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WAVE INTERACTIONS FOR THE PRESSURE GRADIENT EQUATIONS*

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Abstract. In this paper, we solve the Riemann problem for a coupled hyperbolic system of conservation laws, which arises as an intermediate model in the flux splitting method for the computation of Euler equations in gasdynamics. We study the properties of solutions involving shock and rarefaction waves, and establish their existence and uniqueness. We present numerical examples for different initial data, and finally discuss all possible elementary wave interactions; it is noticed that in certain cases the resulting wave pattern after interaction is substantially different from that which arises in isentropic gasdynamics.

Key words. Pressure-gradient equations, wave interactions, Riemann problem, shock, rarefaction wave.

AMS subject classifications. 35L60, 35L65, 76L05, 76N15

1. Introduction.

The exact solution to the Riemann problem is of great significance. For instance, it constitutes the basic building block for the construction of solutions to general initial value problems using the well known random choice method proposed by Glimm [1]. Lax [2] solved the Riemann problem for the case when the initial data consisting of constant states U_l and U_r are such that $||U_l - U_r||$ is sufficiently small; here U is the vector of unknown variables with U_l to the left of $x = 0$ and U_r to the right of $x = 0$ separated by a discontinuity at $x = 0$. Smoller [3] solved the Riemann problem by considering U_l and U_r to be arbitrary constant vectors; for details, the reader is referred to the book by Smoller [4]. Exact solutions of the Riemann problem were proposed by Godunov [5] and Chorin [6]; however, Smoller [4] proposed a rather different approach. Smoller and Temple [7] demonstrated the existence of solutions with shocks for equations describing a perfect fluid in special relativity. Toro [8] presented an efficient solver for computing the exact solution of the Riemann problem for ideal and covolume gases; for detailed methodologies, the reader is referred to the book by Toro [9]. The Riemann problem for kinematical conservation laws and geometrical features of nonlinear wavefronts can be found in Baskar and Prasad [10]. Interaction of shallow water waves and elementary wave interactions in isentropic magnetogasdynamics have been discussed by Raja Sekhar and Sharma [11, 12]. Shen [13] has discussed the wave interactions and stability of the Riemann solutions for chromatography system under the local small perturbations of the Riemann initial data. Concerning compressible duct flows, and two phase flows, we refer to the papers of Andrianov and Warnecke [14, 15].

For interaction of elementary waves in unsteady one-dimensional Euler equations, we refer to Smoller [4], and Chang & Hsiao [16]. The interactions of elementary waves of the scalar conservation laws with discontinuous flux function have been discussed by Wang and Sheng [17]. For an illuminating treatment on Riemann problem, we also refer to an article by Liu [18], Slemrod and Tzavaras [19] and the books of

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Godlewski and Raviart [20], Li−Tsien [21], Dafermos [22], Bressan [23], LeFloch [24] and LeVeque [25].

This paper is devoted to the analysis of the following coupled system of partial differential equations [26, 27]

$$\begin{aligned} &\frac{\partial}{\partial t}(u) + \frac{\partial}{\partial x}(p) = 0, \\ &\frac{\partial}{\partial t}(p + u^2/2) + \frac{\partial}{\partial x}(pu) = 0, \qquad t > 0,\ x \in \mathbb{R} \end{aligned} \tag{1.1}$$

which arises as an intermediate model in the flux splitting technique, used for numerical computation of Euler equations in gasdynamics, and is referred to as the pressure gradient equations in the literature. Here, the unknown quantities are velocity u and pressure p. Recently, the system (1.1) has been studied by Zhang et. al [28] to describe the interaction between two rarefaction waves; we, in the present paper, discuss all possible interactions of elementary waves using a different approach, and observe that when waves belonging to the same family of characteristic curves interact, the resulting wave pattern deviates remarkably from that which appears in the isentropic gasdynamic case. For instance, when a 1-shock wave overtakes another 1-shock wave, the transmitted and reflected waves are respectively 1-shock wave and 2-shock wave, i.e., $S_1S_1 \to S_1S_2$, which is in contrast to the corresponding isentropic gasdynamic case, where the transmitted and reflected waves are respectively 1-shock wave and 2-rarefaction wave, i.e., $S_1S_1 \to S_1R_2$. Similar is the case with other interactions such as S_2S_2, S_2R_2, R_2S_2, S_1R_1 and R_1S_1.

The paper is organized as follows; in Section 2, we show that the system is strictly hyperbolic, and that its characteristic fields are genuinely nonlinear. We establish the existence of shocks and rarefaction waves, and prove the stability conditions for shocks, and discuss how pressure and velocity vary across shocks and rarefaction waves. We show that the characteristic speed increases from left to right for rarefaction waves. In Section 3, we consider the Riemann problem for arbitrary initial data, and show that it is uniquely solvable, and establish the condition for the vacuum state. In Section 4, we discuss numerical results for different initial data. In Section 5, we discuss all possible interactions of elementary waves.

2. Properties of shock and rarefaction waves. For carrying out the characteristic analysis of (1.1), it is convenient to use the primitive variables $U = (p, u)^T$, rather than the vector of conserved variables, where superscript T denotes transposition. Then for smooth solutions, system (1.1) is equivalent to

$$U_t + AU_x = 0, \tag{2.1}$$

where A is 2×2 matrix with elements $A_{11} = A_{22} = 0$, $A_{12} = p$ and $A_{21} = 1$. The eigenvalues of A are $\lambda_1 = -\sqrt{p}$ and $\lambda_2 = \sqrt{p}$ with associated right eigenvectors $\vec{r}_1 = (-\sqrt{p}, 1)^T$ and $\vec{r}_2 = (\sqrt{p}, 1)^T$; thus, the system (2.1) is strictly hyperbolic when $p > 0$. Since, $\nabla\lambda_1 \cdot \vec{r}_1 = 1/2 = \nabla\lambda_2 \cdot \vec{r}_2$, the characteristic fields λ_1 and λ_2 are genuinely nonlinear, and the waves associated with them are either shocks or rarefaction waves, which are the weak solutions of (1.1) or (2.1).

2.1. Shocks. Suppose U is a weak solution of (1.1) or equivalently (2.1) such that U_l and U_r are C^1 and extend continuously to the shock $x = x(t)$. Let $[U] = U_l - U_r$ be the jump discontinuity across the shock and $\sigma = dx/dt$ the shock speed. Then, the following Rankine−Hugoniot jump conditions hold across the shock

$$\sigma[u] = [p], \tag{2.2}$$

$$\sigma[p + u^2/2] = [pu]. \tag{2.3}$$

Since the system is genuinely nonlinear, which corresponds to the strict convexity of the flux function, we require the following Lax conditions for a k-shock with speed σ_k, $k = 1, 2$

$$\lambda_k(U_r) < \sigma_k < \lambda_k(U_l), \quad \lambda_{k-1}(U_l) < \sigma_k < \lambda_{k+1}(U_r) \tag{2.4}$$

for some k.

Lemma 2.1. *Let the states U_l and U satisfy the Rankine–Hugoniot jump conditions (2.2) and (2.3).Let $S_1 = S_1(U_l)$ and $S_2 = S_2(U_l)$ respectively denote 1-shock and 2-shock curves associated with λ_1 and λ_2 characteristic fields. Then the shock curves satisfy*

$$u = u_l \mp (p - p_l)\sqrt{\frac{2}{p + p_l}}, \tag{2.5}$$

indeed, on S_1, we have $\frac{du}{dp} < 0$ and $\frac{d^2u}{dp^2} > 0$, whilst on S_2 we have $\frac{du}{dp} > 0$ and $\frac{d^2u}{dp^2} < 0$.

Proof. The σ-elimination of (2.2) and (2.3) yields (2.5), and then differentiating (2.5) with respect to p, we obtain

$$\frac{du}{dp} = \mp\frac{(p + 3p_l)}{\sqrt{2}(p + p_l)^{3/2}}. \tag{2.6}$$

It is easy to show using (2.6) that $\frac{du}{dp} < 0$ on S_1, and $\frac{du}{dp} > 0$ on S_2. Differentiating (2.6) with respect to p, we get

$$\frac{d^2u}{dp^2} = \pm\frac{(p + 7p_l)}{2\sqrt{2}(p + p_l)^{5/2}}. \tag{2.7}$$

For all values of p we obtain, in view of (2.7), that $\frac{d^2u}{dp^2} > 0$ for 1-shock, and $\frac{d^2u}{dp^2} < 0$ for 2-shock. □

We now show that the shocks satisfy the Lax stability conditions.

Lemma 2.2. *Across 1-shock (respectively, 2-shock), $p > p_l$ and $u < u_l$ (respectively, $p < p_l$ and $u < u_l$) if, and only if, the Lax conditions hold, i.e., 1-shock satisfies*

$$\sigma_1 < \lambda_1(U_l), \quad \lambda_1(U) < \sigma_1 < \lambda_2(U), \tag{2.8}$$

while the 2-shock satisfies

$$\lambda_1(U_l) < \sigma_2 < \lambda_2(U_l), \quad \lambda_2(U) < \sigma_2, \tag{2.9}$$

where σ_1 and σ_2 are propagation speeds of 1-shock and 2-shock respectively.

Proof. First let us consider 1-shock and prove $\sigma_1 < \lambda_1(U_l)$. On 1-shock, $p_l < p$, implying thereby that $p_l < (p_l + p)/2$, which implies that $-\sqrt{(p + p_l)/2} < -\sqrt{p_l}$; for 1-shock, in view of equation (2.5), we obtain

$$\sigma_1 = \frac{p - p_l}{u - u_l} < -\sqrt{p_l} = \lambda_1(U_l). \tag{2.10}$$

Next, since $p_l < p$ on 1-shock, we have $(p + p_l)/2 < p$, which implies that $-\sqrt{p} < -\sqrt{(p+p_l)/2}$, or equivalently

$$\lambda_1(U) = -\sqrt{p} < \frac{p - p_l}{u - u_l} = \sigma_1. \tag{2.11}$$

Since 1-shock speed $\sigma_1 < 0$, and $\lambda_2(U)$ is positive, we have

$$\sigma_1 < \lambda_2(U). \tag{2.12}$$

Therefore 1-shock satisfies Lax conditions; proof for 2-shock follows on similar lines. Conversely, we assume for 1-shock that the left and right hand states satisfy Lax conditions (2.8), and show that $p > p_l$ and $u < u_l$. From (2.8), we get $-\sqrt{p} < -\sqrt{p_l}$ which implies that $p > p_l$. Since 1-shock speed $\sigma_1 < 0$ and $p > p_l$, we obtain, in view of (2.2), that $u < u_l$. The corresponding results for 2-shock are proved in a similar way, and we shall not reproduce the details.

2.2. Rarefaction waves. Here we construct the rarefaction wave curves, and recall that an k rarefaction wave ($k = 1, 2$), connecting the states U_l and U_r, is a solution to (2.1) of the form

$$U(x,t) = \begin{cases} U_l, & \frac{x}{t} \le \lambda_k(U_l) \\ U(\frac{x}{t}), & \lambda_k(U_l) \le \frac{x}{t} \le \lambda_k(U_r) \\ U_r, & \frac{x}{t} \ge \lambda_k(U_r), \end{cases} \tag{2.13}$$

with $\lambda_n(U_l) \le \lambda_n(U_r)$, and where $U(\eta)$ with $\eta = \frac{x}{t}$ is a solution to the system of ordinary differential equations $(A - \eta I)(\dot{p}, \dot{u})^T = 0$, where I is 2×2 identity matrix and an overhead dot denotes differentiation with respect to the variable η. If $(\dot{p}, \dot{u})^T = (0, 0)$ then p and u are constant; but as we are interested in non-constant solutions, we consider $(\dot{p}, \dot{u})^T \neq (0, 0)$ and then it follows that $(\dot{p}, \dot{u})^T$ is an eigenvector of the matrix A corresponding to the eigenvalue η. Since the matrix A has two real and distinct eigenvalues λ_1 and λ_2, there are two families of rarefaction waves, R_1 and R_2 which denote, respectively, 1-rarefaction waves and 2-rarefaction waves.
First we consider 1-rarefaction waves. Since, $(A - \lambda_1 I)(\dot{p}, \dot{u})^T = 0$ with $\lambda_1 = -\sqrt{p}$, we have, $\dot{p} + \sqrt{p}\dot{u} = 0$, implying thereby that

$$\Pi_1 \equiv u + 2\sqrt{p} = constant, \tag{2.14}$$

which represents R_1 curves with Π_1 as the 1-Riemann invariant. Similarly, 2-rarefaction wave curves are given by

$$\Pi_2 \equiv u - 2\sqrt{p} = constant, \tag{2.15}$$

and Π_2 is the 2-Riemann invariant; indeed, the integral curves of the vector fields $\vec{r}_1$ and $\vec{r}_2$ are nothing but the level sets of the Riemann invariants Π_1 and Π_2 respectively. Let U be a k-rarefaction wave of the form (2.13), and let Π be a k-Riemann invariant; here $k = 1, 2$. Since, U is continuous and Π is assumed to be smooth, the function Π : $(x, t) \to \Pi(U)$ is continuous for $t > 0$. Obviously, $\Pi(U)$ is constant for $\frac{x}{t} \le \lambda_k(U_l)$ and $\frac{x}{t} \ge \lambda_k(U_r)$.
Further, since $\eta = \frac{x}{t}$, we have

$$\frac{d\Pi(U)}{d\eta} = \nabla\Pi(U) \cdot \dot{U}. \tag{2.16}$$

As $\dot{U}$ is parallel to $\vec{r}_k$, the right hand side of (2.16) is zero, and thus, it follows that on R_k, the Riemann invariant Π_k is constant.

THEOREM 2.1. *The R_1 curve is convex and monotonic decreasing while R_2 curve is concave and monotonic increasing.*

Proof. The 1-rarefaction curve is given by

$$u = u_l + 2(\sqrt{p_l} - \sqrt{p}), \qquad p \leq p_l \tag{2.17}$$

which on differentiation with respect to p, yields $\frac{du}{dp} = -\frac{1}{\sqrt{p}} < 0$, and subsequently, $\frac{d^2u}{dp^2} = \frac{1}{2p\sqrt{p}} > 0$. and, therefore, u is convex with respect to p for 1-rarefaction waves. In a similar way, we can prove for 2-rarefaction waves. □

LEMMA 2.3. *Across 1-rarefaction waves (respectively, 2-rarefaction waves), $p \leq p_l$ and $u_l \leq u$ (respectively, $p \geq p_l$ and $u \geq u_l$) if, and only if, the characteristic speed increases from left hand state to right hand state.*

Proof. Since $p \leq p_l$ for 1-rarefaction waves, which implies that $\lambda_1(U_l) \leq \lambda_1(U)$. In a similar way, we can prove $\lambda_2(U_l) \leq \lambda_2(U)$ for 2-rarefaction waves.
Conversely, for 1-rarefaction waves, since $\lambda_1(U_l) \leq \lambda_1(U)$, we have $-\sqrt{p_l} \leq -\sqrt{p}$ implying thereby that $p \leq p_l$. Further, since in 1-rarefaction wave region Π_1 is constant, we have $u - u_l = 2(\sqrt{p_l} - \sqrt{p}) \geq 0$; hence, $p \leq p_l$ and $u \geq u_l$. In the same way, one can prove that for 2-rarefaction waves, $p \geq p_l$ and $u \geq u_l$. □

3. The Riemann Problem. We now consider the Riemann problem for the system (1.1), which consists in finding weak solutions with piecewise constant initial data of the form

$$U(x,0) = \begin{cases} U_l, & if\ x < 0, \\ U_r, & if\ x > 0. \end{cases} \tag{3.1}$$

We solve this problem in the class of functions consisting of constant states, separated by either shocks or rarefaction waves. The solution of the Riemann problem consists of at most three constant states (including U_l and U_r), which are separated either by a shock or a rarefaction wave.

THEOREM 3.1. *The curves of shock and rarefaction waves for 1-family, i.e., S_1 and R_1 (respectively 2-family, i.e., S_2 and R_2) have the second order contact at U_l.*

Proof. In order to prove S_1 and R_1 have the second order contact at U_l, we have to show that S_1 and R_1 curves at $p = p_l$, upto second derivatives, are equal. The equation for 1-rarefaction wave is given in (2.17), and we obtain

$$u|_{p=p_l} = u_l, \quad \frac{du}{dp}|_{p=p_l} = -\frac{1}{\sqrt{p_l}}, \quad (\frac{d^2u}{dp^2})|_{p=p_l} = \frac{1}{2p_l\sqrt{p_l}}. \tag{3.2}$$

The equation for 1-shock is given in (2.5) and from (2.6) & (2.7), we get

$$u|_{p=p_l} = u_l, \quad \frac{du}{dp}|_{p=p_l} = -\frac{1}{\sqrt{p_l}}, \quad (\frac{d^2u}{dp^2})|_{p=p_l} = \frac{1}{2p_l\sqrt{p_l}}. \tag{3.3}$$

Thus u, $\frac{du}{dp}$ and $\frac{d^2u}{dp^2}$ at $p = p_l$ have the same value for 1-shock and 1-rarefaction wave curve. Therefore, S_1 and R_1 have the second order contact at U_l. Proof for 2-family follows on similar lines. □

When U_r is sufficiently close to U_l, the existence and uniqueness of the solution of Riemann problem for system (1.1) in the class of elementary waves follow from the general theorem of Lax, which applies to any system of conservation laws that is strictly hyperbolic and genuinely nonlinear in each characteristic field (see [2], [20]). For arbitrary initial data we discuss the existence of solution of the Riemann problem for the system (1.1).

We consider the physical variables as coordinate system; we divide the (p, u)-plane into five disjoint open regions namely I, II, III, IV and V. The first four regions namely I, II, III and IV are separated by the curves S_1, S_2, R_1 and R_2, represented by (2.5), (2.14) and (2.15), respectively; these curves are drawn in Fig. 3a for a given left state U_l, and the regions IV and V are separated by the curve $R_2(U_{l_0})$ where $U_{l_0} = (0, u_l + 2\sqrt{p_l})$. Indeed, we fix U_l and allow U_r to vary; if U_r lies on any of the above five curves, then we have seen how to solve the problem. First we assume that U_r belongs to one of the four open regions I, II, III and IV as shown in Fig. 3a.

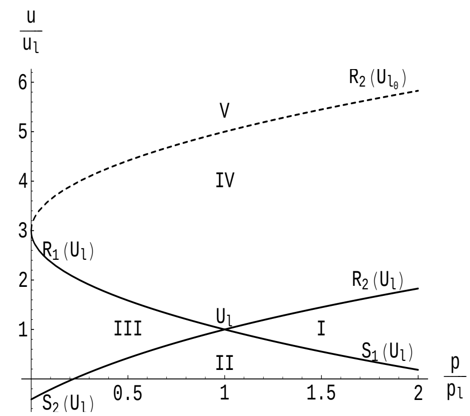

FIG. 3a. *Wave curves in (p, u)-plane*

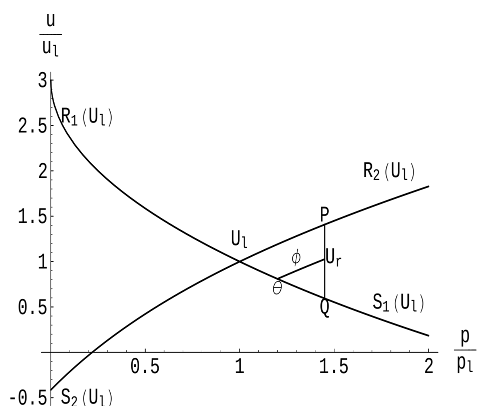

FIG. 3b. *U_r is in region I*

Following ([4]), we define, for $\hat{U} \in \mathbb{R}^+ \times \mathbb{R}$, $S_k(\hat{U}) = \{(p, u) : (p, u) \in S_k(\hat{U})\}$, $R_k(\hat{U}) = \{(p, u) : (p, u) \in R_k(\hat{U})\}$ and $T_k(\hat{U}) = S_k(\hat{U}) \bigcup R_k(\hat{U})$, $k = 1, 2$. For fixed $U_l \in \mathbb{R}^+ \times \mathbb{R}$, we consider the family of curves $\mathcal{S} = \{T_2(\hat{U}) : \hat{U} \in T_1(U_l)\}$. As the (p, u) plane is covered univalently by the family of curves $\mathcal{S}$, i.e., through each point U_r, there passes exactly one curve $T_2(\hat{U})$ of $\mathcal{S}$, the solution to the Riemann problem is given as follows; we connect $\hat{U}$ to U_l on the right by a 1-wave (either shock or rarefaction wave), and then we connect U_r to $\hat{U}$ on the right by a 2-wave (either S_2 or R_2). Indeed, depending on the position of U_r we have different wave configurations.

THEOREM 3.2. *Let U_l, $U_r \in \mathbb{R}^+ \times \mathbb{R}$ with U_l fixed, and U_r is allowed to vary then the Riemann problem is solvable.*

Proof. We are allowing to vary U_r i.e., U_r is in region I, II, III or IV. If $U_r \in I$, then draw a vertical line $p = p_r$ as shown in Fig. 3b, which meets R_2 and S_1 uniquely at $P = (p_1, u_1)$ and $Q = (p_2, u_2)$ respectively. We notice that the subfamily of curves in $\mathcal{S}$, consisting of the set $\{T_2(\hat{U}) \equiv T_2(\hat{p}, \hat{u}) : p_l \leq \hat{p} \leq p_r\}$ induces a continuous mapping $\theta \to \phi(\theta)$ from the arc U_lQ to line segment PQ, see ([4]); indeed, the region I is covered by curves in $\mathcal{S}$. So, let us suppose that (p_m, u_m) is the point which is mapped to U_r. Then

$$u = u_l - (p_m - p_l)\sqrt{\frac{2}{p_m + p_l}} - 2(\sqrt{p_m} - \sqrt{p_r}), \tag{3.4}$$

which on differentiation yields $\frac{du}{dp_m} = -(\frac{(p_m+3p_l)}{\sqrt{2}(p_m+p_l)^{3/2}} + \frac{1}{\sqrt{p_m}}) < 0$, implying thereby that (p_m, u_m) is unique. Similarly, we can prove uniqueness if U_r is in region II, III or IV.
Thus if $U_r \in I$, then the solution to Riemann problem consists of 1-shock and a 2-rarefaction wave connecting U_l to U_r. Suppose U_r is in region II, then the solution consists of shocks S_1 and S_2 joining U_l to U_r. If $U_r \in III$, then the solution of Riemann problem is obtained by connecting U_l to U_r by R_1, followed by S_2. If U_r lies in region IV, then the solution consists of 1-rarefaction wave and 2-rarefaction wave. Thus the set $\{T_2(\hat{U}) : \hat{U} \in T_1(U_l)\}$ covers the region I, II, III and IV in a 1-1 fashion. Therefore, the solution to the Riemann problem is solvable for arbitrary U_r lying in any of the regions I, II, III and IV. □

If $U_r \in V$ or $R_2(U_{l_0})$ then vacuum $(p = 0)$ does occur, and we have the following result:

LEMMA 3.1. *If* $u_r - u_l \geq 2(\sqrt{p_l} + \sqrt{p_r})$ *then the vacuum occurs*

Proof. Across 1-rarefaction wave, 1-Riemann invariant is constant, i.e., $\Pi_1(p_l, u_l) = \Pi_1(p_m, u_m)$ and similarly across 2-rarefaction wave, 2-Riemann invariant is constant, i.e., $\Pi_2(p_m, u_m) = \Pi_2(p_r, u_r)$. So $\Pi_2(p_r, u_r) - \Pi_1(p_l, u_l) = \Pi_2(p_m, u_m) - \Pi_1(p_m, u_m)$ implying thereby that $0 \leq u_r - u_l - 2(\sqrt{p_l} + \sqrt{p_r}) = -4\sqrt{p_m}$, which implies that $p_m = 0$. Hence, vacuum occurs.

4. Numerical examples. For a given left state U_l and a right state U_r, we give numerical algorithm to find the unknown state U_m (see Table 1) in (x, t)-plane.
Case a: For $p_l < p_m$ and $p_m \leq p_r$, we eliminate u_m from (2.5) and (2.15) to obtain

$$u_r - u_l + 2(\sqrt{p_m} - \sqrt{p_r}) + (p_m - p_l)\sqrt{\frac{2}{p_m + p_l}} = 0. \tag{4.1}$$

Case b: For $p_l < p_m$ and $p_r < p_m$, we obtain from (2.5) that

$$u_r - u_l + (p_m - p_l)\sqrt{\frac{2}{p_m + p_l}} - (p_r - p_m)\sqrt{\frac{2}{p_m + p_r}} = 0. \tag{4.2}$$

Case c: For $p_m \leq p_l$ and $p_m \leq p_r$, eliminating u_m from (2.14) and (2.15), we get

$$u_r - u_l - 2(\sqrt{p_l} - 2\sqrt{p_m} + \sqrt{p_r}) = 0. \tag{4.3}$$

Case d: For $p_m \leq p_l$ and $p_m > p_r$, eliminating u_m from (2.14) and (2.5), we get

$$u_r - u_l - 2(\sqrt{p_l} - \sqrt{p_m}) - (p_r - p_m)\sqrt{\frac{2}{p_r + p_m}} = 0. \tag{4.4}$$

Thus, for all the four possible wave patterns (4.1)-(4.4), we obtain a single nonlinear equation

$$f_r(p_m, U_r) + f_l(p_m, U_l) + u_r - u_l = 0, \tag{4.5}$$

where

$$f_l(p_m, U_l) = \begin{cases} (p_m - p_l)\sqrt{\frac{2}{p_l+p_m}}, & \text{if } p_m > p_l, \\ 2(\sqrt{p_m} - \sqrt{p_l}), & \text{if } p_m \le p_l, \end{cases}$$

and

$$f_r(p_m, U_r) = \begin{cases} (p_m - p_r)\sqrt{\frac{2}{p_r+p_m}}, & \text{if } p_m > p_r, \\ 2(\sqrt{p_m} - \sqrt{p_r}), & \text{if } p_m \le p_r. \end{cases}$$

We solve (4.5) for p_m by using Newton-Raphson iterative procedure with a stop criterion when the relative error is less than 10^{-8}; the initial guess for p_m is taken to be the average value of p_l and p_r. Once p_m is known, the solution for the particle velocity u_m can be obtained from (2.5) or (2.14) (respectively, from (2.5) or (2.15)) depending on whether the 1-wave (respectively, 2-wave) is a shock or a rarefaction wave. In case of rarefaction waves, we have to find the solution inside the wave region. For 1-rarefaction wave, the slope of the characteristic from $(0,0)$ to (x,t) is

$$\frac{dx}{dt} = \frac{x}{t} = -\sqrt{p}, \tag{4.6}$$

then p is found from (4.6). Since Π_1 is constant in 1-rarefaction wave region we have

$$u = u_l + 2(\sqrt{p_l} - \sqrt{p}), \tag{4.7}$$

in view of (4.7) we can obtain the particle velocity u. In a similar way, we find the solution inside the 2-rarefaction wave.

Test	p_l	u_l	p_m	u_m	p_r	u_r	Result
1	1.0	1.0	2.280776	0.0	1.0	−1.0	S_1S_2
2	1.0	−0.5	0.5625	0.0	1.0	0.5	R_1R_2
3	0.8	1.1	1.322781	0.592563	1.7	0.9	S_1R_2
4	3.0	0.0	2.187311	0.50619	1.5	0.0	R_1S_2

Table 1

5. Interaction of Elementary Waves. The interaction of elementary waves, obtained from the Riemann problem (3.1), gives rise to new emerging elementary waves. We define the initial function, with two jump discontinuities at x_1 and x_2, as follows.

$$U(x,0) = \begin{cases} U_l, & if\ x \le x_1, \\ U_*, & if\ x_1 < x \le x_2, \\ U_r, & if\ x_2 < x, \end{cases} \tag{5.1}$$

with an appropriate choice of U_* and U_r in terms of U_l and arbitrary x_1 and $x_2 \in \mathbb{R}$. With the above initial data, we have two Riemann problems locally. An elementary wave of the first Riemann problem may interact with an elementary wave of the second Riemann problem, and a new Riemann problem is formed at the time of interaction.

Here, we use the notation $S_2R_1 \to R_1S_2$, which means that a 2-shock wave, S_2, of first Riemann problem (connecting U_l to U_*) interacts with 1-rarefaction wave, R_1, of second Riemann problem (connecting U_* to U_r), and the interaction leads to a new Riemann problem (connecting U_l to U_r via U_m), the solution of which consists of 1-rarefaction wave, R_1, and a 2-shock wave S_2 (i.e., R_1S_2). The possible interactions of elementary waves belonging to different families are R_2R_1, R_2S_1, S_2R_1 and S_2S_1 while the elementary wave interactions belonging to the same family are R_2S_2, S_2R_2, S_1R_1, R_1S_1, S_1S_1 and S_2S_2.

5.1. Interaction of Elementary Waves from Different Families.

(i) Collision of two shocks (S_2S_1). We consider that U_l is connected to U_* by a 2-shock, S_2, of first Riemann problem and U_* is connected to U_r by a 1-shock, S_1, of second Riemann problem. In other words, for a given U_l, we choose U_* and U_r in such a way that $p_* < p_l$, $u_* = u_l + (p_* - p_l)\sqrt{\frac{2}{p_l+p_*}}$ and $p_* < p_r$, $u_r = u_* - (p_r - p_*)\sqrt{\frac{2}{p_r+p_*}}$. Since speed of 2-shock of the first Riemann problem is greater than speed of 1-shock of the second Riemann problem, S_2 overtakes S_1. In order to show that for any arbitrary state U_l, the state U_r lies in the region II (see Fig. 3a), it is sufficient to prove that $(p - p_l)\sqrt{\frac{2}{p+p_l}} + (p_* - p_l)\sqrt{\frac{2}{p_*+p_l}} - (p - p_l)\sqrt{\frac{2}{p+p_l}} < 0$ for $p_* < p_l$ and $p_* < p$.

We have $(p - p_l)\sqrt{\frac{2}{p+p_l}} + (p_* - p_l)\sqrt{\frac{2}{p_*+p_l}} < (p - p_l)\sqrt{\frac{2}{p+p_l}} < (p - p_*)\sqrt{\frac{2}{p+p_l}} < (p - p_*)\sqrt{\frac{2}{p+p_*}}$ (since $p_* < p_l$ and $p_* < p$). Hence $(p - p_l)\sqrt{\frac{2}{p+p_l}} + (p_* - p_l)\sqrt{\frac{2}{p_*+p_l}} - (p - p_l)\sqrt{\frac{2}{p+p_l}} < 0$, i.e., the curve $S_1(U_*)$ lies below the curve $S_1(U_l)$, and therefore U_r lies in the region II. Thus, in view of the results presented in the preceding section, it follows that the interaction result is $S_2S_1 \to S_1S_2$.

(ii) Collision of a shock and rarefaction wave (S_2R_1). Here $U_* \in S_2(U_l)$ and $U_r \in R_1(U_*)$. That is, for a given U_l, we choose U_* and U_r such that $p_* < p_l$, $u_* = u_l + (p_* - p_l)\sqrt{\frac{2}{p_*+p_l}}$ and $p_r \leq p_*$, $u_r = u_* + 2(\sqrt{p_*} - \sqrt{p_r})$. Since 2-shock has greater velocity than 1-rarefaction wave, it follows that S_2 overtakes R_1. Moreover, since for any given U_l, $2(\sqrt{p_l} - \sqrt{p_*}) - (p_* - p_l)\sqrt{\frac{2}{p_*+p_l}} > 0$ for $p_l > p_* > p$, it follows that the curve $R_1(U_*)$ lies below the curve $R_1(U_l)$; hence U_r lies in the region III, and subsequently $S_2R_1 \to R_1S_2$.

(iii) Collision of two rarefaction waves (R_2R_1). We consider $U_* \in R_2(U_l)$ and $U_r \in R_1(U_*)$. In other words, for a given U_l, we choose U_* and U_r such that $p_l \leq p_*$, $u_* = u_l - 2(\sqrt{p_l} - \sqrt{p_*})$ and $p_r \leq p_*$, $u_r = u_* + 2(\sqrt{p_*} - \sqrt{p_r})$. Since the trailing end of 2-rarefaction wave has a greater velocity (bounded above) in (x,t)-plane than that 1-rarefaction wave velocity (bounded above), interaction will take place. Since $p_l < p_*$, therefore $4(\sqrt{p_*} - \sqrt{p_l}) > 0$, it follows that the curve $R_1(U_*)$ lies above the curve $R_1(U_l)$; hence U_r lies in the region IV and the interaction result is $R_2R_1 \to R_1R_2$.

(iv) Collision of a rarefaction wave and a shock (R_2S_1). Here $U_* \in R_2(U_l)$ and $U_r \in S_1(U_*)$, i.e., for a given U_l, we choose U_* and U_r such that $p_l \leq p_*$, $u_* = u_l - 2(\sqrt{p_l} - \sqrt{p_*})$ and $p_* < p_r$, $u_r = u_* - (p_r - p_*)\sqrt{\frac{2}{p_r+p_*}}$. Since 1-shock speed of second Riemann problem is less than the speed of trailing end of 2-rarefaction wave of first Riemann problem in (x,t)-plane, and therefore S_1 penetrates R_2. For any given U_l, we show that $U_r \in I$; for this, it is enough to show that

$$(p - p_l)\sqrt{\frac{2}{p+p_l}} - (p - p_*)\sqrt{\frac{2}{p+p_*}} - 2(\sqrt{p_l} - \sqrt{p_*}) > 0. \tag{5.2}$$

We have $(p-p_l)\sqrt{\frac{2}{p+p_l}} > (p-p_*)\sqrt{\frac{2}{p+p_*}}$ for $p_l < p_*$; hence, the inequality (5.2) follows, implying thereby that the curve $S_1(U_*)$ lies above the curve $S_1(U_l)$, and U_r lies in the region I. Thus the interaction result is $R_2S_1 \to S_1R_2$.

5.2. Interaction of Elementary Waves from Same Family.

(i) 1-rarefaction wave overtakes 1-shock (R_1S_1)**.** In this case, U_l is connected to U_* by 1-rarefaction wave of the first Riemann problem and U_* is connected to U_r by 1-shock of the second Riemann problem. That is, for a given U_l, we choose U_* and U_r in such a way that $p_* \leq p_l$, $u_* = u_l + 2(\sqrt{p_l} - \sqrt{p_*})$ and $p_* < p_r$, $u_r = u_* - (p_r - p_*)\sqrt{\frac{2}{p_*+p_r}}$.
First we show that $R_1(U_l)$ lies below the curve $S_1(U_*)$ for $p_* < p \leq p_l$; in other words, for $p_* < p \leq p_l$

$$2(\sqrt{p} - \sqrt{p_*}) - (p-p_*)\sqrt{\frac{2}{p+p_*}} > 0. \tag{5.3}$$

Let us define $\phi_1(p) = 2(\sqrt{p}-\sqrt{p_*})-(p-p_*)\sqrt{\frac{2}{p+p_*}}$ so that $\phi_1(p_*) = 0$. Differentiating $\phi_1(p)$ with respect to p, we have to show that $\phi_1^{'}(p) > 0$. Let us assume on contrary that $\phi_1^{'}(p) \leq 0$, which implies that $\sqrt{2}(p+p_*)^{\frac{3}{2}} \leq \sqrt{p}(p+3p_*)$; squaring both sides and simplifying it, we obtain $(p^3 - p_*^3) + 6p_*^2(p-p_*) \leq 0$, which is a contradiction since the left hand side of the inequality is strictly positive. Thus, $\phi_1^{'}(p) > 0$, implying thereby that $\phi_1(p) > \phi_1(p_*) = 0$; hence $R_1(U_l)$ lies below the curve $S_1(U_*)$ for $p_* < p \leq p_l$.
Next we prove that $S_1(U_l)$ lies below the curve $S_1(U_*)$ for $p_l < p$; for this it is sufficient to prove that

$$2(\sqrt{p_l} - \sqrt{p_*}) + (p-p_l)\sqrt{\frac{2}{p+p_l}} - (p-p_*)\sqrt{\frac{2}{p+p_*}} > 0, \quad \forall\ p > p_l > p_*. \tag{5.4}$$

Since $2\sqrt{p_l} + (p-p_l)\sqrt{\frac{2}{p+p_l}}$ is a increasing function with respect to p_l for $p_l < p$; and hence $2(\sqrt{p_l} - \sqrt{p_*}) + (p-p_l)\sqrt{\frac{2}{p+p_l}} - (p-p_*)\sqrt{\frac{2}{p+p_*}} > 0$ for $p_* < p_l < p$.
Lastly, we show that $S_1(U_*)$ and $R_2(U_l)$ intersect uniquely at some point $(\tilde{p}_1, \tilde{u}_1)$, where $p_* < p_l < \tilde{p}_1$. To prove this, we define a new function $\phi_2(p) = 4\sqrt{p_l} - 2(\sqrt{p_*} + \sqrt{p}) - (p-p_*)\sqrt{\frac{2}{p+p_*}}$ for $p_* < p_l < p$. Since $\phi_2(p_l) = \phi_1(p_l) > 0$ and $\phi_2(p) < 0$ for large values of p, by virtue of monotonicity and intermediate value property, there exists a unique $\tilde{p}_1$, for $p_l < \tilde{p}_1$, such that $\phi_2(\tilde{p}_1) = 0$. Thus, the intersection of $R_2(U_l)$ and $S_1(U_*)$ is uniquely determined. Indeed, depending on the value of p_r, we distinguish three cases:
a) When $p_r < \tilde{p}_1$, $U_r \in IV$ and the interaction result is $R_1S_1 \to R_1R_2$.
b) When $p_r = \tilde{p}_1$, U_r lies on $R_2(U_l)$ and the interaction result is $R_1S_1 \to R_2$; thus, when two waves of first family interact, they annihilate each other, and give rise to a wave of second family.
c) When $p_r > \tilde{p}_1$, $U_r \in I$ and the interaction result is $R_1S_1 \to S_1R_2$; this means that 1-shock of second Riemann problem, which is strong compared to the 1-rarefaction wave of first Riemann problem, overtakes the trailing end of 1-rarefaction wave, and a reflected rarefaction wave $R_2(U_m, U_r)$, connecting a new constant state U_m on the left to the known state U_r on the right, is produced. The transmitted wave, after interaction, is the 1-shock that joins U_l on the left to the state U_m on the right.

(ii) 1-shock overtakes 1-rarefaction wave (S_1R_1)**.** Here $U_* \in S_1(U_l)$ and $U_r \in R_1(U_*)$. That is, for a given U_l, we choose U_* and U_r such that $p_l < p_*$, $u_* = u_l - (p_* - p_l)\sqrt{\frac{2}{p_*+p_l}}$ and $p_r \leq p_*$, $u_r = u_* + 2(\sqrt{p_*} - \sqrt{p_r})$. In the (x,t) plane the speed of trailing end of 1-rarefaction wave, $\lambda_1(U_*)$, is less than the velocity $\sigma_1(U_l, U_*)$ and therefore 1-rarefaction wave from right overtakes 1-shock from left after a finite time.
First we show that $S_1(U_l)$ lies below the curve $R_1(U_*)$ for $p_l < p < p_*$; for this we need to show that for p lying in the interval (p_l, p_*)

$$(p - p_l)\sqrt{\frac{2}{p+p_l}} - 2\sqrt{p} - ((p_* - p_l)\sqrt{\frac{2}{p_* + p_l}} - 2\sqrt{p_*}) > 0. \tag{5.5}$$

To prove this, we define a new function $\phi_3(p) = (p - p_l)\sqrt{\frac{2}{p+p_l}} - 2\sqrt{p}$ for $p_l < p$. Then one can show that $\phi_3^{'}(p) = \frac{(p+3p_l)}{\sqrt{2}(p+p_l)^{\frac{3}{2}}} - \frac{1}{\sqrt{p}} > 0$ for $p_l < p$, implying thereby that $\phi_3(p) > \phi_3(p_*)$, and thus we prove the inequality (5.5).
Next we show that $R_1(U_*)$ lies above the curve $R_1(U_l)$ for $p \leq p_l < p_*$, i.e., $2(\sqrt{p_*} - \sqrt{p_l}) - (p_* - p_l)\sqrt{\frac{2}{p_*+p_l}} > 0$ for $p \leq p_l < p_*$. Since the left hand side of this inequality, for $p \leq p_l < p_*$, turns out to be $\phi_3(p_l)$, which has already been shown to be positive, the conclusion follows.
Lastly, we show that $R_2(U_l)$ and $R_1(U_*)$ intersect uniquely at some point, say, $(\tilde{p}_2, \tilde{u}_2)$ for $p_l < \tilde{p}_2 < p_*$; for this, it is enough to show that the equation $2(\sqrt{p_l} + \sqrt{p_*} - 2\sqrt{p}) - (p_* - p_l)\sqrt{\frac{2}{p_*+p_l}} = 0$ has a unique root $\tilde{p}_2$ such that $p_* > \tilde{p}_2 > p_l$. To establish this, we define a new function $\phi_4(p) = 2(\sqrt{p_l} + \sqrt{p_*} - 2\sqrt{p}) - (p_* - p_l)\sqrt{\frac{2}{p_*+p_l}}$; since $\phi_4(p_l) > 0$ and $\phi_4(p_*) < 0$, in view of monotonicity and intermediate value property, it follows that the curves $R_1(U_*)$ and $R_2(U_l)$ intersect uniquely. Here again we distinguish three cases depending on the value of p_r:
a) When $p_r < \tilde{p}_2$, $U_r \in IV$ and the interaction result is $S_1R_1 \to R_1R_2$, i.e., 1-rarefaction wave is sufficiently strong compared to 1-shock wave which, after interaction, produces a new elementary wave.
b) When $p_r = \tilde{p}_2$, $U_r \in R_2(U_l)$ and the interaction result is $S_1R_1 \to R_2$, i.e., the interaction of elementary waves of first family gives rise to a new elementary wave of second family.
c) When $p_r > \tilde{p}_2$, $U_r \in I$ and the interaction result is $S_1R_1 \to S_1R_2$.

(iii) 2-shock overtakes 2-rarefaction wave (S_2R_2)**.** The S_2R_2 interaction takes place when $U_* \in S_2(U_l)$ and $U_r \in R_2(U_*)$. In other words, for a given U_l, we choose U_* and U_r in such a way that $p_* < p_l$, $u_* = u_l + (p_* - p_l)\sqrt{\frac{2}{p_l+p_*}}$ and $p_* \leq p_r$, $u_r = u_* - 2(\sqrt{p_*} - \sqrt{p_r})$.
First we show that for $p_* < p < p_l$, $S_2(U_l)$ lies below $R_2(U_*)$, i.e.,

$$2\sqrt{p} - (p - p_l)\sqrt{\frac{2}{p+p_l}} - (2\sqrt{p_*} - (p_* - p_l)\sqrt{\frac{2}{p_* + p_l}}) > 0, \quad \forall\, p \in (p_*, p_l]. \tag{5.6}$$

To prove this we define a new function $\phi_5(p) = 2\sqrt{p} - (p - p_l)\sqrt{\frac{2}{p+p_l}}$ for $p < p_l$. Since $\phi_5^{'}(p) = \frac{1}{\sqrt{p}} - \frac{(p+3p_l)}{(p+p_l)\sqrt{2(p+p_l)}} > 0$, we have $\phi_5(p) > \phi_5(p_*)$, and the inequality (5.6) follows, implying thereby that $S_2(U_l)$ lies below $R_2(U_*)$.

Next we show that the curve $R_2(U_l)$ lies below the curve $R_2(U_*)$ for $p_* < p_l \le p$; for this it is enough to prove $2(\sqrt{p_l}-\sqrt{p_*})+(p_*-p_l)\sqrt{\frac{2}{p_*+p_l}} > 0$ for $p_* < p_l \le p$. We notice that the left hand side of this inequality is $\phi_5(p_l)-\phi_5(p_*)$ which has already been shown to be positive, and hence the curve $R_2(U_l)$ lies below the curve $R_2(U_*)$ for $p_* < p_l \le p$.
Lastly, we show that $R_2(U_*)$ and $R_1(U_l)$ intersect uniquely, say, at $(\tilde{p}_3, \tilde{u}_3)$ for $p_* < \tilde{p}_3 < p_l$.
Let us define $\phi_6(p) = 2(2\sqrt{p}-\sqrt{p_l}-\sqrt{p_*})+(p_*-p_l)\sqrt{\frac{2}{p_*+p_l}}$ for $p_* < p \le p_l$ so that $\phi_6(p_l) > 0$ and $\phi_6(p_*) < 0$. Then, there exists a $\tilde{p}_3$ such that $\phi_6(\tilde{p}_3) = 0$. Thus $R_2(U_*)$ and $R_1(U_l)$ intersect uniquely at $(\tilde{p}_3, \tilde{u}_3)$ as $R_2(U_*)$ and $S_1(U_l)$ are monotone.
Here again the following cases arise:
a) When $p_r < \tilde{p}_3$, $U_r \in III$ and the interaction result is $S_2R_2 \to R_1S_2$; this means that the strength of R_2 is small compared to the elementary wave S_2, and S_2 annihilates R_2 in a finite time. The strength of reflected R_1 wave is small compared to the incident waves S_2 and R_2.
b) When $p_r = \tilde{p}_3$, $U_r \in R_1(U_l)$ and the interaction result is $S_2R_2 \to R_1$.
c) When $p_r > \tilde{p}_3$, $U_r \in IV$ and the interaction result is $S_2R_2 \to R_1R_2$, implying thereby that R_2 is stronger than S_2.

(iv) 2-rarefaction wave overtakes 2-shock **(R_2S_2).** Here $U_* \in R_2(U_l)$ and $U_r \in S_2(U_*)$. Thus, for a given U_l, we choose U_* and U_r such that $p_l \le p_*$, $u_* = u_l - 2(\sqrt{p_l}-\sqrt{p_*})$ and $p_r < p_*$, $u_r = u_* + (p_r - p_*)\sqrt{\frac{2}{p_r+p_*}}$.
Now we show that $R_2(U_l)$ lies below $S_2(U_*)$ for $p_l \le p < p_*$, i.e.,

$$2(\sqrt{p_*}-\sqrt{p})+(p-p_*)\sqrt{\frac{2}{p+p_*}} > 0, \quad \forall\ p_l \le p < p_*. \tag{5.7}$$

To prove this we define a new function $\phi_7(p) = 2(\sqrt{p_*}-\sqrt{p})+(p-p_*)\sqrt{\frac{2}{p+p_*}}$ for $p_l \le p \le p_*$ so that $\phi_7(p_*) = 0$. Since $\frac{d\phi_7(p)}{dp} = \frac{(p+3p_*)}{(p+p_*)\sqrt{2(p+p_*)}} < 0$, implying thereby that $\phi_7(p) > \phi_7(p_*) = 0$. Hence, the result.
Next we show that $S_2(U_l)$ lies below the curve $S_2(U_*)$ for $p < p_l < p_*$; for this, it is sufficient to prove that for $p < p_l < p_*$, the following inequality holds

$$2\sqrt{p_*}+(p-p_*)\sqrt{\frac{2}{p+p_*}}-2\sqrt{p_l}-(p-p_l)\sqrt{\frac{2}{p+p_l}} > 0. \tag{5.8}$$

In order to prove this inequality, we define a new function $\phi_8(p,p_*) = 2\sqrt{p_*}+(p-p_*)\sqrt{\frac{2}{p+p_*}}$ for $p < p_*$; since $\frac{d\phi_8(p,p_*)}{dp_*} = \frac{1}{\sqrt{p_*}} - \frac{(3p+p_*)}{(p+p_*)\sqrt{2(p+p_*)}} > 0$, it follows that $\phi_8(p,p_*) > \phi_8(p,p_l)$, and hence the inequality (5.8).
Lastly, we show that $R_1(U_l)$ and $S_2(U_*)$ intersect uniquely at a point, say, $(\tilde{p}_4, \tilde{u}_4)$, where $\tilde{p}_4 < p_l < p_*$. The proof for this follows on similar lines as discussed earlier.
Here also we encounter three possibilities:
a) When $p_r > \tilde{p}_4$, $U_r \in IV$ and the interaction result is $R_2S_2 \to R_1R_2$; this means that R_2 is strong compared to the elementary wave S_2, and the strength of reflected R_1 is small compared to the incident waves R_2 and S_2.
b) When $p_r = \tilde{p}_4$, $U_r \in R_1(U_l)$ and the interaction result is $R_2S_2 \to R_1$.
c) When $p_r < \tilde{p}_4$, $U_r \in III$ and the interaction result is $R_2S_2 \to R_1S_2$, implying thereby that the elementary wave S_2 is strong compared to R_2.

(v) 1-shock overtakes another 1-shock (S_1S_1)**.** We consider the situation in which U_l is connected to U_* by a 1-shock of first Riemann problem and U_* is connected to U_r by a 1-shock of second Riemann problem. In other words, for a given left state U_l, the intermediate state U_* and the right state U_r are chosen such that $p_l < p_*$ and $u_* = u_l - (p_* - p_l)\sqrt{\frac{2}{p_*+p_l}}$ with Lax stability conditions

$$\sigma_1(U_l, U_*) < \lambda_1(U_l), \quad \lambda_1(U_*) < \sigma_1(U_l, U_*) < \lambda_2(U_*), \tag{5.9}$$

and $p_* < p_r$, $u_r = u_* - (p_r - p_*)\sqrt{\frac{2}{p_*+p_r}}$ with Lax stability conditions

$$\sigma_1(U_*, U_r) < \lambda_1(U_*), \quad \lambda_1(U_r) < \sigma_1(U_*, U_r) < \lambda_2(U_r), \tag{5.10}$$

where $\sigma_1(U_l, U_*)$ is the speed of shock connecting U_l to U_*, and similarly $\sigma_1(U_*, U_r)$ is the speed of shock connecting U_* to U_r. From (5.9) and (5.10) we obtain $\sigma_1(U_*, U_r) < \sigma_1(U_l, U_*)$, i.e., the 1-shock of second Riemann problem overtakes 1-shock of the first Riemann problem after a finite time, and gives rise to a new Riemann problem with data U_l and U_r. In order to solve this problem, we must determine the region in which U_r lies with respect to U_l. We claim that U_r lies in region II so that the solution of the new Riemann problem consists of S_1 and S_2. In other words, to prove our claim, we need to show that $S_1(U_*)$ lies entirely in the region II; to show this we are required to prove that for $p > p_* > p_l$,

$$(p - p_*)\sqrt{\frac{2}{p + p_*}} - (p - p_l)\sqrt{\frac{2}{p + p_l}} - (p_* - p_l)\sqrt{\frac{2}{p_* + p_l}} > 0. \tag{5.11}$$

Let us define a new function $\phi_9(p) = (p-p_*)\sqrt{\frac{2}{p+p_*}} - (p-p_l)\sqrt{\frac{2}{p+p_l}} - (p_*-p_l)\sqrt{\frac{2}{p_*+p_l}}$, so that $\phi_9(p_*) = 0$, and

$$\frac{d\phi_9}{dp} = \sqrt{\frac{2}{p + p_*}} - \frac{p - p_*}{\sqrt{2}(p + p_*)^{\frac{3}{2}}} - (\sqrt{\frac{2}{p + p_l}} - \frac{p - p_l}{\sqrt{2}(p + p_l)^{\frac{3}{2}}}). \tag{5.12}$$

Now we define

$$\phi_{10}(p, p_*) = \sqrt{\frac{2}{p + p_*}} - \frac{(p - p_*)}{\sqrt{2}(p + p_*)^{\frac{3}{2}}}. \tag{5.13}$$

Since $\phi_{10}(p, p_*)$ is an increasing function with respect to second variable p_* for $p > p_*$, we have $\phi_{10}(p, p_*) > \phi_{10}(p, p_l)$, it follows from (5.12) that $\phi_9(p)$ is a increasing function of p, which implies that $\phi_9(p) > \phi_9(p_*) = 0$. Hence, $S_1S_1 \to S_1S_2$.

(vi) 2-shock overtakes another 2-shock (S_2S_2)**.** The analytical proof that U_r lies in the region II, so that $S_2S_2 \to S_1S_2$, is similar to the previous case.

REFERENCES

[1] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations,* Comm. Pure. Appl. Math., 18 (1963), pp. 697–715.

[2] P. D. Lax, *Hyperbolic systems of conservation laws,* II, Comm. Pure. Appl. Math., 10 (1957), pp. 537–566.

[3] J. Smoller, *On the solution of the Riemann problem with general step data for an extended class of hyperbolic systems*, Michigan Math. J., 16 (1969), pp. 201–210.

[4] J. Smoller, *Shock waves and reaction diffusion equations*, Springer Verlag, Second edition, 1994.
[5] S. K. Godunov, *Numerical solution of multidimensional problems in gasdynamics,* Nauka Press, Moscow, 1976.
[6] A. J. Chorin, *Random choice solutions of hyperbolic systems,* J. Comp. Phys., 22 (1976), pp. 517–533.
[7] J. Smoller and B. Temple, *Global solutions of the relativistic Euler equations,* Comm. Math. Phys., 156 (1993), pp. 67–99.
[8] E. F. Toro, *A fast Riemann solver with constant covolume applied to the random choice method*, Int. J. Num. Methods Fluids, 9:4 (1989), pp. 1145–1164.
[9] E. F. Toro, *Riemann solvers for numerical methods for fluid dynamics,* Springer-Verlag, Berlin, 1999.
[10] S. Baskar and P. Prasad, *Riemann problem for kinematical conservation laws and geometrical features of nonlinear wavefronts,* IMA J. Appl. Math., 69:4 (2004), pp. 391–419.
[11] T. Raja Sekhar and V.D. Sharma, *Interaction of shallow water waves*, Stud. Appl. Math, 121:1 (2008), pp. 1–25.
[12] T. Raja Sekhar and V.D. Sharma, *Riemann problem and elementary wave interactions in isentropic magnetogasdynamics*, Nonlinear Anal. Real World Appl., 11:2 (2010), pp. 619–636.
[13] C. Shen, *Wave interactions and stability of the Riemann solutions for the chromatography equations*, J. Math. Anal. Appl., 365:2 (2010), pp. 609–618.
[14] N. Andrianov and G. Warnecke, *On the solution to the Riemann problem for the compressible duct flow,* SIAM J. Appl. Math., 64:3 (2004), pp. 878–901.
[15] N. Andrianov and G. Warnecke, *The Riemann problem for the Baer Nunziato model of two-phase flows,* J. Comput. Phys., 195:2 (2004), pp. 434–464.
[16] T. Chang and L. Hsiao, *The Riemann problem and interaction of waves in gas dynamics*, Longman, Harlow, 1989.
[17] G. Wang and W. Sheng, *Interaction of elementary waves of scalar conservation laws with discontinuous flux function,* J. Shanghai Univ., 10:5 (2006), pp. 381–387.
[18] T. P. Liu, *Admissible solutions of hyperbolic conservation laws*, Mem. Amer. Math. Soc. 30:240 (1981), pp. 1–78.
[19] M. Slemrod and A.E. Tzavaras, *A limiting viscosity approach for the Riemann problem in isentropic gas dynamics*, Indiana Univ. Math. J., 38:4 (1989), pp. 1047–1074.
[20] E. Godlewski and P. A. Raviart, *Numerical approximation of hyperbolic systems of conservation laws,* Springer-Verlag, New York, 1996.
[21] Ta-tsien Li, *Global classical solutions for quasilinear hyperbolic systems,* Research in Applied Mathematics, Masson-Wiley, Paris-Chichester, 1994.
[22] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Springer-Verlag, New York, 2000.
[23] A. Bressan, *Hyperbolic systems of conservation laws*, Oxford Univ. Press, Oxford, 2000.
[24] P. G. LeFloch, *Hyperbolic systems of conservation laws: The theory of classical and nonclassical shock waves,* Lectures in Mathematics, ETH Zurich, Birkhäuser, 2002.
[25] R. J. LeVeque, *Finite volume methods for hyperbolic problems*, Cambridge University Press, Cambridge, UK 2002.
[26] Y. Zheng, *System of conservation laws: Two dimensional Riemann problems*, Boston: Birkhauser, 2001.
[27] J. Li, T. Zhang and S. Yang, *Two-dimensional Riemann problem in gas dynamics,* Essex: Addison Wesley Longman, 1998.
[28] T. Zhang, H. Yang and Y. He, *Interaction between two rarefaction waves for the pressure-gradient equations in the gas dynamics*, Appl. Math. Comput., 199:1 (2008), pp. 231–241.