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CONTINUITY OF A CERTAIN INVARIANT OF A MEASURE ON A CAT(0) SPACE

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Abstract. For a finitely supported probability measure μ on a complete CAT(0) space Y, Izeki and Nayatani defined an invariant $\delta(\mu) \in [0,1]$ in [1]. The supremum of those for all such measures on Y is an invariant of Y, called the Izeki-Nayatani invariant, which plays an important role in the study of fixed-point property of groups. In this paper, we establish continuity of δ on the space of finitely supported probability measures. We prove the lower-semicontinuity of δ with respect to the (L^2-) Wasserstein metric, and continuity with respect to some metric which induces a stronger topology.

1. Introduction

First we set up some notations. Let (Y,d) be a complete CAT(0) space. For any $p,q \in Y$, there is a unique geodesic γ joining p to q, that is an isometric embedding of the closed interval $[0,d(p,q)]$ into Y with $\gamma(0)=p$ and $\gamma(d(p,q))=q$, and we denote its image by $[p,q]$. We denote by $\mathcal{P}(Y)$ the set of all finitely supported probability measures on Y other than measures supported on a single point. For any $\nu \in \mathcal{P}(Y)$, there exists a unique point $\overline{\nu} \in Y$ which minimizes the function

$$x \mapsto \int_Y d(x,y)^2 \nu(dy)$$

defined on Y. This point $\overline{\nu}$ is called the *barycenter* of ν. For detailed accounts of CAT(0) spaces and behaviors of probability measures on them, we refer the reader to [5] and [6]. Throughout this paper, we fix an infinite dimensional Hilbert space $\mathcal{H}$.

Definition 1.1 (Izeki-Nayatani). Let Y be a complete CAT(0) space. For $\mu \in \mathcal{P}(Y)$, we denote by $\Phi(\mu)$ the set of all maps $\phi : \mathrm{supp}\mu \to \mathcal{H}$ from the support of μ

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to $\mathcal{H}$ such that

$$\|\phi(p)\| = d(p, \overline{\mu}), \tag{1.1}$$

$$\|\phi(p) - \phi(q)\| \leq d(p, q) \tag{1.2}$$

for all $p, q \in \mathrm{supp}\mu$. For $\mu = \sum_{i=1}^{m} t_i \mathrm{Dirac}_{p_i} \in \mathcal{P}(Y)$, we define a function $D^{\mu} : \Phi(\mu) \to \mathbb{R}$ by

$$D^{\mu}(\phi) = \frac{\| \sum_{i=1}^{m} t_i \phi(p_i)\|^2}{\sum_{i=1}^{m} t_i \|\phi(p_i)\|^2}, \quad \phi \in \Phi(\mu).$$

Here and henceforth, Dirac_p denotes the Dirac measure at p. For $\mu \in \mathcal{P}(Y)$, we define $\delta(\mu)$ as

$$\delta(\mu) = \inf_{\phi \in \Phi(\mu)} D^{\mu}(\phi).$$

And we define the *Izeki-Nayatani* invariant $\delta(Y)$ of Y as

$$\delta(Y) = \sup_{\mu \in \mathcal{P}(Y)} \delta(\mu).$$

By definition, we have $\delta(\mu) \in [0, 1]$ for all $\mu \in \mathcal{P}(Y)$. We can say that the Izeki-Nayatani invariant measures a sort of singularity of a CAT(0) space. And it plays an important role in the study of fixed-point property of groups; we refer the reader to [1], [2], [3], [4], and [7].

However, computation of this invariant is generally hard. To estimate the Izeki-Nayatani invariants of various CAT(0) spaces, and to understand this invariant better, it must be helpful if continuity of δ is guaranteed. In this paper, we formulate some continuity results for $\delta : \mathcal{P}(Y) \to [0, 1]$.

Recall that the (L^2-) *Wasserstein distance* $d^W(\mu, \nu)$ between

$$\mu = \sum_{i=1}^{m} t_i \mathrm{Dirac}_{p_i} \in \mathcal{P}(Y)$$

and

$$\nu = \sum_{j=1}^{n} s_j \mathrm{Dirac}_{q_j} \in \mathcal{P}(Y)$$

is defined by

$$d^W(\mu, \nu)^2 = \inf_{\pi} \int_{Y \times Y} d(x, y)^2 d\pi(x, y),$$

where the infimum is taken over all measures

$$\pi = \sum_{1 \leq i \leq m, 1 \leq j \leq n} T_{ij} \mathrm{Dirac}_{(p_i, q_j)} \tag{1.3}$$

on $Y \times Y$ such that $\sum_{i=1}^{m} T_{ij} = s_j$ for all $1 \leq j \leq m$ and $\sum_{j=1}^{n} T_{ij} = t_i$ for all $1 \leq i \leq n$. Such a measure π is called a *coupling* of μ and ν, so we can restate that the infimum is taken over all couplings of μ and ν. The Wasserstein distance makes

$\mathcal{P}(Y)$ a metric space. The Wasserstein distance can be formulated in more general setting, and plays a significant role in the theory of optimal transport. For more information about this distance, we refer the reader to [8].

In Section 2, we prove the lower-semicontinuity of δ with respect to d^W:

Theorem 1.2. *Let* (Y, d) *be a complete* CAT(0) *space. Then* $\delta : \mathcal{P}(Y) \to [0, 1]$ *is a lower-semicontinuous function on* $(\mathcal{P}(Y), d^W)$.

In Section 3, we introduce a new metric d_{HW} on $\mathcal{P}(Y)$, which induces a stronger topology on $\mathcal{P}(Y)$ than d^W, and prove the continuity of δ with respect to this metric.

2. Lower-semicontinuity with respect to d^W

In this section, we prove Theorem 1.2. But before starting the proof, we define an invariant of a measure, which plays an important role in our proof.

Definition 2.1. Let (Y, d) be a CAT(0) space, and let $\nu \in \mathcal{P}(Y)$. We set

$$S_\nu = \left\{(p, q) \in \mathrm{supp}\nu \times \mathrm{supp}\nu \;\middle|\; p \notin [\overline{\nu}, q],\ q \notin [\overline{\nu}, p]\right\},$$

and define a positive real number L_ν as

$$L_\nu = \min\left\{d(p, q)^2 - (d(p, \overline{\nu}) - d(q, \overline{\nu}))^2 \;\middle|\; (p, q) \in S_\nu\right\}.$$

Because of the triangle inequality, we have

$$d(p, q)^2 - (d(p, \overline{\nu}) - d(q, \overline{\nu}))^2 \geq 0$$

for any $p, q \in \mathrm{supp}\nu$, and the equality holds if and only if $p \in [\overline{\nu}, q]$ or $q \in [\overline{\nu}, p]$. Therefore, we have $L_\nu > 0$ for any $\nu \in \mathcal{P}(Y)$.

Proof of Theorem 1.2. Let

$$\nu = \sum_{j=1}^{n} s_j \mathrm{Dirac}_{q_j}$$

be an arbitrary measure in $\mathcal{P}(Y)$, and let $J = \{1, \ldots, n\}$. Let

$$\mu^{(N)} = \sum_{i=1}^{m^{(N)}} t_i^{(N)} \mathrm{Dirac}_{p_i^{(N)}}, \quad N = 1, 2, 3, \ldots$$

be an arbitrary sequence of measures in $\mathcal{P}(Y)$ which converges to ν in $(\mathcal{P}(Y), d^W)$, and let $I^{(N)} = \{1, \ldots, m^{(N)}\}$ for each $N \in \mathbb{N}$. Then what we have to show is that

$$\liminf_{N \to \infty} \delta(\mu^{(N)}) \geq \delta(\nu). \tag{2.1}$$

Because the sequence of the barycenters $\{\overline{\mu^{(N)}}\}$ converges to $\overline{\nu}$ in Y, for any $\eta > 0$, we can find a positive real number N_η such that $N \geq N_\eta$ implies

$$d^W(\mu^{(N)}, \nu) < \eta^2, \quad d(\overline{\mu^{(N)}}, \overline{\nu}) < \eta. \tag{2.2}$$

Then, to prove (2.1), it is sufficient to show that for any sufficiently small $\eta > 0$, any $N \geq N_\eta$ and any $\phi \in \Phi(\mu^{(N)})$, there exists $\tilde{\phi}_\eta \in \Phi(\nu)$ such that

$$D^{(\mu^{(N)})}(\phi) \geq D^\nu(\tilde{\phi}_\eta) - F(\eta), \tag{2.3}$$

where F is some function converging to 0 when $\eta \to 0$.

In the proceeding argument, assume that $\eta > 0$ is an arbitrary positive real number such that

$$\eta < \min\left\{ \frac{1}{2} d(q_j, q_{j'}) \,\middle|\, j, j' \in J,\ \ j \neq j' \right\},$$
$$\eta^2 < \min\{s_j \mid j \in J\}.$$

And suppose N be an arbitrary integer such that $N \geq N_\eta$.

Let

$$I_j^{(N)} = \{i \in I^{(N)} \mid d(p_i^{(N)}, q_j) \leq \eta\}$$

for any $j \in J$, and

$$I_0^{(N)} = \{i \in I^{(N)} \mid \forall j \in J;\ d(p_i^{(N)}, q_j) > \eta\}.$$

Then $I_0^{(N)}, I_1^{(N)}, \ldots, I_n^{(N)}$ satisfy the following three conditions.

- $I^{(N)} = I_0^{(N)} \cup I_1^{(N)} \cup \cdots \cup I_n^{(N)}$;
- If $j \neq j'$ then $I_j^{(N)} \cap I_{j'}^{(N)} = \phi$;
- For every $j \in J$, $I_j^{(N)} \neq \phi$.

The first two are obvious. The last one is shown as follows: If $I_j^{(N)}$ were empty, then for any coupling $\pi = \sum_{i,j'} T_{ij'} \mathrm{Dirac}_{(p_i^{(N)}, q_{j'})}$ of $\mu^{(N)}$ and ν, we would have

$$\begin{aligned}\int_{Y \times Y} d(x,y)^2 d\pi(x,y) &\geq \sum_{i \in I^{(N)}} T_{ij} d(p_i^{(N)}, q_j)^2 \\ &\geq s_j \eta^2 > \eta^4.\end{aligned}$$

But this contradicts the fact that $d^W(\mu^{(N)}, \nu) < \eta^2$.

Now, we will construct $\tilde{\phi}_\eta \in \Phi(\nu)$ in (2.3) in three steps. As the first step, we construct a vector $A_j \in \mathcal{H}$ for each $j \in J$. For each $j \in J$, since $I_j^{(N)}$ is nonempty, we can choose some $i_0 \in I_j^{(N)}$. If $p_{i_0} \neq \overline{\mu^{(N)}}$, let

$$A_j = \frac{d(\overline{\nu}, q_j)}{d(\overline{\mu^{(N)}}, p_{i_0}^{(N)})} \phi(p_{i_0}^{(N)}) \in \mathcal{H},$$

and if $p_{i_0} = \overline{\mu^{(N)}}$, let A_j be an arbitrary vector of length $d(\overline{\nu}, q_j)$. Then, by the second inequality of (2.2) and the assumption on η, we have

$$\begin{aligned}
\|\phi(p_i^{(N)}) - A_j\| &\le \|\phi(p_i^{(N)}) - \phi(p_{i_0}^{(N)})\| + \|\phi(p_{i_0}^{(N)}) - A_j\| \\
&\le d(p_i^{(N)}, p_{i_0}^{(N)}) + \Big| \|\phi(p_{i_0}^{(N)})\| - \|A_j\| \Big| \\
&\le d(p_i^{(N)}, p_{i_0}^{(N)}) + \Big| d(p_{i_0}^{(N)}, \overline{\mu^{(N)}}) - d(q_j, \overline{\nu}) \Big| \\
&\le d(p_i^{(N)}, p_{i_0}^{(N)}) + \Big| d(p_{i_0}^{(N)}, \overline{\mu^{(N)}}) - d(\overline{\mu^{(N)}}, q_j) \Big| + \Big| d(\overline{\mu^{(N)}}, q_j) - d(\overline{\nu}, q_j) \Big| \\
&\le d(p_i^{(N)}, q_j) + d(q_j, p_{i_0}^{(N)}) + d(p_{i_0}^{(N)}, q_j) + d(\overline{\mu^{(N)}}, \overline{\nu}) \\
&\le 4\eta
\end{aligned}$$

for any $i \in I_j$. For any $j, j' \in J$, we have

$$\begin{aligned}
\|A_j - A_{j'}\| &\le \|A_j - \phi(p_i^{(N)})\| + \|\phi(p_i^{(N)}) - \phi(p_{i'}^{(N)})\| + \|\phi(p_{i'}^{(N)}) - A_{j'}\| \\
&\le 8\eta + d(p_i^{(N)}, q_j) + d(q_j, q_{j'}) + d(q_{j'}, p_{i'}^{(N)}) \\
&\le 10\eta + d(q_j, q_{j'}).
\end{aligned}$$

In the preceding inequality, i and i' are arbitrary elements of I_j and $I_{j'}$ respectively.

Before moving to the next step, we set up some notations related to ν. We first divide J into "branches". We define a set $\tilde{J} \subset J$, "representatives of branches" , by declaring $j \in \tilde{J}$ if and only if $q_j \neq \overline{\nu}$ and there is no $j' \in J$ other than j itself such that $q_{j'}$ is on the geodesic segment $[\overline{\nu}, q_j]$ joining $\overline{\nu}$ to q_j. Let k be the cardinality of $\tilde{J}$, and we denote elements of $\tilde{J}$ as $j_1, \dots, j_k$. We define subsets $J_0, \dots, J_k$ of J as follows:

$$\begin{aligned}
J_l &= \{j \in J \mid q_{j_l} \in [\overline{\nu}, q_j]\}, \quad 1 \le l \le k, \\
J_0 &= \{j \in J \mid q_j = \overline{\nu}\}.
\end{aligned}$$

It follows immediately that $J = \cup_{l=0}^k J_l$, and that $l \neq l'$ implies $J_l \cap J_{l'} = \phi$. And we claim that $j, j' \in J \backslash J_0$ must be contained in the same J_l for some $l \in \{1, \dots, k\}$ whenever $q_j \in [\overline{\nu}, q_{j'}]$ or $q_{j'} \in [\overline{\nu}, q_j]$. We define $K_\nu > 0$ and $k_\nu > 0$ as

$$K_\nu = \max\{d(\overline{\nu}, q_j) \mid j \in J\}, \quad k_\nu = \min\{d(\overline{\nu}, q_j) \mid j \in J \backslash J_0\}.$$

As the second step, we define

$$B_j = \frac{d(q_j, \overline{\nu})}{d(q_{j_l}, \overline{\nu})} A_{j_l},$$

for any $j \in J_l$ ($l = 1, \dots, k$), and $B_j = 0$ for $j \in J_0$. Using the cosine formula for the triangle spanned by A_{j_l} and A_j, we have

$$\begin{aligned}\cos\angle(A_j, B_j) &\geq \frac{d(\overline{\nu}, q_j)^2 + d(\overline{\nu}, q_{j_l})^2 - (10\eta + d(\overline{\nu}, q_j) - d(\overline{\nu}, q_{j_l}))^2}{2d(\overline{\nu}, q_j)d(\overline{\nu}, q_{j_l})} \\ &= 1 - \frac{100\eta^2 - 20\eta(d(\overline{\nu}, q_{j_l}) - d(\overline{\nu}, q_j))}{2d(\overline{\nu}, q_j)d(\overline{\nu}, q_{j_l})}.\end{aligned}$$

Then it follows that

$$\|A_j - B_j\| \leq \sqrt{\frac{K_\nu}{k_\nu}(100\eta^2 + 20\eta K_\nu)}.$$

This is still true in the case of $j \in J_0$. Hence, we have

$$\begin{aligned}\|B_j - B_{j'}\|^2 &\leq (\|B_j - A_j\| + \|A_j - A_{j'}\| + \|A_{j'} - B_{j'}\|)^2 \\ &\leq d(q_j, q_{j'})^2 + f(\eta),\end{aligned}$$

for any $j, j' \in J$, where

$$\begin{aligned}f(\eta) = 100\eta^2 + 4\frac{K_\nu}{k_\nu}(100\eta^2 + 20\eta K_\nu) + 40\eta\sqrt{\frac{K_\nu}{k_\nu}(100\eta^2 + 20\eta K_\nu)} \\ + \max_{j,j' \in J} d(q_j, q_{j'}) \left(20\eta + 4\sqrt{\frac{K_\nu}{k_\nu}(100\eta^2 + 20\eta K_\nu)}\right).\end{aligned}$$

Now we come to the final step. Let $E \in \mathcal{H}$ be a unit vector orthogonal to the subspace spanned by $B_1, \dots, B_n$. For $0 < \theta < \frac{\pi}{2}$, we define

$$B_j^\theta = \sin\theta \cdot \|B_j\| E + \cos\theta \cdot B_j$$

for any $j \in J$. Then for any $0 < \theta < \frac{\pi}{2}$ and $j \in J$,

$$\|B_j^\theta\| = \|B_j\|,$$

and for any $j, j' \in J$,

$$\|B_j - B_{j'}\|^2 - \|B_j^\theta - B_{j'}^\theta\|^2 = \sin^2\theta \cdot \{\|B_j - B_{j'}\|^2 - (\|B_j\| - \|B_{j'}\|)^2\}. \quad (2.4)$$

Assuming that $\eta > 0$ is taken to be small enough if necessary, we define

$$\theta_\eta = \sin^{-1}\sqrt{\frac{f(\eta)}{L_\nu}}.$$

We define a map $\tilde{\phi}_\eta : \mathrm{supp}\nu \to \mathcal{H}$ by

$$\tilde{\phi}_\eta(q_j) = B_j^{\theta_\eta}, \quad j \in J.$$

We have to confirm $\tilde{\phi}_\eta \in \Phi(\nu)$. The condition (1.1) is obvious, so we examine the condition (1.2) by considering three cases separately.

Case I: $(q_j, q_{j'}) \notin S_\nu$. In this case, B_j and $B_{j'}$ are parallel vectors by definition, so we have

$$\begin{aligned}\|B_j^{\theta_\eta} - B_{j'}^{\theta_\eta}\| &= \|B_j - B_{j'}\| = |\|B_j\| - \|B_{j'}\|| \\ &= |d(q_j, \overline{\nu}) - d(q_{j'}, \overline{\nu})| \le d(q_j, q_{j'}).\end{aligned}$$

Case II: $(q_j, q_{j'}) \in S_\nu$ and $\|B_j - B_{j'}\|^2 - (\|B_j\| - \|B_{j'}\|)^2 < L_\nu$. In this case, we have

$$\begin{aligned}\|B_j^{\theta_\eta} - B_{j'}^{\theta_\eta}\|^2 &\le \|B_j - B_{j'}\|^2 \\ &< L_\nu + (d(q_j, \overline{\nu}) - d(q_{j'}, \overline{\nu}))^2 \le d(q_j, q_{j'})^2.\end{aligned}$$

Case III: $(q_j, q_{j'}) \in S_\nu$ and $\|B_j - B_{j'}\|^2 - (\|B_j\| - \|B_{j'}\|)^2 \ge L_\nu$. In this case, by (2.4) and the definition of θ_η, we have

$$\|B_j^{\theta_\eta} - B_{j'}^{\theta_\eta}\|^2 \le d(q_j, q_{j'})^2.$$

Hence $\tilde{\phi}_\eta \in \Phi(\nu)$.

Let

$$\begin{aligned}F(\eta) &= D^\nu(\tilde{\phi}_\eta) - D^{(\mu^{(N)})}(\phi) \\ &= \frac{\left\| \sum_{j\in J} s_j B_j^{\theta_\eta} \right\|^2}{\sum_{j\in J} s_j \left\| B_j^{\theta_\eta} \right\|^2} - \frac{\| \sum_{j=0}^n \sum_{i\in I_j^{(N)}} t_i^{(N)} \phi(p_i^{(N)}) \|^2}{\sum_{j=0}^n \sum_{i\in I_j^{(N)}} t_i^{(N)} \|\phi(p_i^{(N)})\|^2}.\end{aligned} \tag{2.5}$$

To complete our proof, it is sufficient to show that $F(\eta)$ tends to 0 when $\eta \to 0$. And, since the limit

$$\lim_{\eta\to 0} \sum_{j\in J} s_j \left\| B_j^{\theta_\eta} \right\|^2 = \sum_{j\in J} s_j d\left(\overline{\nu}, q_j\right)^2$$

exists, it is sufficient to prove the following:

- **(i):** $\lim_{\eta\to 0} \sum_{i\in I_0^{(N)}} t_i^{(N)} \phi(p_i^{(N)}) = 0$;
- **(ii):** $\lim_{\eta\to 0} \sum_{i\in I_0^{(N)}} t_i^{(N)} \left\| \phi(p_i^{(N)}) \right\|^2 = 0$;
- **(iii):** For every $j \in J$, $\lim_{\eta\to 0} \left\| \sum_{i\in I_j^{(N)}} t_i^{(N)} \phi(p_i^{(N)}) - s_j B_j^{\theta_\eta} \right\| = 0$;
- **(iv):** For every $j \in J$, $\lim_{\eta\to 0} \left(\sum_{i\in I_j^{(N)}} t_i^{(N)} \|\phi(p_i^{(N)})\|^2 - s_j \|B_j^{\theta_\eta}\|^2 \right) = 0$.

Let $l_i = \min\{d(p_i^{(N)}, q_j) \mid j \in J\}$ for $i \in I_0^{(N)}$. To prove the above assertions, we first show the following:

- **(a):** $\lim_{\eta\to 0} \sum_{i\in I_0^{(N)}} t_i^{(N)} = 0$;
- **(b):** For each $i \in I_0^{(N)}$, $\lim_{\eta\to 0} \sum_{i\in I_0^{(N)}} t_i^{(N)} l_i^2 = 0$;
- **(c):** For every $j \in J$, $\lim_{\eta\to 0} \sum_{i\in I_j^{(N)}} t_i^{(N)} = s_j$.
- **(d):** For any $j \in J$ and $i \in I_j^{(N)}$, $\lim_{\eta\to 0} \|\phi(p_i^{(N)}) - B_j^{\theta_\eta}\| = 0$.

Because $d(p_i^{(N)}, q_j) > \eta$ for any $i \in I_0^{(N)}$ and $j \in J$, we have

$$\sum_{i \in I_0^{(N)}} t_i^{(N)} = \frac{1}{\eta^2} \sum_{i \in I_0^{(N)}} t_i^{(N)} \eta^2 \leq \frac{1}{\eta^2} d^W(\mu^{(N)}, \nu)^2 \leq \eta^2.$$

This implies **(a)**. And **(b)** follows from the following:

$$\sum_{i \in I_0^{(N)}} t_i^{(N)} l_i^2 \leq d^W(\mu^{(N)}, \nu)^2 \leq \eta^4.$$

Next we prove **(c)**. Fix an arbitrary $j \in J$ and let

$$\pi = \sum_{i,j'} T_{ij'} \mathrm{Dirac}_{(p_i^{(N)}, q_{j'})}$$

be any coupling of $\mu^{(N)}$ and ν. Then we have

$$\sum_{i \in I_j^{(N)}, j' \in J \setminus \{j\}} T_{ij'} d(p_i^{(N)}, q_{j'})^2 + \sum_{i \in I^{(N)} \setminus I_j^{(N)}} T_{ij} d(p_i^{(N)}, q_j)^2 \geq |\sum_{i \in I_j^{(N)}} t_i^{(N)} - s_j| \eta^2.$$

Therefore,

$$\eta^4 > d^W(\mu^{(N)}, \nu)^2 \geq |\sum_{i \in I_j^{(N)}} t_i^{(N)} - s_j| \eta^2.$$

This implies **(c)**. Finally, **(d)** is obvious from our construction of $B_j^{\theta_\eta}$.

Now **(i)** follows from **(a)** and **(b)** since

$$\begin{aligned}
\|\sum_{i \in I_0^{(N)}} t_i^{(N)} \phi(p_i^{(N)})\| &\leq \sum_{i \in I_0^{(N)}} t_i^{(N)} d(p_i^{(N)}, \overline{\mu^{(N)}}) \\
&\leq \sum_{i \in I_0^{(N)}} t_i^{(N)} (d(p_i^{(N)}, \overline{\nu}) + \eta) \\
&\leq \sum_{i \in I_0^{(N)}} t_i^{(N)} (K_\nu + l_i + \eta) \\
&\leq (K_\nu + \eta) \sum_{i \in I_0^{(N)}} t_i^{(N)} + \left(\sum_{i \in I_0^{(N)}} t_i^{(N)} \right)^{\frac{1}{2}} \left(\sum_{i \in I_0^{(N)}} t_i^{(N)} l_i^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

(ii) also follows from **(a)** and **(b)** since

$$\sum_{i \in I_0^{(N)}} t_i^{(N)} \|\phi(p_i^{(N)})\|^2 \leq \sum_{i \in I_0^{(N)}} t_i^{(N)} (K_\nu + l_i + \eta)^2.$$

(iii) follows from **(c)** and **(d)** because

$$\| \sum_{i \in I_j^{(N)}} t_i^{(N)} \phi(p_i^{(N)}) - s_j B_j^{\theta_\eta} \| \leq$$

$$\sum_{i \in I_j^{(N)}} t_i^{(N)} \| \phi(p_i^{(N)}) - B_j^{\theta_\eta} \| + \left| \sum_{i \in I_j^{(N)}} t_i^{(N)} - s_j \right| \| B_j^{\theta_\eta} \|.$$

Finally **(iv)** follows from **(c)** and **(d)** because

$$| \sum_{i \in I_j^{(N)}} t_i^{(N)} \| \phi(p_i^{(N)}) \|^2 - s_j \| B_j^{\theta_\eta} \|^2 | \leq$$

$$\sum_{i \in I_j^{(N)}} t_i^{(N)} | \| \phi(p_i^{(N)}) \|^2 - \| B_j^{\theta_\eta} \|^2 | + \left| \sum_{i \in I_j^{(N)}} t_i^{(N)} - s_j \right| \| B_j^{\theta_\eta} \|^2.$$

Now, the proof is completed. □

Remark 2.2. One simple application of Theorem 1.2 is the possibility to restrict the set of measures over which we take supremum when we define the Izeki-Nayatani invariant $\delta(Y)$: Let Y be a complete CAT(0) space and $U \subset \mathcal{P}(Y)$ be a dense subset in $(\mathcal{P}(Y), d^W)$. Then our theorems guarantees that

$$\delta(Y) = \sup_{\mu \in U} \delta(\mu).$$

For example, Let $\mathcal{P}_0(Y) \subset \mathcal{P}(Y)$ be a subset of all $\mu \in \mathcal{P}(Y)$ of the form $\mu = \sum_{i=1}^m \frac{1}{m} \mathrm{Dirac}_{p_i}$. Then, it is obvious that $\mathcal{P}_0(Y)$ is dense in $\mathcal{P}(Y)$ with respect to d^W. So we have

$$\delta(Y) = \sup_{\mu \in \mathcal{P}_0(Y)} \delta(\mu).$$

3. Continuity with respect to d_{HW}

To establish the continuity of δ on $\mathcal{P}(Y)$ we define another distance d_{HW} on $\mathcal{P}(Y)$ by declaring

$$d_{HW}(\mu, \nu) = \max \left\{ d^W(\mu, \nu),\ d_H(\mathrm{supp}\mu, \mathrm{supp}\nu) \right\}$$

for any $\mu, \nu \in \mathcal{P}(Y)$. Here, d_H denotes the Hausdorff distance. Recall that the Hausdorff distance between closed subsets A and B of Y is defined by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B),\ \sup_{b \in B} d(b, A) \right\}.$$

The distance d_{HW} makes $\mathcal{P}(Y)$ a metric space, and induces a topology which is stronger than the one induced by d^W. Then with respect to this topology, we can also prove the upper-semicontinuity of δ by the argument similar to that in the proof of Theorem 1.2.

Theorem 3.1. *Let (Y, d) be a complete* CAT(0) *space. Then $\delta : \mathcal{P}(Y) \to [0,1]$ is a continuous function on $(\mathcal{P}(Y), d_{HW})$.*

Proof. The lower-semicontinuity follows from Theorem 1.2. We prove the upper-semicontinuity. We proceed as in the previous section. Let $\nu = \sum_{j=1}^{n} s_j \mathrm{Dirac}_{q_j}$ be an arbitrary measure in $\mathcal{P}(Y)$, and let

$$\mu^{(N)} = \sum_{i=1}^{m^{(N)}} t_i^{(N)} \mathrm{Dirac}_{p_i^{(N)}}, \quad N = 1, 2, 3, \dots$$

be an arbitrary sequence of measures in $\mathcal{P}(Y)$ which converges to ν in $(\mathcal{P}(Y), d_{HW})$. Then, for any $\eta > 0$, we can find a positive real number N'_η such that $N \geq N'_\eta$ implies

$$d_{HW}(\mu^{(N)}, \nu) < \eta^2, \quad d(\overline{\mu^{(N)}}, \overline{\nu}) < \eta. \tag{3.1}$$

Now, what we have to show is that for any sufficiently small $\eta > 0$, any natural number $N \geq N'_\eta$ and any $\varphi \in \Phi(\nu)$ there exists $\tilde{\varphi_\eta} \in \Phi(\mu^{(N)})$ such that

$$D^{\mu^{(N)}}(\tilde{\varphi_\eta}) \leq D^{\nu}(\varphi) + G(\eta), \tag{3.2}$$

where G is some function converging to 0 when $\eta \to 0$.

As in the previous section, assume that η is an arbitrary positive real number such that

$$\eta < \min \left\{ \frac{1}{2} d(q_j, q_{j'}) \;\middle|\; j, j' \in J,\ j \neq j' \right\},$$
$$\eta^2 < \min \{ s_j \mid j \in J \}.$$

And let N be an arbitrary integer such that $N \geq N_\eta$ and let $\varphi \in \Phi(\nu)$. Let $I_0^{(N)}, \dots, I_n^{(N)}$ be as in the previous section. Then the following four assertions hold:

- $I^{(N)} = I_0^{(N)} \cup I_1^{(N)} \cup \dots \cup I_n^{(N)}$;
- If $j \neq j'$ then $I_j^{(N)} \cap I_{j'}^{(N)} = \phi$;
- For every $j \in J$, $I_j^{(N)} \neq \phi$;
- $I_0^{(N)} = \phi$.

The first three are shown by the same argument as in the previous section, and the last one follows immediately from the fact that $d_H(\mathrm{supp}\mu^{(N)}, \mathrm{supp}\nu) < \eta$.

Now, we will construct $\tilde{\varphi_\eta} \in \Phi(\mu^{(N)})$ in (3.2) in two steps. As the first step, we construct a vector $C_i \in \mathcal{H}$ for each $i \in I^{(N)}$ as follows: For $i \in I_j^{(N)}$, let

$$C_i = \frac{d(\overline{\mu^{(N)}}, p_i^{(N)})}{d(\overline{\nu}, q_j)} \varphi(q_j)$$

if $q_j \neq \overline{\nu}$, and let C_i be an arbitrary vector of length $d(\overline{\mu^{(N)}}, p_i^{(N)})$ if $q_j = \overline{\nu}$. Then the second inequality of (3.1) and the assumption on η imply

$$\begin{aligned}\|\varphi(q_j) - C_i\| &= |d(\overline{\mu^{(N)}}, p_i^{(N)}) - d(\overline{\nu}, q_j)| \\ &\leq d(\overline{\mu^{(N)}}, \overline{\nu}) + d(p_i^{(N)}, q_j) < 2\eta.\end{aligned}$$

Hence for any $i \in I_j^{(N)}$ and $i' \in I_{j'}^{(N)}$, if $j \neq j'$, we have

$$\begin{aligned}\|C_i - C_{i'}\| &\leq \|C_i - \varphi(q_j)\| + \|\varphi(q_j) - \varphi(q_{j'})\| + \|\varphi(q_{j'}) - C_{i'}\| \\ &\leq 4\eta + d(q_j, p_i^{(N)}) + d(p_i^{(N)}, p_{i'}^{(N)}) + d(p_{i'}^{(N)}, q_{j'}) \\ &\leq 6\eta + d(p_i^{(N)}, p_{i'}^{(N)}).\end{aligned}$$

Thus, with the fact that $d_H(\mathrm{supp}\mu^{(N)}, \mathrm{supp}\nu) < \eta$ we have

$$\|C_i - C_{i'}\|^2 \leq d(p_i^{(N)}, p_{i'}^{(N)})^2 + g(\eta), \tag{3.3}$$

where $g(\eta)$ is some function converging to 0 when $\eta \to 0$. And in the case $j = j'$, we have

$$\begin{aligned}\|C_i - C_{i'}\| &= |d(\overline{\mu^{(N)}}, p_i^{(N)}) - d(\overline{\mu^{(N)}}, p_{i'}^{(N)})| \\ &\leq d(p_i^{(N)}, p_{i'}^{(N)}).\end{aligned}$$

Let us proceed to the second step. First, we set up one notation. We define a subset $T^{(N)} \subset \mathrm{supp}\mu^{(N)} \times \mathrm{supp}\mu^{(N)}$ by declaring $(p_i^{(N)}, p_{i'}^{(N)}) \in T^{(N)}$ if and only if $i \in I_j^{(N)}$ and $i' \in I_{j'}^{(N)}$ for some j, j' such that $(q_j, q_{j'}) \in S_\nu$. And let

$$L'_{(N)} = \min \Big\{ d(p_i^{(N)}, p_{i'}^{(N)})^2 - \Big(d(p_i^{(N)}, \overline{\mu^{(N)}}) - d(p_{i'}^{(N)}, \overline{\mu^{(N)}})\Big)^2 \;\Big|\; (p_i^{(N)}, p_{i'}^{(N)}) \in T^{(N)} \Big\}.$$

Then observe that for all sufficiently small $\eta > 0$ we have

$$L'_{(N)} \geq \frac{L_\nu}{2}. \tag{3.4}$$

This follows from the fact that for any $(p_i^{(N)}, p_{i'}^{(N)}) \in T^{(N)}$, there is $(q_j, q_{j'}) \in S_\nu$ such that

$$d(p_i^{(N)}, q_j) < \eta, \quad d(p_{i'}^{(N)}, q_{j'}) < \eta, \quad d(\overline{\mu^{(N)}}, \overline{\nu}) < \eta.$$

Now we assume that $\eta > 0$ is sufficiently small so that (3.4) holds.

Let $E \in \mathcal{H}$ be a unit vector orthogonal to the subspace spanned by $C_1, \ldots, C_{m^{(N)}}$. For $0 < \theta < \frac{\pi}{2}$, we define

$$C_i^\theta = \sin\theta \cdot \|C_i\| E + \cos\theta \cdot C_i$$

for any $i \in I^{(N)}$. Then for any $0 < \theta < \frac{\pi}{2}$ and $i \in I^{(N)}$,

$$\|C_i^\theta\| = \|C_i\| = d(\overline{\mu^{(N)}}, p_i^{(N)}),$$

and for any $i, i' \in I^{(N)}$,

$$\|C_i - C_{i'}\|^2 - \|C_i^\theta - C_{i'}^\theta\|^2 = \sin^2\theta\{\|C_i - C_{i'}\|^2 - (\|C_i\| - \|C_{i'}\|)^2\}. \tag{3.5}$$

Assuming that $\eta > 0$ is taken to be small enough if necessary, we set

$$\vartheta_\eta = \sin^{-1}\sqrt{\frac{2g(\eta)}{L_\nu}},$$

and define a map $\tilde{\varphi}_\eta : \mathrm{supp}\mu^{(N)} \to \mathcal{H}$ by

$$\tilde{\varphi}_\eta(p_i^{(N)}) = C_i^{\vartheta_\eta}, \quad i \in I^{(N)}.$$

We want to confirm $\tilde{\varphi}_\eta \in \Phi(\mu^{(N)})$. The condition (1.1) is obvious, so we examine the condition (1.2) by considering three cases separately.

Case I: $(p_i^{(N)}, p_{i'}^{(N)}) \notin T^{(N)}$. In this case, C_i and $C_{i'}$ must be parallel vectors, so we have

$$\begin{aligned}\|C_i^{\vartheta_\eta} - C_{i'}^{\vartheta_\eta}\| &= \|C_i - C_{i'}\| = |\|C_i\| - \|C_{i'}\|| \\ &= \left| d(p_i^{(N)}, \overline{\mu^{(N)}}) - d(p_{i'}^{(N)}, \overline{\mu^{(N)}}) \right| \le d(p_i^{(N)}, p_{i'}^{(N)}).\end{aligned}$$

Case II: $(p_i^{(N)}, p_{i'}^{(N)}) \in T^{(N)}$ and $\|C_i - C_{i'}\|^2 - (\|C_i\| - \|C_{i'}\|)^2 < L'_{(N)}$. In this case, we have

$$\begin{aligned}\|C_i^{\vartheta_\eta} - C_{i'}^{\vartheta_\eta}\|^2 &\le \|C_i - C_{i'}\|^2 \\ &< L'_{(N)} + (d(p_i^{(N)}, \overline{\mu^{(N)}}) - d(p_{i'}^{(N)}, \overline{\mu^{(N)}}))^2 \le d(p_i^{(N)}, p_{i'}^{(N)})^2.\end{aligned}$$

Case III: $(p_i^{(N)}, p_{i'}^{(N)}) \in T^{(N)}$ and $\|C_i - C_{i'}\|^2 - (\|C_i\| - \|C_{i'}\|)^2 \ge L'_{(N)}$. In this case, by (3.3), (3.4), (3.5) and the definition of ϑ_η, we have

$$\|C_i^{\vartheta_\eta} - C_{i'}^{\vartheta_\eta}\|^2 \le d(p_i^{(N)}, p_{i'}^{(N)})^2.$$

Hence $\tilde{\varphi}_\eta \in \Phi(\mu^{(N)})$.

Let

$$\begin{aligned}G(\eta) &= D^{(\mu^{(N)})}(\varphi) - D^\nu(\tilde{\varphi}_\eta) \\ &= \frac{\|\sum_{j=0}^n \sum_{i \in I_j^{(N)}} t_i^{(N)} \varphi(p_i^{(N)})\|^2}{\sum_{j=0}^n \sum_{i \in I_j^{(N)}} t_i^{(N)} \|\varphi(p_i^{(N)})\|^2} - \frac{\left\|\sum_{j \in J} s_j C_j^{\vartheta_\eta}\right\|^2}{\sum_{j \in J} s_j \left\|C_j^{\vartheta_\eta}\right\|^2}.\end{aligned}$$

It is sufficient to show that $G(\eta)$ tends to 0 when $\eta \to 0$. It is quite obvious that this follows from the same argument by which we show that $F(\eta)$ tends to 0 in the previous section, so our proof is completed. □

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