On the immersibility of almost parallelizable manifolds

By

T. WATABE and T. HIROSE

(Received, December 15, 1963)

Introduction

M. W. Hirsch has shown that an almost parallelizable n-manifold is immersible in Euclidean $(n+k)$ - space R^{n+k} if $n<2k$ [1]. He has proved the result by making use of a result due to M. Kervaire that the Smale invariant of an immersion of n-sphere in R^{n+k} vanishes if $n\leqq 2k-2$ [4]. In this paper we shall prove the following;

Proposition 1;

An almost parallelizable n-manifold is immersible in R^{n+1} *if* $n\not\equiv 0$ *(mod 4).*

Proposition 2;

If $n\equiv 0$ *(mod 4), an almost parallelizable n - manifold is in general not immersible in* R^{n+1}. *In particular, the Hirsch's result is best possible for* $n=4$.

The authors wish to express their thanks to Prof. K. Aoki and Prof. T. Kaneko for their many valuable suggestions and several discussions.

§ 1. Definitions and Lemmas

In the following discussion, all manifolds are considered as connected, orientable, C^∞ manifolds. By *immersion* $f: M^n\to R^p$ we mean a C^∞ map whose Jacobian matrix has rank $n=dim\ M^n$ at each point. A homeomorphic immersion will be called *imbedding*. A manifold M^n will be called *parallelizable* if its tangent bundle is trivial, we say M^n is *almost parallelizable* if $M^n - x$ is parallelizable for some $x\in M^n$. M^n will be called *π-manifold* if M^n is imbedded in R^{n+k} $(k>n)$ with trivial normal bundle ν^k.

Since a non - closed (i.e. non - compact or with boundary) almost parallelizable manifold is parallelizable, hence it is immersible in R^{n+1}. (Theorem 6.3 of [2]). Therefore we may consider only closed manifolds. First we consider the condition for that an orientable n-manifold is immersible in R^{n+1}.

Lemma 1;

Let M^n *be an orientable manifold. Then it is necessary and sufficient for* M^n *to be immersible in* R^{n+1} *is that it is a π-manifold.*

(**Proof**) If M^n is a π-manifold then M^n is immersible in R^{n+1}, according to M. W. Hirsch [2]. Let f be an immersion of M^n into R^{n+1}, and g be any imbedding of M^n in R^{n+k} $(k>n)$. By a result due to M. W. Hirsch [2], g is regularly homotopic to $i\cdot f$ where i denotes the inclusion map $i: R^{n+1}\to R^{n+k}$, hence the normal bundle induced by g is trivial, since the normal bundle of f is trivial. Thus M^n is a π-manifold.

Next we consider the principal $SO(k)$ bundle associated to the normal bundle ν induced by an imbedding of an almost parallelizable manifold M^n in R^{n+k} $(k>n)$. By definition, $\nu|M^n-x$ (x is some point of M^n), the restriction of ν to M^n-x. is trivial. Hence ν admits a cross section over M^n-x. Let o_n denote the obstruction to the extension over M^n of the cross section over M^n-x. o_n is an element of π_{n-1} $(SO(k))$. The following lemma is well known.

Lemma 2;

Let $J: \pi_{n-1}(SO(k))\to\pi_{n+k-1}(S^k)$ *be the Hopf -Whitehead homomorphism. Then* $J(o_n)=0$.

In his paper [3], M. Kervaire has obtained the relation between o_n and pontryagin class when $n\equiv 0$ (mod 4). His result is as follows:

Lemma 3;

Let the fundamental class of M^{4k} *be* M^{4k}, *the* $4k$ - *dimensional Pontryagin class* p_k *of the bundle* ξ *takes on* M^{4k} *the value given by the formula;*

$$p_k(\xi)[M^{4k}]=a_k\cdot o_n\cdot(2k-1)! \quad \text{up to sign,}$$

where a_k *is equal to* 1 *or* 2 *according to whether* k *is even or odd respectively.*

Moreover J. Milnor and M. Kervaire have proved in [5];

Lemma 4;

There exist an almost parallelizable manifold M_0^{4k} *with*

$$p_k[M_0^{4k}]=j_k\cdot a_k\cdot(2k-1)!$$

where p_k *is the* $4k$ *dimensional Pontryagin class of* M_0^{4k}, j_k *denotes the order of the finite cyclic group* $J\pi_{4k-1}(SO(m))$, $(m>4k-1)$, *and* a_k *is equal to* 1 *or* 2 *according to whether* k *is even or odd respectively.*

Recently J. F. Admas has proved in [6] that;

Lemma 5;

If $n\equiv 1, 2$ *(mod 8), then* $J: \pi_{n-1}$ $(SO(k))\to\pi_{n+k-1}(S^k)$ $(k>n)$ *is injective.*

§ 2. Proof of the Propositions

In the case of $n\not\equiv 0$ (*mod* 4), $\pi_{n-1}(SO(k))=0$ or Z_2 according to whether $n\equiv 3,5,6,7$ (*mod* 8) or $n\equiv 1,2$ (*mod* 8) respectively. From this it follows that in the case of $n\equiv 3,5,6,7$ (*mod* 8), $o_n=0$. In the case $n\equiv 1,2$ (*mod* 8) o_n is also zero, since it belongs to the kernel of J (Lemma 2) which is injective [1]. Hence M^n is a π-manifold, so that it is immersible in R^{n+1} [2], this complete the proof of the Proposition 1.

In the case $n\equiv 0$ (*mod* 4), by lemma 4, there exists an almost parallelizable $4k$ - dimensional manifold M_0^{4k} with $p_k\,[M_0^{4k}]\neq 0$ [3]. By Whitney dulity, the $4k$ - dimensional Pontryagin class of $p_k\,(\nu)$ of the normal bundle ν coincides with $4k$ - dimensional Pontryagin class of M_0^{4k}, up to sign, therefore $p_k\,(\nu)\,[M_0^{4k}]\neq 0$. Hence it follows from lemma 3, that $o_{4k}\neq 0$ [4]. This means M_0^{4k} is not a π-manifold, so that it is not immersible in R^{n+1}.

In particular we can prove that M_0^4 is not immersible in R^6, for suppose M_0^4 be immersed in R^6, then the normal bundle induced by this immersion is trivial since the second Stifel-Whitney class $w_2(\nu)=0$ hence by lemma 6.4 in [2], M_0^4 is immersible in R^5, which is a contradiction. In other words, the Hirsch's result is best possible for $n=4$.

References

[1] M. W. Hirsch: *Immersions of almost parallelizable manifolds*, Proc. Amer. Math. Soc., Vol. 12, (1961), pp. 845–846.

[2] ——————: *Immersions of manifolds*, Trans. Amer. Math. Soc., Vol. 93, (1959), pp. 242–276.

[3] M. Kervaire: *On the Pontryagin class of certain SO(n) - bnndle*, Amer. J. Math., Vol. 80, (1958), pp. 632–638.

[4] ——————: *Sur l'invariant de Smale d'un plongement*, Comm. Math. Helv., Vol. 34, (1960), pp. 127–139.

[5] J. Milnor and M. Kervaire: *Bernoulli numbers, homotopy gronps and a theorem of Rohlin*, Proc. International Congress of Math. (1958), pp. 454–458.

[6] J. F. Adams: *Vector fields on spheres*, Ann. of Math. Vol. 75, (1962), pp. 603–632.