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A strong vectorial Ekeland's variational principle

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Abstract: Using a concept of approximately efficient point introduced by Tanaka [8], we present a certain vectorial version of Ekeland's variational principle.

Key words: Ekeland's variational principle, approximately efficient solution, vector-valued function

1 Introduction

Since Ekeland [2] in 1972, the variational principle and its equivalent formulations have been one of the main subjects in many fields of nonlinear functional analysis, convex analysis, and optimization.

Theorem 1.1 (Ekeland[2]). *Let (X, d) be a complete metric space and $f : X \to (-\infty, \infty]$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. Let $\varepsilon > 0$ and $u \in X$ satisfy*

$$f(u) \leq \inf_{x \in X} f(x) + \varepsilon.$$

Then there exists some point $v \in X$ such that

(i) $f(v) \leq f(u)$,

(ii) $d(u, v) \leq 1$,

(iii) *for each* $w \neq v$, $\quad f(v) - \varepsilon d(v, w) < f(w)$.

We are interested in generalizing the variational principle to that of vector-valued function. Gopfert, Tammer and Zalinescu (G-T-Z)[4, 5] proved several minimal point theorems and their corresponding variants for vectorial versions of Ekeland's variational principle. In their papers, they used the concept of ε-efficient point, which is an approximate concept introduced by Loridan [7]. On the other hand, Tanaka introduced another concept called by approximately efficient solution point.

Definition 1.2 (Tanaka[8]). Let S be a nonempty subset of Y and $\varepsilon > 0$. A point $y \in Y$ is said to be a lower ε-approximately efficient point of S with respect to C if $y \in S$ and $(y - C) \cap (S \setminus B_\varepsilon(\hat{y})) = \emptyset$, where $B_\varepsilon(\hat{y}) = \{y \in Y | \|y - \hat{y}\| \leq \varepsilon\}$.

The existence of such solutions guarantees a kind of lower boundedness of S which is different from the Loridan's type lower boundedness. Based on this property we present a specification of G-T-Z's vectorial Ekeland's variational principle and obtain detail properties of this.

We give the preliminary terminology and notation used throughout this paper. Let X be a complete metric space and Y a normed space. For a set $A \subset Y$, cor$A$ and int$A$ denote the algebraic interior and the topological interior of A, respectively. We assume that a nonempty set $C \subset Y$ is a solid closed convex cone, that is,

(a) $\text{int}C \neq \emptyset$,

(b) $\text{cl}C = C$,

(c) $C + C \subseteq C$,

(d) $\lambda C \subseteq C$ for all $\lambda \in [0, \infty)$.

A cone C is said to be pointed if $C \cap (-C) = \{0\}$. If a pointed convex cone $C \subseteq Y$ is given, we can define an ordering in Y by "$x \leq_C y$ when $y - x \in C$." This ordering is compatible with the vector structure of Y, that is, for every $x \in Y$ and $y \in Y$,

(i) $x \leq_C y$ implies that $x + z \leq_C y + z$ for all $z \in Y$;

(ii) $x \leq_C y$ implies that $\alpha x \leq_C \alpha y$ for all $\alpha \geq 0$.

We denote $x \leq_{\text{int}C} y$ when $y - x \in \text{int}C$, $B_\varepsilon(\hat{y}) = \{y \in Y | \|y - \hat{y}\| \leq \varepsilon\}$ for any $\varepsilon > 0$ and $f(X) = \bigcup_{x \in X}\{f(x)\}$ for a function $f : X \to Y$.

Tammer and Weidner introduced the following nonlinear scalarizing function, which takes values in $\mathbb{R}$ in the setting of this paper because C is solid.

Lemma 1.3 (Lemma 7 in [5]). *Let C be a convex cone. We take $k^0 \in C \setminus (-\text{cl}C)$ and define $h_{C,k^0} : Y \to [-\infty, \infty]$ by*

$$h_{C,k^0}(y) = \inf\{t \in \mathbb{R} \,|\, y \in tk^0 - C\}.$$

Then the function h_{C,k^0} has the following five properties:

(i) *h_{C,k^0} is proper ($h_{C,k^0} \not\equiv +\infty$ and $h_{C,k^0}(y) > -\infty$ for every $y \in Y$),*

(ii) h_{C,k^0} *is sublinear* ($h_{C,k^0}(\lambda y_1 + \mu y_2) \le \lambda h_{C,k^0}(y_1) + \mu h_{C,k^0}(y_2)$ *for every* $y_1, y_2 \in Y$ *and* $\lambda, \mu \ge 0$),

(iii) h_{C,k^0} *is increasing with respect to* $\le_C$ ($y_1 \le_C y_2$ *implies* $h_{C,k^0}(y_1) \le h_{C,k^0}(y_2)$),

(iv) $\{y \in Y | h_{C,k^0}(y) \le t\} = tk^0 - C$,

(v) $h_{C,k^0}(y + \lambda k^0) = h_{C,k^0}(y) + \lambda$ *for every* $y \in Y$ *and* $\lambda \in \mathbb{R}$.

Moreover, if $k^0 \in \mathrm{cor}C$ *then* h_{C,k^0} *is finitely valued,* $\{y \in Y | h_{C,k^0}(y) < t\} = tk^0 - \mathrm{cor}C$ *and* $h_{C,k^0}(y_1) < h_{C,k^0}(y_2)$ *if* $y_2 - y_1 \in \mathrm{cor}C$. *Furthermore, if* C *is closed, then* h_{C,k^0} *is lower semicontinuous.*

As a corollary of the above lemma, Gerth(Tammer) and Weidner presented the following nonconvex separation theorem.

Lemma 1.4 (Theorem 2.3.6 in [4]). *Assume that* Y *is a topological vector space,* C *a solid closed convex cone and* $A \subset Y$ *a nonempty set such that* $A \cap (-\mathrm{int}C) = \emptyset$. *Then* h_{C,k^0} *is a finite-valued continuous function such that*

$$h_{C,k^0}(-y) < 0 \le h_{C,k^0}(x) \quad \forall x \in A,\ y \in \mathrm{int}C,$$

moreover, $h_{C,k^0}(x) > 0$ *for all* $x \in \mathrm{int}A$.

The above two lemmas play important roles in this paper.

2 Main result

We obtain the following vectorial Ekeland's variational principle.

Theorem 2.1. *Let* $f : X \to Y$ *be a vector-valued function,* $x_0 \in X$, $\varepsilon > 0$ *and* $k^0 \in \mathrm{int}C$. *Assume that* f *satisfies* $(f(X) \setminus B_\varepsilon(f(x_0))) \cap (f(x_0) - \mathrm{int}C) = \emptyset$ *and that*

(H) $\{x' \in X \mid f(x') + d(x, x')k^0 \le_C f(x)\}$ *is closed for every* $x \in X$.

Moreover we also assume $\varepsilon' > 0$ *satisfies* $(-\varepsilon' k^0 - \mathrm{int}C) \cap B_\varepsilon(0) = \emptyset$. *Then there exists* $\bar{x} \in X$ *such that*

(i) $f(\bar{x}) \le_{\mathrm{int}C} f(x_0)$

(ii) $||f(\bar{x}) - f(x_0)|| \le \varepsilon$

(iii) $d(x_0, \bar{x}) \leq \varepsilon'$

(iv) *if for some* $x \in X$, $f(x) + d(x, \bar{x})k^0 \leq_C f(\bar{x})$ *then* $x = \bar{x}$.

Proof. First of all, $(h_{C,k^0} \circ f)(x)$ is bounded from below on X for all $x \in X$. By Proposition 1 in [8] and the assumption of Theorem 2.1, we have that a point $f(x_0)$ is a Loridan's ε'-efficient point of $f(X)$, that is, $f(X) \cap (f(x_0) - \varepsilon' k^0 - \text{int}C) = \emptyset$. By Lemma 1.4, we have

$$h_{C,k^0}(-y) < 0 \leq h_{C,k^0}(f(x) - f(x_0) + \varepsilon' k^0)$$

for all $x \in X$, $y \in \text{int}C$. Using (ii) and (v) of Lemma 1.3, we have

$$-\infty < h_{C,k^0}(-y) - h_{C,k^0}(-f(x_0)) - \varepsilon' < h_{C,k^0}(f(x)).$$

We consider the following set-valued map $F : X \to 2^X$

$$F(x) := \{y \in X | f(y) + d(x, y)k^0 \leq_C f(x)\}.$$

By condition (H), $F(x)$ is a closed set for each $x \in X$ and F has the following properties:

(a) $x \in F(x)$ (reflexivity),

(b) if $y \in F(x)$ then $F(y) \subset F(x)$ (transitivity).

Property (a) is easy. To prove property (b), we take $y \in F(x)$ and suppose that $z \in F(y)$. Then we have that

$$f(y) + d(x, y)k^0 \leq_C f(x) \quad \text{and} \quad f(z) + d(y, z)k^0 \leq_C f(y).$$

By the compatibility of the ordering $\leq_C$ to the vector structure, the triangle inequality on d and $k^0 \in C$, we have that

$$f(z) + d(x, z)k^0 \leq_C f(x),$$

which implies $z \in F(x)$.

Next, using (iii) and (v) of Lemma 1.3, we have that $y \in F(x)$ implies

$$h_{C,k^0}(f(y)) + d(x, y) \leq h_{C,k^0}(f(x)),$$

and hence

$$d(x, y) \leq h_{C,k^0}(f(x)) - \inf_{z \in F(x)} h_{C,k^0}(f(z))$$

for all $y \in F(x)$, which implies the following upper bound on the diameter of $F(x)$

$$\mathrm{Diam}(F(x)) \leq 2(h_{C,k^0}(f(x)) - \inf_{z \in F(x)} h_{C,k^0}(f(z))). \tag{2.1}$$

For each $n = 1, 2, \ldots$, by definition of the infimum, there exists $x_{n+1} \in F(x_n)$ such that $h_{C,k^0}(f(x_{n+1})) \leq \inf_{z \in F(x_n)} h_{C,k^0}(f(z)) + 2^{-n}$. Since $F(x_{n+1}) \subset F(x_n)$ by property (b), we have

$$\inf_{z \in F(x_n)} h_{C,k^0}(f(z)) \leq \inf_{z \in F(x_{n+1})} h_{C,k^0}(f(z)).$$

On the other hand, since we always have $\inf_{z \in F(y)} h_{C,k^0}(f(z)) \leq h_{C,k^0}(f(y))$ by property (a), we obtain the inequalities

$$0 \leq h_{C,k^0}(f(x_{n+1})) - \inf_{z \in F(x_{n+1})} h_{C,k^0}(f(z)) \leq 2^{-n}. \tag{2.2}$$

By combining (2.1) and (2.2), we get $\mathrm{Diam}(F(x_{n+1})) \leq 2 \cdot 2^{-n}$. Consequently, it follows that the sequence of diameters of the closed sets $F(x_n)$ converges to 0. By Cantor's theorem, we have that

$$\bigcap_{n=0}^{\infty} F(x_n) = \{\bar{x}\}.$$

Since $\bar{x}$ belongs to $F(x_0)$, we have that

$$f(\bar{x}) + d(x_0, \bar{x})k^0 \leq_C f(x_0) \tag{2.3}$$

and hence

$$f(x_0) - f(\bar{x}) \in C + d(x_0, \bar{x})k^0 \subset \mathrm{int}C,$$

which shows that the condition (i) holds. Since $\bar{x}$ belongs to all the $F(x_n)$, we have that $F(\bar{x}) \subset F(x_n)$ and consequently that

$$F(\bar{x}) = \{\bar{x}\}.$$

Thus, we deduce that the condition (iv) holds. Moreover, by condition (i), that is, $f(\bar{x}) \in f(x_0) - \mathrm{int}C$, and assumption $(f(X) \setminus B_\varepsilon(f(x_0))) \cap (f(x_0) - \mathrm{int}C) = \emptyset$, we have that

$$f(\bar{x}) \in B_\varepsilon(f(x_0)),$$

therefore condition (ii) holds. To prove condition (iii), we suppose that $d(x_0, \bar{x}) > \varepsilon'$. Then we have that $(d(x_0, \bar{x}) - \varepsilon')k^0 + C \subset \mathrm{int}C$. By condition (2.3) we have that

$$f(\bar{x}) \in f(x_0) - d(x_0, \bar{x})k^0 - C \subset f(x_0) - \varepsilon' k^0 - \mathrm{int}C,$$

which is a contradiction. □

Remark 1. Note that the case of $Y = \mathbb{R}$, $C = \mathbb{R}_+ = [0, \infty)$, $k^0 = \varepsilon \in \mathbb{R}_+ \setminus \{0\}$ and $\varepsilon' = 1$ in Theorems 2.1 becomes Theorem 1.1. We also note that the pointedness of C is not needed to prove Theorem 2.1.

Remark 2. In Theorem 2.1, the solidness of C is used to ensure that the constructed functional h_{C,k^0} takes finite values. If we set

$$Y = \mathbb{R}^2, \quad C = \{(x,x) | x \in \mathbb{R}\}, \quad k^0 = (1,1), \quad a = (2,2), \quad b = (1,0).$$

We have that

$$h_{C,k^0}(a) = 2 \quad \text{but} \quad h_{C,k^0}(b) = \infty.$$

This fact guarantees the lower boundedness of the function in Theorem 2.1.

3 Conclusions

Gopfert, Tammer and Zalinescu[5] obtained a vectorial Ekeland's variational principle with an estimate of $d(x_0, \bar{x})$. In this paper, we assume the existence of approximately efficient solution point introduced by Tanaka and obtain a vectorial Ekeland's principle not only an estimate of $d(x_0, \bar{x})$ but also an estimate of $||f(\bar{x}) - f(x_0)||$.

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