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MINIMAL SUBMANIFOLDS IMMERSED IN A COMPLEX SPACE FORM

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ABSTRACT. We study a minimal submanifold M of a complex space form $M^m(c)$ of constant holomorphic sectional curvature c. We give a Simons' type formula for $|A|^2$, the square of the length of the second fundamental form A of a minimal submanifold M of $M^m(c), c > 0$. As applications, we prove some pinching theorems in terms of $|A|^2$. We also give some fundamental results of submanifolds of $M^m(c)$ related to the structure induced from the action of the almost complex structure of $M^m(c)$. We do not assume the condition that the immersion is compatible with the standard fibration.

1. Introduction

In 1968, Simons [10] gave the integral formula for the square of the length of the second fundamental form A of a compact n-dimensional minimal submanifold M in a real space form $M^m(k)$ of constant curvature k. The specific expression of the formula is the following:

$$\frac{1}{2}\Delta|A|^2 = nk|A|^2 - \sum_{a,b}(\mathrm{tr}A_{v_a}A_{v_b})^2 + \sum_{a,b}\mathrm{tr}[A_{v_a}, A_{v_b}]^2 + |\nabla A|^2,$$

where $\{v_1, \cdots, v_p\}$ is an orthonormal basis of normal vector space and $p = m - n$ is the codimension of M. Here we denote by $|\cdot|$ the length of a tensor with respect to the Riemannian metric g on M and by $[\ ,\ ]$ the commutator.

As an application, Simons proved that if the second fundamental form A of a compact n-dimensional minimal submanifold M in S^{n+p} satisfies $|A|^2 < n/(2-1/p)$, then M is totally geodesic. Moreover, Chern, do Carmo and Kobayashi [4] proved that if the second fundamental form A satisfies $|A|^2 = n/(2-1/p)$, then M is a Clifford hypersurface or a Veronese surface in S^4.

The Simons' type formula was studied by many authors under different situations, and many interesting results are given. For the special submanifolds of complex space forms, we see the following results:

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For the study of complex submanifolds in a complex space form, Ogiue [7] and Tanno [11] showed the Simons' type formula for the square of the length of the second fundamental form.

In 1974, Chen-Ogiue [3] made the Simons' type formula for minimal totally real submanifolds in a complex space form $M^m(c)$.

Simons' type formula for generic submanifolds and CR submanifolds in $M^m(c)$ was given by Yano-Kon [13] under some additional conditions on the normal curvature and the mean curvature vector field. Using this formula, they proved some results of the pinching problem for the square of the length of the second fundamental form.

For the general submanifolds of a complex space form, a direct extension of Simons methods for the sphere to the complex projective space CP^m as an ambient space has some difficulties (see Lawson [6]). So many authors push known theorems on the sphere down to CP^m by using the following commutative diagram:

$$\begin{array}{ccc} N & \longrightarrow & S^{2m+1} \\ \downarrow & & \downarrow \\ M & \longrightarrow & CP^m, \end{array}$$

where $\pi : S^{2n+1} \longrightarrow CP^n$ is the standard fibration and N and M are submanifolds of S^{2n+1} and CP^n, respectively (cf. [6], [8], [13]).

In this paper, we compute the Simons' type formula and its useful modification for general submanifolds in a complex space form $M^m(c)$, $c > 0$, and give pinching theorems in terms of the square of the length of the second fundamental form without the assumption that the existence of the above commutative diagram for the standard fibration.

In section 2, we prepare some definitions and basic formulas for submanifolds in a complex space form $M^m(c)$. In section 3 and section 4, we give a Simons' type formula for submanifolds in a complex space form. Using the formula, we prove some pinching theorems for the square of the length of the second fundamental form or the scalar curvature. In the last section, we give some results for submanifolds in a complex space form with semi-flat normal connection.

2. Preliminaries

Let $M^m(c)$ denote the complex space form of complex dimension m (real dimension $2m$) with constant holomorphic sectional curvature c. We denote by J the almost complex structure of $M^m(c)$. The Hermitian metric of $M^m(c)$ is denoted by g.

Let M be a real n-dimensional Riemannian manifold immersed in $M^m(c)$. We denote by the same g the Riemannian metric on M induced from that of $M^m(c)$, and by p the codimension of M, that is, $p = 2m - n$. We denote by $\tilde{\nabla}$ the Levi-Civita connection in $M^m(c)$ and by ∇ the connection induced on M. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y), \qquad \tilde{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the normal connection. A normal vector field V on M is said to be parallel if $D_X V = 0$ for any vector field X tangent to M. We call both A and B the *second fundamental form* of M that are related by $g(B(X,Y),V) = g(A_V X, Y)$. The second fundamental form B is symmetric. A_V can be considered as a symmetric (n,n)-matrix.

For the second fundamental form B, we define ∇B, the covariant derivative of B, by

$$(\nabla_X B)(Y,Z) = D_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

for any vector fields X, Y and Z tangent to M. If $\nabla_X B = 0$ for all X, then the second fundamental form B of M is said to be *parallel*. This is equivalent to the condition $\nabla_X A = 0$ for all X, where $\nabla_X A$ is defined by

$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{D_X V} Y - A_V(\nabla_X Y).$$

We notice the relation

$$g((\nabla_X B)(Y,Z),V) = g((\nabla_X A)_V Y, Z).$$

The *mean curvature vector* μ of M is defined to be $\mu = (1/n)\mathrm{tr}B$, where $\mathrm{tr}B$ is the trace of B, that is $\mathrm{tr}B = \sum_i B(e_i, e_i)$, $\{e_i\}$ being an orthonormal basis for the tangent space $T_x(M)$ at x. If $\mu = 0$, then M is said to be *minimal*.

We next give some notations and fundamental formulas on M induced from the action of the almost complex structure J of $M^m(c)$ to the tangent space of M.

For any vector field X tangent to M, we put

$$JX = PX + FX,$$

where PX is the tangential part of JX and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on the tangent bundle $T(M)$.

For any vector field V normal to M, we put

$$JV = tV + fV,$$

where tV is the tangential part of JV and fV the normal part of JV. Then P and f are skew-symmetric with respect to g and $g(FX,V) = -g(X,tV)$. We also have $P^2 = -I - tF$, $FP + fF = 0$, $Pt + tf = 0$ and $f^2 = -I - Ft$. We notice that $|FP| = |fF| = |Pt| = |tf|$, where $|\cdot|$ denotes the length of a tensor with respect to g.

We define the covariant derivatives of P, F, t and f by $(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y$, $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$, $(\nabla_X t)V = \nabla_X(tV) - tD_X V$ and $(\nabla_X f)V = D_X(fV) - fD_X V$, respectively. We then have

$$(\nabla_X P)Y = A_{FY}X + tB(X,Y), \quad (\nabla_X F)Y = -B(X,PY) + fB(X,Y),$$
$$(\nabla_X t)V = -PA_V X + A_{fV}X, \quad (\nabla_X f)V = -FA_V X - B(X,tV).$$

The Riemannian curvature tensor $\tilde{R}$ of a complex space form $M^m(c)$ is given by

$$\begin{aligned}\tilde{R}(X,Y)Z =& \frac{c}{4}\Big(g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY + 2g(X,JY)JZ\Big)\end{aligned}$$

for any vector fields X, Y and Z of $M^m(c)$. Then the *equation of Gauss* and the *equation of Codazzi* are given respectively by

$$\begin{aligned}R(X,Y)Z =& \frac{c}{4}\Big(g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX - g(PX,Z)PY \\ & - 2g(PX,Y)PZ\Big) + A_{B(Y,Z)}X - A_{B(X,Z)}Y\end{aligned}$$

and

$$\begin{aligned}&(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &= \frac{c}{4}\Big(g(PY,Z)FX - g(PX,Z)FY + 2g(X,PY)FZ\Big).\end{aligned}$$

We define the curvature tensor $R^\perp$ of the normal bundle $T(M)^\perp$ of M by

$$R^\perp(X,Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}Z,$$

where X and Y are vector fields tangent to M and V is a vector field normal to M. Then we have the *equation of Ricci*:

$$\begin{aligned}&g(R^\perp(X,Y)U,V) + g([A_V,A_U]X,Y) \\ &= \frac{c}{4}\Big(g(FY,U)g(FX,V) - g(FX,U)g(FY,V) \\ &\quad + 2g(X,PY)g(fU,V)\Big),\end{aligned}$$

where [,] denotes the commutator and $[A_V, A_U] = A_V A_U - A_U A_V$.

Definition 2.1. Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. If the normal curvature tensor $R^\perp$ of M satisfies

$$R^\perp(X,Y)U = \frac{1}{2}cg(X,PY)fU$$

for any vector fields X and Y tangent to M and any vector field U normal to M, then the normal connection of M is said to be *semi-flat*.

If the normal curvature tensor $R^\perp$ of M vanishes identically, then the normal connection of M is said to be *flat*.

We denote by S the Ricci tensor field of M. Then we have

$$\begin{aligned} S(X,Y) =& \frac{c}{4}\Big((n-1)g(X,Y) + 3g(PX,PY)\Big) \\ &+ \sum_a \mathrm{tr}A_a g(A_aX,Y) - \sum_a g(A_a^2X,Y), \end{aligned}$$

where $\{v_1, \cdots v_p\}$ is an orthonormal frame for the normal space $T_x(M)^\perp$ at x and A_a is the second fundamental form in the direction of v_a. From this the scalar curvature r of M is given by

$$r = \frac{c}{4}\Big((n-1)n - 3\mathrm{tr}P^2\Big) + \sum_a (\mathrm{tr}A_a)^2 - |A|^2,$$

where $|A|^2 = \sum_a \mathrm{Tr}A_a^2$.

Next we define the notion of *CR submanifold* (Bejancu [1]).

Definition 2.2. A submanifold M of a Kähler manifold $\tilde{M}$ with almost complex structure J is called a CR submanifold of $\tilde{M}$ if there exists a differentiable distribution $\mathcal{D} : x \longrightarrow \mathcal{D}_x \subset T_x(M)$ on M satisfying the following conditions:

(i) H is holomorphic, i.e., $J\mathcal{D}_x = \mathcal{D}_x$ for each $x \in M$, and

(ii) the complementary orthogonal distribution $\mathcal{D}^\perp : x \longrightarrow \mathcal{D}_x^\perp \subset T_x(M)$ is anti-invariant, i.e., $J\mathcal{D}_x^\perp \subset T_x(M)^\perp$ for each $x \in M$.

If $JT_x(M)^\perp \subset T_x(M)$ for any point x of M, then we call M a *generic submanifold* of $\tilde{M}$. Any real hypersurface of $M^m(c)$ is obviously a generic submanifold of $M^m(c)$.

If $JT_x(M) \subset T_x(M)$ for any point x of M, then F and t vanish identically, and we call M a *complex submanifold* of $\tilde{M}$. If $JT_x(M) \subset T_x(M)^\perp$ for any point x of M, then we call M a *totally real submanifold* of $\tilde{M}$. Then P vanishes identically.

Lemma 2.3. *Let M be a CR submanifold of a Kähler manifold $\tilde{M}$. Then*

$$\begin{aligned} &FP = 0, \quad fF = 0, \quad tf = 0, \quad Pt = 0, \\ &P^3 + P = 0, \quad f^3 + f = 0. \end{aligned}$$

We use the following theorem (see [13]).

Theorem 2.4. *In order for a submanifold M of a Kähler manifold $\tilde{M}$ to be a CR submanifold, it is necessary and sufficient that $FP = 0$.*

Example. Let S^{2m+1} be a $(2m+1)$-dimensional unit sphere and N be a $(n+1)$-dimensional submanifold immersed in S^{2m+1}. With respect to the standard fibration $\pi : S^{2m+1} \longrightarrow CP^m$, we consider the following commutative diagram (cf. [6], [9], [13])

$$\begin{array}{ccc} N & \longrightarrow & S^{2m+1} \\ \downarrow & & \downarrow \\ M & \longrightarrow & CP^m. \end{array}$$

We denote by (ϕ, ξ, η, G) the contact metric structure on S^{2m+1}. The horizontal lift with respect to the connection η will be denoted by $*$. Then $(JX)^* = \phi X^*$ and $G(X^*, Y^*) = g(X, Y)^*$ for any vectors X and Y tangent to CP^m. A submanifold N in S^{2m+1} is tangent to the totally geodesic fibre of π and the structure vector field ξ is tangent to N.

Let α be the second fundamental form of N in S^{2m+1}. Then we have the relations of the second fundamental form α of N and B of M:

$$\alpha(X^*, Y^*) = B(X, Y)^*, \quad \alpha(\xi, \xi) = 0.$$

Moreover, we have

$$(\nabla_{X^*}\alpha)(Y^*, Z^*) = [(\nabla_X B)(Y, Z) + g(PX, Y)FZ + g(PX, Z)FY]^*,$$
$$(\nabla_{X^*}\alpha)(Y^*, \xi) = [fB(X, Y) - B(X, PY) - B(Y, PX)]^*,$$
$$(\nabla_{X^*}\alpha)(\xi, \xi) = -2(FPX)^*$$

for any vectors X, Y and Z tangent to M. From the third equation, we see that if the second fundamental form α of N is parallel, then $FP = 0$ and M is a CR submanifold of CP^m by Theorem 2.4.

We denote by $\mu' = (1/(n+1))\mathrm{tr}\alpha$ the mean curvature vector field of N, and by $\mu = (1/n)\mathrm{tr}B$ the mean curvature vector field of M. Then we have

$$\mu' = \frac{n}{n+1}\mu^*, \quad D'_{X^*}\mu' = \frac{n}{n+1}(D_X\mu)^*, \quad D'_\xi\mu' = (f\mu)^*,$$

where D' is the normal connection of N. Thus the mean curvature vector field μ' of N is parallel if and only if the mean curvature vector field μ of M is parallel and $f\mu = 0$.

Let $K^\perp$ be the curvature tensor of the normal bundle of N. Then we have

$$G(K^\perp(X^*, Y^*)V^*, U^*) = [g(R^\perp(X, Y)V, U) - 2g(X, PY)g(fV, U)]^*,$$
$$G(K^\perp(X^*, \xi)V^*, U^*) = g((\nabla_X f)V, U)^*$$

for any vectors X and Y tangent to M and any vectors V and U normal to M. Therefore, the normal connection of N in S^{2m+1} is flat if and only if the normal connection of M is semi-flat and $\nabla f = 0$ (see [8], [9], [13]).

We put

$$N = S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k), \quad n+1 = \sum_{i=1}^k m_i, \quad 1 = \sum_{i=1}^k r_i^2,$$

where $m_1, \cdots, m_k$ are odd numbers. Then $n+k$ is also odd. The second fundamental form α of N is parallel in S^{2m+1}. We can see that $M = \pi(N)$ is a generic submanifold in $CP^{(n+k-1)/2}$ with flat normal connection. Especially, $\pi(S^1(r_1) \times S^n(r_2))$ is called a geodesic hypersphere in $CP^{(n+1)/2}$. Moreover, M is a CR submanifold in CP^m $(m > (n+k-1)/2)$ with semi-flat normal connection and $\nabla f = 0$.

If $r_i = (m_i/(n+1))^{1/2}$ $(i = 1, \cdots, k)$, then M is a generic minimal submanifold in $CP^{(n+k-1)/2}$. Then we have $|A|^2 = \sum_a \mathrm{tr} A_a^2 = (n-1)q$, $q = k-1$.

If M is a complex submanifold in CP^m, the normal connection of M is semi-flat if and only if M is totally geodesic (see [5]).

3. Laplacian

In the following, we put $\nabla_i = \nabla_{e_i}$ and $D_i = D_{e_i}$, where $\{e_i\}$ being an orthonormal basis of $T_x(M)$. We use the following (see Simons [10])

Lemma 3.1. *Let M be a submanifold of a locally symmetric Riemannian manifold $\bar{M}$. If the mean curvature vector field of M is parallel, then*

$$\begin{aligned}
&g((\nabla^2 B)(X,Y),V) = \sum_i g((\nabla_i \nabla_i B)(X,Y),V) \\
&= \sum_i \Big(2g(\bar{R}(e_i,Y)B(X,e_i),V) + 2g(\bar{R}(e_i,X)B(Y,e_i),V) \\
&\quad - g(A_V X, \bar{R}(e_i,Y)e_i) - g(A_V Y, \bar{R}(e_i,X)e_i) + g(\bar{R}(e_i,B(X,Y))e_i,V) \\
&\quad + g(\bar{R}(B(e_i,e_i),X)Y,V) - 2g(A_V e_i, \bar{R}(e_i,X)Y) \Big) \\
&\quad + \sum_a \Big(\mathrm{tr} A_a g(A_V A_a X,Y) - \mathrm{tr} A_a A_V g(A_a X,Y) + 2g(A_a A_V A_a X,Y) \\
&\quad - g(A_a^2 A_V X,Y) - g(A_V A_a^2 X,Y) \Big).
\end{aligned}$$

We notice that $g((\nabla^2 B)(X,Y),V) = g((\nabla^2 A)_V X,Y)$ for any vectors X, Y tangent to M and any vector V normal to M. Since a complex space form $M^m(c)$ is locally symmetric, using the expression of the curvature tensor $\tilde{R}$ of $M^m(c)$, we obtain

$$
\begin{aligned}
&g(\nabla^2 A, A)\\
&= \frac{nc}{4}|A|^2 - \frac{3c}{4}\sum_{a,b} \mathrm{tr}A_a A_b g(tv_a, tv_b) - \frac{c}{4}\sum_a (\mathrm{tr}A_a)^2\\
&\quad - \frac{3c}{2}\sum_a \mathrm{tr}P^2 A_a^2 + \frac{3c}{2}\sum_a (\mathrm{tr}A_a P)^2 + \frac{3c}{4}\sum_{a,b} \mathrm{tr}A_b g(A_a tv_a, tv_b)\\
&\quad + c\sum_{a,b}\Big(g(A_a tv_b, A_b tv_a) - g(A_a tv_a, A_b tv_b)\Big) - 2c\sum_a \mathrm{tr}A_a A_{fa} P\\
&\quad + \sum_{a,b}\Big(\mathrm{tr}A_b \mathrm{tr}A_a^2 A_b - (\mathrm{tr}A_a A_b)^2 + 2\mathrm{tr}(A_a A_b)^2 - 2\mathrm{tr}A_a^2 A_b^2\Big),
\end{aligned}
$$

where we put $A_{fa} = A_{fv_a}$. Substituting equations:

$$
\begin{aligned}
\sum_{a,b} \mathrm{tr}A_a A_b g(tv_a, tv_b) &= |A|^2 - \sum_{a,b} \mathrm{tr}A_a A_b g(fv_a, fv_b)\\
&= |A|^2 - \sum_a \mathrm{tr}A_{fa}^2,
\end{aligned}
$$

$$
2\sum_{a,b}(\mathrm{tr}A_a^2 A_b^2 - \mathrm{tr}(A_a A_b)^2) = -\sum_{a,b} \mathrm{tr}[A_a, A_b]^2,
$$

$$
2\sum_a (\mathrm{tr}(A_a P)^2 - \mathrm{tr}A_a^2 P^2) = \sum_a |[P, A_a]|^2
$$

into the equation above, we have the following theorems.

Theorem 3.2. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$ with parallel mean curvature vector field. Then we have*

$$
\begin{aligned}
&g(\nabla^2 A, A)\\
&= \frac{(n-3)c}{4}|A|^2 + \frac{3c}{4}\sum_a \mathrm{tr}A_{fa}^2 - \frac{c}{4}\sum_a (\mathrm{tr}A_a)^2 + \frac{3c}{4}\sum_{a,b} \mathrm{tr}A_b g(A_a tv_a, tv_b)\\
&\quad + c\sum_{a,b}(g(A_a tv_b, A_b tv_a) - g(A_a tv_a, A_b tv_b)) - 2c\sum_a \mathrm{tr}A_a A_{fa} P\\
&\quad + \frac{3c}{4}\sum_a |[P, A_a]|^2 + \sum_{a,b} \mathrm{tr}[A_a, A_b]^2 + \sum_{a,b}(\mathrm{tr}A_b \mathrm{tr}A_a^2 A_b - (\mathrm{tr}A_a A_b)^2).
\end{aligned}
$$

When M is minimal, $\mathrm{tr}A_a = 0$ for all a. Thus Theorem 3.2 reduces to

Theorem 3.3. *Let M be an n-dimensional minimal submanifold of a complex space form $M^m(c)$. Then we have*

$$\begin{aligned}
&g(\nabla^2 A, A)\\
&= \frac{(n-3)c}{4}|A|^2 + \frac{3c}{4}\sum_a \mathrm{tr} A_{fa}^2\\
&\quad + c\sum_{a,b}(g(A_a t v_b, A_b t v_a) - g(A_a t v_a, A_b t v_b)) - 2c\sum_a \mathrm{tr} A_a A_{fa} P\\
&\quad + \frac{3c}{4}\sum_a |[P, A_a]|^2 + \sum_{a,b}\mathrm{tr}[A_a, A_b]^2 - \sum_{a,b}(\mathrm{tr} A_a A_b)^2.
\end{aligned}$$

Next, we compute the Laplacian for the square of the length of F of an n-dimensional submanifold M immersed in $M^m(c)$.

Lemma 3.4. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$ with parallel mean curvature vector field. Then we have*

$$\begin{aligned}
\Delta|F|^2 =& 3c|Pt|^2 + 4\sum_a \mathrm{tr} A_{fa}^2 - 4\sum_a \mathrm{tr} A_a A_{fa} P\\
&- 4\sum_{a,b} g(A_a t v_b, A_a t v_b) + 4\sum_{a,b} g(A_a t v_b, A_b t v_a).
\end{aligned}$$

Proof. First we compute

$$\begin{aligned}
\frac{1}{2}\Delta|F|^2 &= \frac{1}{2}\sum_{i,j}\nabla_j\nabla_j g(Fe_i, Fe_i)\\
&= \sum_{i,j}\nabla_j g((\nabla_j F)e_i, Fe_i)\\
&= \sum_{j,a}(\nabla_j g(A_a e_j, Ptv_a) + \nabla_j g(A_{fa} e_j, tv_a))\\
&= \sum_{j,a}(g((\nabla_j A)_a e_j, Ptv_a) + g(A_{D_j v_a} e_j, Ptv_a) + g(A_a e_j, (\nabla_j P)tv_a)\\
&\quad + g(A_a e_j, P(\nabla_j t)v_a) + g(A_a e_j, PtD_j v_a) + g((\nabla_j A)_{fa} e_j, tv_a)\\
&\quad + g(A_{D_j f v_a} e_j, tv_a) + g(A_{fa} e_j, (\nabla_j t)v_a) + g(A_{fa} e_j, tD_j v_a)).
\end{aligned}$$

Since the mean curvature vector field of M is parallel, using the equation of Codazzi, we have

$$\begin{aligned}\sum_j g((\nabla_j A)_a e_j, X) &= \sum_j g((\nabla_j A)_a X, e_j)\\ &= \sum_j g((\nabla_X A)_a e_j, e_j) - \frac{3c}{4} g(PX, tv_a)\\ &= -\frac{3c}{4} g(PX, tv_a).\end{aligned}$$

Moreover, using formulas for ∇P and ∇t, we obtain our equation.

4. Integral formulas

In this section we give integral formulas for a compact submanifold in a complex space form $M^m(c)$, $c > 0$, with respect to the square of the length of the second fundamental form A.

We notice that second fundamental form A_V can be considered as a symmetric (n, n)-matrix for any vector V normal to M. For an orthonormal basis $\{e_i\}$ of the tangent space $T_x(M)$ and an orthonormal basis $\{v_a\}$ of the normal space $T_x(M)^\perp$, we put $A_a e_i = \sum_k h^a_{ik} e_k$. Let H_a, $a = 1, \cdots, p$, be a symmetric $(n+1, n+1)$-matrix defined as

$$H_a = \left(\begin{array}{ccc|c} & & & \mu^a_1 \\ & A_a & & \vdots \\ & & & \mu^a_n \\ \hline \mu^a_1 & \cdots & \mu^a_n & 0 \end{array}\right) = \left(\begin{array}{ccc|c} h^a_{11} & \cdots & h^a_{1n} & \mu^a_1 \\ \vdots & \ddots & \vdots & \vdots \\ h^a_{n1} & \cdots & h^a_{nn} & \mu^a_n \\ \hline \mu^a_1 & \cdots & \mu^a_n & 0 \end{array}\right),$$

where $\mu^a_i = -(\sqrt{c}/2) g(tv_a, e_i)$. In the following, we put $|H|^2 = \sum_a \mathrm{tr} H_a^2$.

The main purpose of this section is to prove the following

Theorem 4.1. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$, $c > 0$ with parallel mean curvature vector field. Then*

$$\begin{aligned}&- g(\nabla^2 A, A) - \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2)\\ &+ \frac{3c}{4}\sum_a (\mathrm{tr} A_{fa}^2 + |[P, A_a]|^2 - 4\mathrm{tr} A_a A_{fa} P) + \frac{3c^2}{4}|FP|^2\\ &= -\sum_{a,b} \mathrm{tr}[H_a, H_b]^2 + \sum_{a.b} (\mathrm{tr} H_a H_b)^2 - \frac{(n+1)c}{4}|H|^2 + \frac{c}{4}\Delta|F|^2\\ &\quad + \frac{c}{4}\sum_a (\mathrm{tr} H_a)^2 - \sum_{a,b} \mathrm{tr} H_b \mathrm{tr} H_a^2 H_b + \sum_{a,b} \mathrm{tr} H_b \mathrm{tr}((H_a H_b - H_b H_a) H_a E),\end{aligned}$$

where

$$E = \left(\begin{array}{ccc|c} & & & 0 \\ & 0 & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array}\right).$$

Remark. In Theorem 4.1, if the mean curvature vector field μ of M satisfies $f\mu = 0$, then $\sum_{a,b} \mathrm{tr}H_b \mathrm{tr}((H_aH_b - H_bH_a)H_aE) = 0$. For the condition $f\mu = 0$, see Example in section 2.

Before we prove Theorem 4.1, by the consequence of this theorem, we state the following theorems.

Theorem 4.2. *Let M be an n-dimensional minimal submanifold of a complex space form $M^m(c)$, $c > 0$. Then*

$$\begin{aligned}
&- g(\nabla^2 A, A) - \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2) \\
&+ \frac{3c}{4}\sum_a (\mathrm{tr}A_{fa}^2 + |[P, A_a]|^2 - 4\mathrm{tr}A_aA_{fa}P) + \frac{3c^2}{4}|FP|^2 \\
&= -\sum_{a,b} \mathrm{tr}[H_a, H_b]^2 + \sum_{a.b}(\mathrm{tr}H_aH_b)^2 - \frac{(n+1)c}{4}|H|^2 + \frac{c}{4}\Delta|F|^2.
\end{aligned}$$

If M is compact, then $\int_M |\nabla A|^2 = -\int_M g(\nabla^2 A, A)$ (see [10]). Thus we have

Theorem 4.3. *Let M be an n-dimensional compact submanifold of a complex space form $M^m(c)$, $c > 0$, with parallel mean curvature vector field. Then*

$$\begin{aligned}
&\int_M \Big(|\nabla A|^2 - \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2) \\
&+ \frac{3c}{4}\sum_a (\mathrm{tr}A_{fa}^2 + |[P, A_a]|^2 - 4\mathrm{tr}A_aA_{fa}P) + \frac{3c^2}{4}|FP|^2\Big) \\
&= \int_M \Big(-\sum_{a,b} \mathrm{tr}[H_a, H_b]^2 + \sum_{a,b}(\mathrm{tr}H_aH_b)^2 - \frac{(n+1)c}{4}|H|^2 \\
&\quad + \frac{c}{4}\sum_a (\mathrm{tr}H_a)^2 - \sum_{a,b} \mathrm{tr}H_b \mathrm{tr}H_a^2 H_b \\
&\quad + \sum_{a,b} \mathrm{tr}H_b \mathrm{tr}((H_aH_b - H_bH_a)H_aE)\Big).
\end{aligned}$$

Theorem 4.4. *Let M be an n-dimensional compact minimal submanifold of a complex space form $M^m(c)$, $c > 0$. Then*

$$\begin{aligned}
&\int_M \Big(|\nabla A|^2 - \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2) \\
&+ \frac{3c}{4}\sum_a (\mathrm{tr}A_{fa}^2 + |[P, A_a]|^2 - 4\mathrm{tr}A_a A_{fa} P) + \frac{3c^2}{4}|FP|^2 \Big) \\
&= \int_M \Big(-\sum_{a,b} \mathrm{tr}[H_a, H_b]^2 + \sum_{a,b} (\mathrm{tr}H_a H_b)^2 - \frac{(n+1)c}{4}|H|^2 \Big).
\end{aligned}$$

To prove Theorem 4.1, we prepare some lemmas.

Lemma 4.5. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$, $c > 0$. Then*

$$\begin{aligned}
-\sum_{a,b} \mathrm{tr}[H_a, H_b]^2 = &\sum_{a,b} \Big(-\mathrm{tr}[A_a, A_b]^2 \\
&+ c(g(A_a tv_b, A_a tv_b) - g(A_a tv_b, A_b tv_a)) \\
&+ c(g(A_a tv_a, A_b tv_b) - g(A_a tv_b, A_b tv_a)) \\
&+ \frac{c^2}{8}(g(tv_a, tv_a) g(tv_b, tv_b) - g(tv_a, tv_b)^2) \Big).
\end{aligned}$$

Proof. By the straightforward computation, we have

$$\begin{aligned}
&-\sum_{a,b} \mathrm{tr}[H_a, H_b]^2 \\
&= 2\sum_{a,b} \mathrm{tr}H_a^2 H_b^2 - 2\sum_{a,b} \mathrm{tr}(H_a H_b)^2 \\
&= 2\sum_{a,b} \Big(\sum_{i,j,k,l} h_{ik}^a h_{kj}^a h_{jl}^b h_{li}^b + 2\sum_{i,j,l} h_{jl}^b h_{li}^b \mu_i^a \mu_j^a + \sum_{i,j,k} h_{ik}^a h_{kj}^a \mu_j^b \mu_i^b \\
&\quad + \sum_{i,j} \mu_i^a \mu_j^a \mu_j^b \mu_i^b + 2\sum_{j,k,l} h_{jk}^a h_{lj}^b \mu_k^a \mu_l^b + (\sum_k (\mu_k^a)^2)(\sum_l (\mu_l^b)^2) \\
&\quad - \sum_{i,j,k,l} h_{ik}^a h_{kj}^b h_{jl}^a h_{li}^b - \sum_{i,j,l} h_{jl}^a h_{li}^b \mu_i^a \mu_j^b - \sum_{i,j,k} h_{ik}^a h_{kj}^b \mu_j^a \mu_i^b \\
&\quad - \sum_{i,j} \mu_i^a \mu_j^b \mu_j^a \mu_i^b - 2\sum_{j,k,l} h_{jk}^a h_{lj}^b \mu_k^b \mu_l^a - (\sum_k \mu_k^a \mu_k^b)^2 \Big).
\end{aligned}$$

Since $A_a e_i = \sum_k h^a_{ik} e_k$ and $\mu^a_i = -(\sqrt{c}/2) g(tv_a, e_i)$, we have

$$-\sum_{a,b} \mathrm{tr}[A_a, A_b]^2 = 2\sum_{a,b}\Big(\sum_{i,j,k,l} h^a_{ik}h^a_{kj}h^b_{jl}h^b_{li} - \sum_{i,j,k,l} h^a_{ik}h^b_{kj}h^a_{jl}h^b_{li}\Big),$$

$$\sum_{a,b} g(A_a tv_b, A_a tv_b) = \sum_{a,b} g(A_a tv_b, e_i) g(A_a tv_b, e_i)$$
$$= \frac{4}{c}\sum_{a,b}\sum_{i,k,l} h^a_{ik}h^a_{il}\mu^b_k\mu^b_l,$$

$$\sum_{a,b} g(A_a tv_b, A_b tv_a) = \frac{4}{c}\sum_{a,b}\sum_{i,k,l} h^a_{ik}h^b_{il}\mu^b_k\mu^a_l,$$

$$\sum_{a,b} g(A_a tv_a, A_b tv_b) = \frac{4}{c}\sum_{a,b}\sum_{i,k,l} h^a_{ik}h^b_{il}\mu^a_k\mu^b_l,$$

$$\sum_{a,b}\big(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2\big)$$
$$= \frac{16}{c^2}\sum_{a,b}\Big(\big(\sum_k \mu^a_k\big)^2\big(\sum_l \mu^b_l\big)^2 - \big(\sum_k \mu^a_k\mu^b_k\big)^2\Big).$$

From these equations we have our equation. □

We also have

Lemma 4.6. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$, $c > 0$. Then*

$$\sum_{a,b}(\mathrm{tr}H_aH_b)^2 = \sum_{a,b}(\mathrm{tr}\, A_aA_b)^2 + c|A|^2 - c\sum_a \mathrm{tr}A^2_{fa} + \frac{c^2}{4}|Ft|^2.$$

Lemma 4.7. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$, $c > 0$. Then*

$$|H|^2 = |A|^2 + \frac{c}{2}|t|^2.$$

Lemma 4.8. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$, $c > 0$. Then*

$$\frac{c}{4}\sum_a(\mathrm{tr}A_a)^2 - \frac{3c}{4}\sum_{a,b}\mathrm{tr}A_b g(A_a tv_a, tv_b) - \sum_{a,b}\mathrm{tr}A_b\mathrm{tr}A^2_aA_b$$
$$= \frac{c}{4}\sum_a(\mathrm{tr}H_a)^2 - \sum_{a,b}\mathrm{tr}H_b\mathrm{tr}H^2_aH_b + \sum_{a,b}\mathrm{tr}H_b\mathrm{tr}((H_aH_b - H_bH_a)H_aE).$$

Proof. From the definition of H_a, we have $\mathrm{tr}H_a = \mathrm{tr}A_a$. Next, by the straightforward computation, we have the following equations

$$\sum_a \mathrm{tr}H_a^2H_b = \sum_a(\sum_{i,j,k} h_{ik}^a h_{kj}^a h_{ji}^b + \sum_{i,j} h_{ji}^b \mu_i^a \mu_j^a + 2\sum_{i,j} h_{ji}^a \mu_i^a \mu_j^b),$$

$$\sum_a \mathrm{tr}((H_aH_b - H_bH_a)H_aE) = \sum_a(\sum_{i,j} h_{ji}^b \mu_i^a \mu_j^a - \sum_{i,j} h_{ji}^a \mu_i^a \mu_j^b).$$

Thus we have

$$\begin{aligned}&\sum_{a,b} \mathrm{tr}H_b \mathrm{tr}H_a^2H_b + \sum_{a,b} \mathrm{tr}H_b \mathrm{tr}((H_aH_b - H_bH_a)H_aE)\\ &= \sum_{a,b}(\mathrm{tr}H_b)(\sum_{i,j,k} h_{ik}^a h_{kj}^a h_{ji}^b + 3\sum h_{ji}^a \mu_i^a \mu_j^b).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\frac{3c}{4}\sum_{a,b}(\mathrm{tr}A_b)g(A_atv_a, tv_b) + \sum_{a,b} \mathrm{tr}A_b \mathrm{tr}A_a^2A_b\\ &= \sum_{a,b}(\mathrm{tr}H_b)(3\sum_{i,j} h_{ji}^a \mu_i^a \mu_j^b + \sum_{i,j,k} h_{ik}^a h_{kj}^a h_{ji}^b).\end{aligned}$$

From these equations, we have our equation. □

From Theorem 3.2 and Lemmas 3.4, 4.5–4.8, we have Theorem 4.1.

5. Pinching theorems

We give some pinching theorems with respect to the square of the length of the second fundamental form A and the square of the length of H. First, we prepare some inequalities.

Lemma 5.1. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. Then*

$$|\nabla A|^2 \geq \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2).$$

Proof. We put

$$T_1(X, Y, Z) = (\nabla_X B)(Y, Z) + \frac{c}{4}(g(PX, Y)FZ + g(PX, Z)FY).$$

Then

$$\begin{aligned}|T_1^2| =&|\nabla B|^2 + \frac{c^2}{8}\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) + \frac{c^2}{8}\sum_i g(FPe_i, FPe_i)\\ &+ c\sum_{i,j} g((\nabla_i B)(Pe_i, e_j), Fe_j).\end{aligned}$$

From the equation of Codazzi, we obtain

$$\begin{aligned}\sum_{i,j} & g((\nabla_i B)(Pe_i, e_j), Fe_j) \\ &= \sum_{i,j} g((\nabla_j B)(e_i, Pe_i), Fe_j) - \frac{c}{4}\sum_{i,a} g(Pe_i, Pe_i) g(tv_a, tv_a) \\ &\quad - \frac{c}{4}\sum_{i} g(FPe_i, FPe_i).\end{aligned}$$

Since B is symmetric and P is skew-symmetric, the first term in the right hand side of the equation vanishes. So we have our assertion. □

Lemma 5.2. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$ with parallel mean curvature vector field. If the equality*

$$|\nabla A|^2 = \frac{c^2}{8}\left(|P|^2|t|^2 + |FP|^2\right)$$

holds, then M is a CR submanifold or $c = 0$.

Proof. By the proof of Lemma 5.1, the equation holds if and only if $T_1 = 0$. Suppose that $T_1 = 0$. Then we have

$$D_X(\mathrm{tr}B) = \sum_i (\nabla_X B)(e_i, e_i) = -\frac{c}{2} FPX.$$

Since the mean curvature vector field of M is parallel, we see that $D_X(\mathrm{tr}B) = 0$. When $c \neq 0$, we have $FP = 0$. Then, from Theorem 2.4, M is a CR submanifold. □

Lemma 5.3. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. Then*

$$\sum_a \mathrm{tr}A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \mathrm{tr}A_a A_{fa} P \geq 0.$$

Proof. We put

$$T_2(X, Y) = fB(X, Y) - B(X, PY) - B(PX, Y).$$

Then we have

$$\begin{aligned}|T_2|^2 &= \sum_{i,j} |fB(e_i, e_j) - B(e_i, Pe_j) - B(Pe_i, e_j)|^2 \\ &= \sum_a \mathrm{tr}A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \mathrm{tr}A_a A_{fa} P.\end{aligned}$$

Thus we have our inequality. □

Remark. From the consideration of Example in section 2 and Lemma 5.2, we see that the conditions $T_1 = 0$, $T_2 = 0$ and $FP = 0$ for a submanifold M of CP^m correspond to the notion of the second fundamental form α of a submanifold of S^{2m+1} is parallel. Moreover, if $T_2 = 0$, we see that $f\mu = 0$. When M is a generic submanifold, the condition $T_1 = 0$ was studied by Yano-Kon [12].

Lemma 5.4. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$ with parallel mean curvature vector field. If $T_1 = 0$ and $T_2 = 0$, then $|A|^2$ and $|H|^2$ are constant.*

Proof. Since $T_1 = 0$, Lemma 5.1 implies

$$(\nabla_X B)(Y,Z) = -\frac{c}{4}\big(g(PX,Y)FZ + g(PX,Z)FY\big).$$

Moreover, by Lemma 5.2, M is a CR submanifold, and hence $|t|$ is constant. We notice that

$$|A|^2 = \sum_{a,i} g(A_a e_i, A_a e_i) = \sum_{i,j} g(B(e_i,e_j), B(e_i,e_j)) = |B|^2.$$

Then we have

$$\nabla_X |A|^2 = 2\sum_{i,j} g((\nabla_X B)(e_i,e_j), B(e_i,e_j)) = c\sum_a g(A_a PX, tv_a).$$

Since $T_2 = 0$, we also have $fB(X,Y) = B(PX,Y) + B(X,PY)$. Hence we obtain $\sum_a g(A_a X, tfv_a) = \sum_a g(A_a PX, tv_a) + \sum_a g(A_a X, Ptv_a)$. From Lemmas 2.3 and 4.7, we see that $|A|^2$ and $|H|^2$ are constant. □

We need the following lemma (see Chern-do Carmo-Kobayashi [4]).

Lemma 5.5. *Let A and B be symmetric (n,n)-matrices. Then*

$$-\mathrm{tr}(AB - BA)^2 \leq 2\mathrm{tr}A^2\mathrm{tr}B^2,$$

and the equality holds for non-zero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\bar{A}$ and $\bar{B}$ respectively, while

$$\bar{A} = \left(\begin{array}{cc|c} 0 & 1 & \\ & & 0 \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \qquad \bar{B} = \left(\begin{array}{cc|c} 1 & 0 & \\ & & 0 \\ 0 & -1 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right).$$

Moreover, if A_1, A_2 and A_3 are (n,n)-symmetric matrices and if

$$-\mathrm{tr}(A_iA_j - A_jA_i)^2 = 2\mathrm{tr}A_i^2\mathrm{tr}A_j^2, \qquad 1 \leq i,j \leq 3,$$

then at least one of the matrices A_i must be zero.

Using these lemmas, we prove following

Theorem 5.6. *Let M be an n-dimensional compact minimal submanifold of a complex space form $M^m(c)$, $c > 0$. If H satisfies*

$$|H|^2 \leq \frac{(n+1)c}{8-4/p},$$

then M is a totally geodesic complex submanifold $M^{n/2}(c)$ or a real hypersurface of $M^m(c)$ with $|A|^2 = (n-1)c/4$.

Proof. Using Lemma 5.5, for a suitable choice of an orthonormal basis $\{v_a\}$, we have

$$\begin{aligned}
&\sum_{a,b}(\mathrm{tr}H_aH_b)^2 - \sum_{a,b}\mathrm{tr}[H_a, H_b]^2 \\
&\leq \sum_{a}(\mathrm{tr}H_a^2)^2 + 2\sum_{a\neq b}\mathrm{tr}H_a^2\mathrm{tr}H_b^2 \\
&= 2(\sum_{a}\mathrm{tr}H_a^2)^2 - \sum_{a}(\mathrm{tr}H_a^2)^2 \\
&= (2-\frac{1}{p})(\sum_{a}\mathrm{tr}H_a^2)^2 - \frac{1}{p}\sum_{a>b}(\mathrm{tr}H_a^2 - \mathrm{tr}H_b^2)^2 \\
&\leq (2-\frac{1}{p})|H|^4.
\end{aligned}$$

From Theorem 4.4, Lemmas 5.1 and 5.3, we obtain

$$\begin{aligned}
0 &\leq \int_M \Big(|\nabla A|^2 - \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2) \\
&\quad + \frac{3c}{4}\sum_{a}(\mathrm{tr}A_{fa}^2 + |[P, A_a]|^2 - 4\mathrm{tr}A_aA_{fa}P) + \frac{3c^2}{4}|FP|^2 \Big) \\
&\leq \int_M \Big((2-\frac{1}{p})|H|^2 - \frac{(n+1)c}{4} \Big)|H|^2.
\end{aligned}$$

Thus we see that if $|H|^2 \leq (n+1)c/(8-4/p)$, then $FP = 0$ and M is a CR submanifold by Theorem 2.4. Moreover, we have $|\nabla A|^2 = (c^2/8)(n-q)q$, where $q = |t|^2 = \sum_a g(tv_a, tv_a)$. Then Lemmas 5.1 and 5.3 imply that $T_1 = 0$ and $T_2 = 0$. Therefore, by Lemma 5.4, $|A|^2$ and $|H|^2$ are constant. Consequently we see that $|H|^2 = (n+1)c/(8-4/p)$ or $|H|^2 = 0$.

Suppose that $|H|^2 = 0$. From Lemma 4.7, we have $A_a = 0$ and $tv_a = 0$ for all v_a. Thus M is a totally geodesic complex submanifold, that is, M is a complex space form $M^{n/2}(c)$ of $M^m(c)$.

Next we suppose that $|H|^2 = (n+1)c/(8-4/p)$. Since $\sum_{a>b}(\mathrm{tr}H_a^2 - \mathrm{tr}H_b^2)^2 = 0$, we have $\mathrm{tr}H_a^2 = \mathrm{tr}H_b^2$ for any $a \neq b$. Thus, from Lemma 5.5, we have $p = 1$ or $p = 2$.

Suppose that $p = 2$. If $\dim\mathcal{D}^\perp = 0$, then M is a complex submanifold of $M^m(c)$. Hence we have $PA_a + A_aP = 0$ and $A_{fa} = PA_a$ (c.f. [12]). On the other hand, we obtain $\mathrm{tr}A_{fa}^2 + ||[P, A_a]||^2 - 4\mathrm{tr}A_aA_{fa}P = 0$. Thus we see that $A_a = 0$ for all a and that M is a totally geodesic complex submanifold $M^{n/2}(c)$ of $M^m(c)$.

If there exist vector fields $X \in \mathcal{D}^\perp$ and $V \in N$, where N is the orthogonal complementary of $J\mathcal{D}_x^\perp$ in $T_x(M)^\perp$, then $JX \in J\mathcal{D}^\perp$ and $JV \in N$. So we have $\dim T_x(M)^\perp \geq 3$. This is a contradiction. Thus we see that if $\dim\mathcal{D}^\perp \neq 0$, then $\dim N = 0$, that is, M is a generic submanifold of $M^m(c)$.

Suppose that $\dim\mathcal{D}^\perp \neq 0$. Since M is generic, we have $fv = 0$ for any $v \in T_x(M)^\perp$. Then, we obtain

$$\sum_a (\mathrm{tr}A_{fa}^2 + ||[P, A_a]||^2 - 4\mathrm{tr}A_aA_{fa}P) = \sum_a ||[P, A_a]||^2 = 0,$$

that is, $A_aP = PA_a$ for all v_a. Changing the order of the orthonormal basis $\{e_i\}$ of $T_x(M)$, we suppose that $e_1, e_2 \in \mathcal{D}_x^\perp$, $e_3, \cdots, e_n \in \mathcal{D}_x$ and $v_a = Je_a$ $(a = 1, 2)$. Since $A_aP = PA_a$ for all a, we have

$$g(A_atV, PX) = -g(A_aPtV, X) = 0$$

for any tangent vector field X and normal vector field V. So we have $g(A_ae_i, e_j) = h_{i,j}^a = h_{j,i}^a = 0$ for $i = 1, 2$ and $j \geq 3$. Since $\mathrm{rank}H_a = 2$ and $\mathrm{tr}A_a = 0$ for $a = 1, 2$, the matrices H_a $(a = 1, 2)$ are represented as

$$H_1 = \left(\begin{array}{cc|c|c} 0 & h_{12}^1 & & \sqrt{c}/2 \\ h_{12}^1 & 0 & 0 & 0 \\ \hline & & & 0 \\ 0 & & 0 & \vdots \\ & & & 0 \\ \hline \sqrt{c}/2 & 0 & 0 \ \ldots \ 0 & 0 \end{array}\right)$$

and

$$H_2 = \left(\begin{array}{cc|c|c} 0 & h_{12}^2 & & 0 \\ h_{12}^2 & 0 & 0 & \sqrt{c}/2 \\ \hline & & & 0 \\ 0 & & 0 & \vdots \\ & & & 0 \\ \hline 0 & \sqrt{c}/2 & 0 \ \ldots \ 0 & 0 \end{array}\right).$$

By Lemma 5.5, there exist an orthogonal matrix $T = (t_{ij})$ and scalars α and β such that $TH_1T^{-1} = \alpha\bar{A}$ and $TH_2T^{-1} = \beta\bar{B}$. By the straightforward computation, we have $t_{11} = 0, t_{12} = 0, t_{21} = 0$ and $t_{22} = 0$. Hence we obtain $A_a = 0$ $(a = 1, 2)$.

On the other hand, from Lemma 4.7 and $\sum_a g(tv_a, tv_a) = p = 2$, we have

$$|A|^2 = |H|^2 - c = \frac{(n-5)c}{6}.$$

Consequently, we have $n = 5$ and hence $2m = 7$. Thus is a contradiction. Hence we see that if $|H|^2 = (n+1)c/(8-4/p)$, then M is a real hypersurface with $|A|^2 = (n-1)c/4$. Thus we have our theorem. □

From Theorem 5.6, we have

Theorem 5.7. *Let M be an n-dimensional compact minimal submanifold of a complex space form $M^m(c), c > 0$. If the second fundamental form A satisfies*

$$|A|^2 \leq \frac{c}{4}\Big(\frac{n+1}{2-1/p} - 2p\Big),$$

then M is a totally geodesic complex submanifold $M^{n/2}(c)$ or a real hypersurface of $M^m(c)$ with $|A|^2 = (n-1)c/4$.

Proof. Since $p \geq |t|^2$, we have

$$|A|^2 \leq \frac{c}{4}\Big(\frac{n+1}{2-1/p} - 2|t|^2\Big).$$

From Lemma 4.7, we obtain $|H|^2 \leq (n+1)c/(8-4/p)$. Thus, from Theorem 5.6, we have our conclusion. □

Remark. If M is a real hypersurface of $M^m(c)$ with $|A|^2 = (n-1)c/4$, we see that $PA_a = A_aP$. Then M has at most three constant principal curvatures. When the ambient manifold $M^m(c)$ is $CP^{(n+1)/2}$ of constant holomorphic sectional curvature 4, if the second fundamental form A of a compact minimal real hypersurface M of $CP^{(n+1)/2}$ satisfies $|A|^2 = n-1$, then M is $\pi(S^{2p+1}(((2p+1)/(n+1))^{1/2}) \times S^{2q+1}(((2q+1)/(n+1))^{1/2}))$, $2(p+q) = n$.

Corollary 5.8. *Let M be an n-dimensional compact minimal submanifold of $M^m(c)$. If the scalar curvature r of M satisfies*

$$r \geq \frac{c}{4}\Big(n(n+2) - \frac{n+1}{2-1/p}\Big),$$

then M is a totally geodesic complex submanifold $M^{n/2}(c)$.

Proof. Since the scalar curvature r of M is given by

$$r = \frac{c}{4}\Big((n-1)n + 3|P|^2\Big) - |A|^2,$$

Lemma 4.7 implies

$$\begin{aligned} r =& \frac{c}{4}\left(n(n-1)+3|P|^2\right)+\frac{c}{2}|t|^2-|H|^2 \\ =& \frac{c}{4}\left(n(n+2)-|t|^2\right)-|H|^2 \\ \leq& \frac{n(n+2)c}{4}-|H|^2. \end{aligned}$$

Hence we see that if r satisfies the inequality in the statement, then $|H|^2 \leq (n+1)c/(8-4/p)$. By the proof of Theorem 5.6, M is a totally geodesic complex submanifold $M^{n/2}(c)$ or a real hypersurface with $|H|^2=(n+1)c/4$ of $M^m(c)$. When M is a real hypersurface with $|H|^2=(n+1)c/4$, we have $r=(n^2+n-2)c/4$. This is a contradiction. Thus we have our conclusion. □

6. Semi-flat normal connection

Let M be a n-dimensional submanifold of a complex space form $M^m(c)$. We consider the condition that the normal connection of M is *semi-flat*, that is, the normal curvature tensor $R^\perp$ of M satisfies $R^\perp(X,Y)U=(c/2)g(X,PY)fU$ for any vector fields X and Y tangent to M and any vector field U normal to M. We put

$$\begin{aligned} S_1(X,Y) =& g([A_V,A_U]X,Y)-\frac{1}{4}c(g(FY,U)g(FX,V) \\ =& -g(FX,U)g(FY,V)). \end{aligned}$$

By the equation of Ricci, we see that the normal connection of M is semi-flat if and only if $S_1=0$. Thus we have the following two lemmas.

Lemma 6.1. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. The normal connection of M is semi-flat if and only if the following equation holds*

$$\begin{aligned} &-\sum_{a,b}\mathrm{tr}[A_a,A_b]^2-c\sum_{a,b}g([A_a,A_b]tv_a,tv_b) \\ &+\frac{1}{8}c^2\sum_{a,b}(g(tv_a,tv_a)g(tv_b,tv_b)-g(tv_a,tv_b)^2)=0. \end{aligned}$$

Lemma 6.2. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. If the normal connection of M is semi-flat, then*

$$-\sum_{a,b} \mathrm{tr}[A_a, A_b]^2 = \frac{c^2}{8}\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2),$$

$$\sum_{a,i} g([A_{fa}, A_a]e_i, Pe_i) = 2\sum_a \mathrm{tr}A_a A_{fa} P = \frac{c}{2}\sum_a g(tfv_a, tfv_a),$$

$$\begin{aligned}\sum_{a,b} g([A_a, A_b]tv_a, tv_b) &= \sum_{a,b}(g(A_a tv_b, A_b tv_a) - g(A_a tv_a, A_b tv_b)) \\ &= \frac{c}{4}\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2).\end{aligned}$$

Proposition 6.3. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. Then we have*

$$|S_1|^2 = -\sum_{a,b} \mathrm{tr}[H_a, H_b]^2 - \frac{c}{2}|\nabla f|^2.$$

Proof. From Lemmas 4.5 and 6.2, we have

$$\begin{aligned}|S_1|^2 &= -\sum_{a,b} \mathrm{tr}[A_a, A_b]^2 - c\sum_{a,b} g([A_a, A_b]tv_a, tv_b) \\ &\quad + \frac{c^2}{8}\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2) \\ &= -\sum_{a,b} \mathrm{tr}[H_a, H_b]^2 - c\sum_{a,b}(g(A_a tv_b, A_a tv_b) - g(A_a tv_b, A_b tv_a)).\end{aligned}$$

Since $(\nabla_X f)V = -FA_V X - B(X, tV)$, we obtain

$$|\nabla f|^2 = 2\sum_{a,b}(g(A_a tv_b, A_a tv_b) - g(A_a tv_b, A_b tv_a)).$$

From these equations we have our result. □

Theorem 6.4. *Let M be an n-dimensional compact minimal submanifold with semi-flat normal connection of a complex space form $M^m(c)$, $c > 0$. If $|H|^2 \leq (n-2)c/4$, then M is a totally geodesic complex submanifold $M^{n/2}(c)$ of $M^m(c)$.*

Proof. From Lemmas 3.4 and 6.2, we have

$$\Delta|F|^2 = 2c|Pt|^2 + 4\sum_a \mathrm{tr}A_{fa}^2 - 2|\nabla f|^2.$$

Hence, from Theorem 4.2 and Lemma 6.2, we have

$$\begin{aligned}&\sum_{a,b}(\mathrm{tr}H_aH_b)^2-\frac{(n-2)c}{4}|H|^2\\&=-g(\nabla^2A,A)-\frac{c^2}{8}(|P|^2|t|^2+|FP|^2)+\frac{3c}{4}(|H|^2-\sum_a\mathrm{tr}A_{fa}^2)\\&\quad+\frac{c}{4}|[P,A_a]|^2+\frac{c}{2}(\sum_a\mathrm{tr}A_{fa}^2+\sum_a|[P,A_a]|^2-4\sum_a\mathrm{tr}A_aA_{fa}P).\end{aligned}$$

Thus we have, by Lemmas 5.1 and 5.3,

$$\int_M\Big(\sum_{a,b}(\mathrm{tr}H_aH_b)^2-\frac{(n-2)c}{4}|H|^2\Big)\geq 0.$$

We now choose an orthonormal basis $\{v_a\}$ such that $\mathrm{tr}H_aH_b=0$ for $a\neq b$. Then $\sum_{a,b}(\mathrm{tr}H_aH_b)^2=\sum_a(\mathrm{tr}H_a^2)^2\leq(\sum_a\mathrm{tr}H_a^2)^2$. Hence we have

$$\int_M\Big(|H|^2-\frac{(n-2)c}{4}\Big)|H|^2\geq 0.$$

From Lemma 5.2, M is a CR submanifold of $M^m(c)$. By a similar method of the proof in Theorem 5.6, we see that if $|H|^2\leq(n-2)c/4$, then $|H|^2=(n-2)c/4$ or $|H|^2=0$. When $|H|^2=0$, M is a totally geodesic complex space form $M^{n/2}(c)$ of $M^m(c)$. We suppose that $|H|^2=(n-2)c/4$. Then, we have

$$\sum_a|[P,A_a]|^2=0,\qquad |H|^2=|A|^2+\frac{c}{2}|t|^2=\sum_a\mathrm{tr}A_{fa}^2.$$

Since $|A|^2=\sum_a\mathrm{tr}A_a^2\geq\sum_a\mathrm{tr}A_{fa}^2$, we have $t=0$. Thus M is a complex submanifold of $M^m(c)$. Then we generally see that $PA_a+A_aP=0$ for all a. Combining this to $PA_a=A_aP$, we have $PA_a=0$, and hence $A_a=0$, $n=2$. Consequently, M is a totally geodesic complex space form $M^{n/2}(c)$ of $M^n(c)$. □

From Theorem 6.4, we have the following results.

Theorem 6.5. *Let M be an n-dimensional compact minimal submanifold with semi-flat normal connection of $M^m(c)$. If $|A|^2\leq(n-2p-2)c/4$, then M is a totally geodesic complex submanifold $M^{n/2}(c)$ of $M^m(c)$.*

Corollary 6.6. *Let M be an n-dimensional compact minimal submanifold with semi-flat normal connection of $M^m(c)$. If the scalar curvature r of M satisfies $r\geq(n^2+n+2)c/4$, then M is a complex space form $M^{n/2}(c)$ of $M^m(c)$.*

We next prove a reduction theorem of the codimension of a submanifold of a complex space form.

Theorem 6.7. *Let M be an n-dimensional submanifold with semi-flat normal connection of a complex space form $M^m(c), c > 0$. If $\nabla f = 0$, then M is a totally geodesic complex submanifold of $M^m(c)$ or a generic submanifold of some $M^{n+q}(c)$ in $M^m(c)$.*

Proof. From the assumptions, Lemma 3.4 implies

$$\Delta|F|^2 = 2c|Pt|^2 + 4\sum_a \mathrm{tr} A_{fa}^2.$$

Moreover, we see that $|f|^2$ is constant by $\nabla f = 0$. Then $|t|^2$ and $|F|^2$ are also constant. Hence we have $A_{fa} = 0$ and $Pt = 0$. This means that M is a CR submanifold. If $t = 0$, M is a totally geodesic complex submanifold, that is, complex space form $M^{n/2}(c)$. If $t \neq 0$, then we have $g(D_X V, fU) = -g(V, (\nabla_X f)U) = 0$ for any vector field V in $FT(M)$. Thus $D_X V$ is in $FT(M)$. Therefore, $FT(M)$ is the parallel subbundle in the normal bundle $T(M)^\perp$. From this and $A_{fa} = 0$, we have our assertion (see [2, Lemma 5.9]). □

Remark. In [13, Theorem 3.14, p.236], it was proved that if an n-dimensional compact minimal CR submanifold M of CP^m with semi-flat normal connection and $\nabla f = 0$ satisfies $|A|^2 \leq (n-1)q$, then M is $CP^{n/2}$, or M is a generic minimal submanifold of some $CP^{(n+q)/2}$ in CP^m and is $\pi(S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k))$, $n+1 = \sum_{i=1}^k m_i$, $1 = \sum_{i=1}^k r_i^2$, $q = k-1$, where $m_1, \cdots, m_k$ are odd numbers. Then $n+k$ is also odd.

From Proposition 6.3, we see that $H_a H_b = H_b H_a$ for all a and b if and only if the normal connection of M is semi-flat and $\nabla f = 0$.

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