^t(\nabla_t\varphi_{ks}-\nabla_k\varphi_{ts})-\varphi_k{}^t(\nabla_t\varphi_{js}-\nabla_j\varphi_{ts})]=0 \text{ i.e. } N_{jks}u^s=0\ [5].$$

Consequently, we have, from (5.1)

$$(5.4)\qquad -Q_{jk}\overset{\text{def}}{=}\nabla_j u_k+\varphi_j{}^r\varphi_k{}^t\nabla_r u_t+\frac{u^s}{2}[\varphi_j{}^t(\nabla_t\varphi_{ks}-\nabla_k\varphi_{ts})+\varphi_k{}^t(\nabla_t\varphi_{js}-\nabla_j\varphi_{ts})]=0,$$
$$\nabla^j Q_{jk}=0.$$

On the other hand, we have, from $[(\nabla^j\varphi_s{}^l)\varphi_l{}^k-(\nabla^l\varphi_s{}^j)\varphi_l{}^k]\times$(5.2),

$$(5.5)\qquad \nabla^j u^k[\varphi_k{}^l(\nabla_l\varphi_{js}-\nabla_j\varphi_{ls})+\varphi_j{}^l(\nabla_l\varphi_{ks}-\nabla_k\varphi_{ls})]+2u_b\nabla^j\varphi^{lb}(\nabla_j\varphi_{ls}-\nabla_l\varphi_{js})=0.$$

Next, we shall go to the converse.

Writing out the square of P_{jk}, we get

$$\frac{1}{2}P_{jk}P^{jk}=(\nabla^j u^k)\nabla_j u_k+\varphi_j{}^r\varphi_k{}^t(\nabla_r u_t)\nabla^j u^k+\varphi_j{}^t(\nabla_t\varphi_k{}^s)u_s\nabla^j u^k-\varphi_j{}^t(\nabla_k\varphi_t{}^s)u_s\nabla^j u^k$$

$$+\varphi^{kb}\nabla^a\varphi_k{}^s(\nabla_a u_b)u_s-\varphi^{kb}(\nabla_k\varphi^{as})u_s\nabla_a u_b-\nabla_k\varphi^{as}(\nabla_a\varphi^{kb})u_s u_b$$
$$+\frac{1}{2}\nabla^a\varphi_k{}^s(\nabla_a\varphi^{kb})u_s u_b+\frac{1}{2}\nabla_k\varphi^{as}(\nabla^k\varphi_a{}^b)u_s u_b$$

and therefore we have

$$(5.6)\qquad \frac{1}{2}P_{jk}P^{jk}+\nabla^j(u^k Q_{jk})=\frac{1}{2}P_{jk}P^{jk}+u^k\nabla^j Q_{jk}+(\nabla^j u^k)Q_{jk}$$
$$=u^k[\nabla^j Q_{jk}+\frac{1}{2}\varphi_j{}^t(\nabla_t\varphi_{lk}-\nabla_l\varphi_{tk})\nabla^j u^l-\frac{1}{2}\varphi_l{}^t(\nabla_t\varphi_{jk}-\nabla_j\varphi_{tk})\nabla^j u^l$$
$$+u_b(\nabla_a\varphi_{lk}-\nabla_l\varphi_{ak})\nabla^a\varphi^{lb}+\nabla^j u^l(\nabla_t\varphi_{jk}-\nabla_j\varphi_{tk})\varphi_l{}^t]$$
$$=\frac{u^k}{2}[2\nabla^j Q_{jk}+\nabla^j u^l\{\varphi_l{}^t(\nabla_t\varphi_{jk}-\nabla_j\varphi_{tk})+\varphi_j{}^t(\nabla_t\varphi_{lk}-\nabla_l\varphi_{tk})\}$$
$$+2u_b(\nabla_a\varphi_{lk}-\nabla_l\varphi_{ak})\nabla^a\varphi^{lb}].$$

Hence, if X_n is compact, then we obtain, by Green's theorem,

$$(5.7)\qquad \int_{X_n}[\frac{1}{2}u^k\{2\nabla^j Q_{jk}+(\nabla^j u^l)\varphi_l{}^t(\nabla_t\varphi_{jk}-\nabla_j\varphi_{tk})+(\nabla^j u^l)\varphi_j{}^t(\nabla_t\varphi_{lk}-\nabla_l\varphi_{tk})$$
$$+2u_b(\nabla_a\varphi_{lk}-\nabla_l\varphi_{ak})\nabla^a\varphi^{lb}\}-\frac{1}{2}P_{jk}P^{jk}]d\sigma=0.$$

Thus, from (5.4), (5.5) and (5.7), we have the following

THEOREM 5.1. *In a compact almost-Hermitian space, a necessary and sufficient condition that a covariant vector* u_i *be almost-analytic is that* u_i *satisfy any one of the following two conditions:*

(1) $\nabla^j Q_{jk}=0,\ (\nabla^j u^l)[\varphi_l{}^t(\nabla_t\varphi_{jk}-\nabla_j\varphi_{tk})+\varphi_j{}^t(\nabla_t\varphi_{lk}-\nabla_l\varphi_{tk})]$
$+2u_b(\nabla_j\varphi_{lk}-\nabla_l\varphi_{jk})\nabla^j\varphi^{lb}=0,$

(2) $2\nabla^j Q_{jk}+(\nabla^j u^l)[\varphi_l{}^t(\nabla_t\varphi_{jk}-\nabla_j\varphi_{tk})+\varphi_j{}^t(\nabla_t\varphi_{lk}-\nabla_l\varphi_{tk})]$
$+2u_b(\nabla_j\varphi_{lk}-\nabla_l\varphi_{jk})\nabla^j\varphi^{lb}=0$

where $Q_{jk}=-\nabla_j u_k-\varphi_j{}^r\varphi_k{}^t\nabla_r u_t-\frac{u^s}{2}[\varphi_j{}^t(\nabla_t\varphi_{ks}-\nabla_k\varphi_{ts})+\varphi_k{}^t(\nabla_t\varphi_{js}-\nabla_j\varphi_{ts})].$

In this place, using (2.7), $\nabla^j Q_{jk}$ can be written in the form

$$(5.8)\qquad \nabla^j Q_{jk}=-\nabla^j\nabla_j u_k+u^s R^*{}_{ks}-\varphi_j{}^r(\nabla^j\varphi_k{}^t)\nabla_r u_t-\varphi_k{}^t(\nabla^j\varphi_j{}^r)\nabla_r u_t$$
$$-\frac{1}{2}\nabla^j[u^s\varphi_j{}^t(\nabla_t\varphi_{ks}-\nabla_k\varphi_{ts})+u^s\varphi_k{}^t(\nabla_t\varphi_{js}-\nabla_j\varphi_{ts})].$$

If the space X_n is an *O-space, then we get, by (3.2) and (3.1),

$$(5.9)\qquad \varphi_l{}^t(\nabla_t\varphi_{jk}-\nabla_j\varphi_{tk})+\varphi_j{}^t(\nabla_t\varphi_{lk}-\nabla_l\varphi_{tk})=\frac{1}{2}N_{ljk}+\frac{1}{2}N_{jlk}=0,$$

$$(5.10)\qquad \nabla^j\varphi^{lb}(\nabla_j\varphi_{ls}-\nabla_l\varphi_{js})=\frac{1}{2}(\nabla_j\varphi_{ls}-\nabla_l\varphi_{js})(\nabla^j\varphi^{lb}-\nabla^l\varphi^{jb})$$

$$= -\frac{1}{2}\varphi_j{}^a\varphi_l{}^c(\nabla_a\varphi_{cs}-\nabla_c\varphi_{as})(\nabla^j\varphi^{lb}-\nabla^l\varphi^{jb})$$

$$= \frac{1}{8}N_{jcs}N^{jcb}.$$

By virtue of (5.9) and (3.3), (5.8) turns to

$$\nabla^j Q_{jk} = -\nabla^j\nabla_j u_k + u^s R^*{}_{ks} - \varphi_j{}^r(\nabla^j\varphi_k{}^t)\nabla_r u_t. \tag{5.11}$$

Thus, from the condition (1) in Theorem 5.1, we have the following

THEOREM 5.2. *In a compact *O-space, a necessary and sufficient condition that a covariant vector u_i be almost-analytic is that u_i satisfy*

$$\nabla^j\nabla_j u_k - u^s R^*{}_{ks} + \varphi_j{}^a(\nabla^j\varphi_k{}^b)\nabla_a u_b = 0,\quad u^r N_{jir}N^{ji}{}_k = 0$$

and if the rank of the matrix $\|N_{jir}N^{ji}{}_k\|$ is n, then there exists no covariant almost-analytic vector other than the zero vector.

We next assume that the space X_n is an almost-Kählerian space. On taking account of (3.7), we have the following

THEOREM 5.3. *In a compact almost-Kählerian space, a necessary and sufficient condition that a covariant vector u_i be almost-analytic is that u_i satisfy*

$$\nabla^j\nabla_j u_k - u^s R^*{}_{ks} + \frac{1}{2}N_{kji}\nabla^i u^j = 0,\quad u^r N_{jir}N^{ji}{}_k = 0.$$

Perhaps this is a different result from the one obtained by S. Tachibana [3].

Finally, let X_n be a K-space. Now $u^r N_{jir}N^{ji}{}_k = 0$ is equivalent to $u^r N_{jir} = 0$ and by (3.8) this last equation turns to

$$u^r\nabla_r\varphi_{jl} = 0. \tag{5.12}$$

Operating ∇^j to (5.12) and multiplying by $\varphi_s{}^l$, we get

$$\varphi_s{}^l(\nabla^j u^r)\nabla_r\varphi_{jl} + u^r\varphi_s{}^l\nabla^j\nabla_r\varphi_{jl} = 0$$

or by (2.6) and (3.5)

$$-\varphi_l{}^j(\nabla^l\varphi_s{}^r)\nabla_j u_r + u^r(R_{sr} - R^*{}_{sr}) = 0 \tag{5.13}$$

and moreover, by (3.9), (5.13) can also be written as

$$-\varphi_l{}^j(\nabla^l\varphi_s{}^r)\nabla_j u_r + u^r(\nabla_r\varphi_{jl})\nabla_s\varphi^{jl} = 0. \tag{5.14}$$

From (5.14), it follows that if $u^r\nabla_r\varphi_{jl} = 0$, then $\varphi_l{}^j(\nabla^l\varphi_s{}^r)\nabla_j u_r = 0$.
But in K-space, $u^r N_{jir}N^{ji}{}_k = 0$ *i.e.* $u^r(R_{rk} - R^*{}_{rk}) = 0$ or $u^r(\nabla_r\varphi_{jl})\nabla_k\varphi^{jl} = 0$ is equivalent to $u^r\nabla_r\varphi_{jl} = 0$ and therefore, according to theorem 5.2, we have the following

THEOREM 5.4. *In a compact K-space, a necessary and sufficient condition that a covariant vector u_i be almost-analytic is that u_i satisfy*

$$\nabla^j \nabla_j u_k - u^s R_{ks} = 0, \quad u^r (R_{rk} - R^*_{rk}) = 0$$

and in a K-space, if the rank of the matrix $\|R_{rk} - R^*_{rk}\|$ *is* n, *then there exists no covariant almost-analytic vector other than the zero vector.*

THEOREM 5.5. *In a compact K-space, a harmonic vector* u_i *is almost-analytic if and only if* $u^r (R_{rk} - R^*_{rk}) = 0$.

These theorems in *O-space can also be obtained from the following Theorem 5.6.

That is, since for a covariant almost-analytic vector u_i, (5.3) holds good, (5.1) is equivalent to

$$(5.15) \quad \nabla_j u_k + \varphi_j{}^r \varphi_k{}^t \nabla_r u_t + \frac{1}{2} u^s [\varphi_j{}^t (\nabla_t \varphi_{ks} - \nabla_k \varphi_{ts}) + \varphi_k{}^t (\nabla_t \varphi_{js} - \nabla_j \varphi_{ts})] = 0,$$
$$N_{jks} u^s = 0.$$

Thus, in an *O-space, by virtue of (5.9), we have the following

THEOREM 5.6. *In an *O-space, a vector* u_i *is covariant almost-analytic if and only if*

$$^*O_{jk}^{ab} \nabla_a u_b = 0, \quad N_{jks} u^s = 0$$

Theoem 5.1. was written following the advice of Mr. S. Koto, to whom I want to express my hearty thanks.

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