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VERBLUNSKY PARAMETERS AND LINEAR SPECTRAL TRANSFORMATIONS*

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Abstract. In this paper we analyze the behavior of Verblunsky parameters for Hermitian linear functionals deduced from canonical linear spectral transformations of a quasi-definite Hermitian linear functional. Some illustrative examples are studied.

Key words. Quasi-definite Hermitian linear functionals, orthogonal polynomials, Christoffel transformation, Uvarov transformation, Geronimus transformation, Verblunsky parameters.

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1. Introduction and preliminary results.

Spectral transformations appear in the literature related to bispectral problems, self-similar reductions, factorization of matrices, and Darboux transforms on Jacobi matrices (see [1], [2], [14], [20], [23]). They are connected with perturbations of linear functionals in the linear space of polynomials with complex coefficients, from the point of view of Jacobi matrices as a representation of the multiplication operator in terms of orthogonal polynomial bases, LU factorizations of such matrices, and rational bilinear transformations of Stieltjes functions.

The extension to other contexts has been started in [3] where polynomial perturbations of bilinear functionals have been considered. In such a situation, the representation of the multiplication operator with respect to an orthogonal polynomial basis is a Hessenberg matrix.

In the case of Hermitian linear functionals defined by probability measures supported on the unit circle, some linear spectral transforms have been introduced in the literature. In particular, polynomial and rational perturbations have been considered in [8], [9], [11], and [13] where explicit expressions for polynomials orthogonal with respect to the perturbed measure have been obtained in terms of the orthogonal polynomials with respect to the initial probability measure.

Later on, following the matrix approach to the canonical transformations associated with the spectral measures of Jacobi matrices (see [2] and [23] for the case of Christoffel, Uvarov and Geronimus transformations), as well as the analysis of the generators of the group of linear and rational spectral transformations done in [24], some canonical perturbations of probability measures supported on the unit circle have been analyzed in [4], [5], [15]. More precisely, these contributions focused the attention on the connection between the Hessenberg matrices associated with a probability measure and the perturbed linear functional, respectively, in terms of their QR factorization instead of the LU factorization. Here, and taking into account that the Hessenberg matrix is the representation of the multiplication operator in terms of an orthogonal basis, the main tool is based on the relation between the corresponding orthogonal bases.

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In order to have a self contained presentation, we will introduce the basic background concerning orthogonal polynomials on the unit circle (OPUC).

Suppose $\mathcal{L}$ is a linear functional in the linear space Λ of the Laurent polynomials with complex coefficients $\left(\Lambda = span\,\{z^n\}_{n\in\mathbb{Z}}\right)$ such that $\mathcal{L}$ is Hermitian, i. e. $c_n = \langle \mathcal{L}, z^n\rangle = \overline{\langle \mathcal{L}, z^{-n}\rangle} = \bar{c}_{-n}$, $n \in \mathbb{Z}$. Then, in the linear space $\mathbb{P}$ of polynomials with complex coefficients, a bilinear functional associated with $\mathcal{L}$ can be introduced as follows (see [7], [12])

$$\langle p(z), q(z)\rangle_{\mathcal{L}} = \left\langle \mathcal{L}, p(z)\bar{q}(z^{-1})\right\rangle \tag{1}$$

where $p, q \in \mathbb{P}$.

In terms of the canonical basis $\{z^n\}_{n\geqslant 0}$ of $\mathbb{P}$, the Gram matrix associated with this bilinear functional is

$$\mathbf{T} = \begin{bmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}, \tag{2}$$

i.e., a Toeplitz matrix [10].

The linear functional $\mathcal{L}$ is said to be quasi-definite if the principal leading submatrices of $\mathbf{T}$ are non-singular. If such submatrices have a positive determinant, then the linear functional is said to be positive definite. Every positive definite linear functional $\mathcal{L}$ has an integral representation

$$\langle \mathcal{L}, p(z)\rangle = \int_{\mathbb{T}} p(z)d\sigma(z), \tag{3}$$

where σ is a nontrivial probability Borel measure supported on the unit circle (see [7], [10], [12], [19]), assuming $c_0 = 1$.

The analytic function defined by

$$F(z) = c_0 + 2\sum_{n=1}^{\infty} c_{-n}z^n \tag{4}$$

is called the Carathéodory function associated with $\mathcal{L}$. If $\mathcal{L}$ is positive definite, then $F(z)$ is analytic on the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ and $\mathfrak{Re}F(z) > 0$ for every $z \in \mathbb{D}$. Furthermore, $F(z)$ has the integral representation

$$F(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} d\sigma(w).$$

If $\mathcal{L}$ is a quasi-definite linear functional then a unique sequence of monic polynomials $\{\Phi_n\}_{n\geqslant 0}$ such that

$$\langle \Phi_n, \Phi_m\rangle_{\mathcal{L}} = \mathbf{k}_n \delta_{n,m}, \tag{5}$$

can be introduced, where $\mathbf{k}_n \neq 0$ for every $n \geqslant 0$. It is said to be the monic orthogonal polynomial sequence associated with $\mathcal{L}$.

Let σ be a non trivial probability measure supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then there exists a sequence $\{\varphi_n\}_{n\geqslant 0}$ of orthonormal polynomials

$$\varphi_n(z) = \kappa_n z^n + ..., \quad \kappa_n > 0,$$

such that

$$\int_{-\pi}^{\pi} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}d\sigma(\theta) = \delta_{m,n}, \quad m, n \geqslant 0. \tag{6}$$

The corresponding monic polynomials are then defined by

$$\Phi_n(z) = \frac{\varphi_n(z)}{\kappa_n}.$$

These polynomials satisfy the following recurrence relations (see [7], [10], [19], [21])

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n = 0, 1, 2, ... \tag{7}$$

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) + \overline{\Phi_{n+1}(0)}z\Phi_n(z), \quad n = 0, 1, 2, ... \tag{8}$$

Here $\Phi_n^*(z) = z^n\overline{\Phi}_n(1/z)$ is the reversed polynomial associated with $\Phi_n(z)$ (see [19]), and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$ are called reflection (or Verblunsky) parameters. Notice that $|\Phi_n(0)| < 1$ for every $n \geqslant 1$. For quasi-definite linear functionals, the Verblunsky parameters satisfy $|\Phi_n(0)| \neq 1$, $n \geqslant 1$ (see [7]).

It is well known ([19]) that given a nontrivial probability measure σ supported on the unit circle, there exists a unique sequence of Verblunsky parameters $\{\Phi_n(0)\}_{n\geqslant 1}$ associated with σ. The converse is also true, i.e., given a sequence of complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$, with $\Phi_n(0) \in \mathbb{D}$, there exists a nontrivial probability measure on the unit circle such that those numbers are the associated Verblunsky parameters.

The family of Verblunsky parameters provides a qualitative information about the measure and the corresponding sequence of orthogonal polynomials.

The measure σ can be uniquely decomposed into

$$d\sigma(\theta) = \omega(\theta)\frac{d\theta}{2\pi} + d\sigma_s(\theta).$$

Definition 1 ([19], [21]). *Suppose the Szegő condition,*

$$\int_{\mathbb{T}} \log(\omega(\theta))\frac{d\theta}{2\pi} > -\infty, \tag{9}$$

holds. Then, the Szegő function, D(z), is defined by

$$D(z) = \exp\left(\frac{1}{4\pi}\int \frac{e^{i\theta} + z}{e^{i\theta} - z}\log(\omega(\theta))d\theta\right). \tag{10}$$

The Szegő condition (9) is equivalent to $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$. This is known in the literature as the Szegő theorem (see [19]).

On the other hand, the sequence of Verblunsky parameters $\{\Phi_n(0)\}_{n\geqslant 1}$ is said to be of bounded variation if

$$\sum_{n=0}^{\infty} |\Phi_{n+1}(0) - \Phi_n(0)| < \infty$$

holds.

Let $K_n(z,y)$ be the *n-th reproducing kernel polynomial* associated with $\{\varphi_n\}_{n\geqslant 0}$, defined by

$$K_n(z,y) = \sum_{j=0}^{n} \overline{\varphi_j(y)}\varphi_j(z) = \sum_{j=0}^{n} \frac{\overline{\Phi_j(y)}\Phi_j(z)}{\mathbf{k}_j},$$

with $\mathbf{k}_j = \|\Phi_j\|^2 = (\kappa_j(\sigma))^{-2}$. Notice that the last expression is also valid in the quasi-definite case. There is a direct formula to compute $K_n(z,y)$,

THEOREM 2 (Christoffel-Darboux Formula). *For any $n \geqslant 0$ and $z, y \in \mathbb{C}$ with $\bar{y}z \neq 1$,*

$$K_n(z,y) = \sum_{j=0}^{n} \frac{\overline{\Phi_j(y)}\Phi_j(z)}{\mathbf{k}_j} = \frac{\overline{\Phi_{n+1}^*(y)}\Phi_{n+1}^*(z) - \overline{\Phi_{n+1}(y)}\Phi_{n+1}(z)}{\mathbf{k}_{n+1}(1-\bar{y}z)},$$

$$= \frac{\overline{\Phi_n^*(y)}\Phi_n^*(z) - \bar{y}z\overline{\Phi_n(y)}\Phi_n(z)}{\mathbf{k}_n(1-\bar{y}z)}.$$

For a class of perturbations of the measure σ, some properties of the perturbed measure $\tilde{\sigma}$ have been studied in ([4], [8], [14], [16]). In particular, the relation between the corresponding families of orthogonal polynomials as well as necessary and sufficient conditions for the quasi-definite (positive definite) character of the new measure $\tilde{\sigma}$, assuming the quasi-definite (positive definite) character of σ, among others. We point out the following canonical cases.

(*i*) If $d\tilde{\sigma} = |z-\alpha|^2 d\sigma$, $|z| = 1$, $\alpha \in \mathbb{C}$, then the so-called canonical Christoffel transformation appears.

(*ii*) If $d\tilde{\sigma} = d\sigma + \boldsymbol{m}\delta(z-\alpha)$, $|\alpha| = 1$, $\boldsymbol{m} \in \mathbb{R}$, then the so-called canonical Uvarov transformation appears.

(*iii*) If $d\tilde{\sigma} = d\sigma + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1})$, $\boldsymbol{m} \in \mathbb{C}$, $|\alpha| < 1$, then a more general case of the Uvarov transformation appears.

(*iv*) If $d\tilde{\sigma} = \frac{1}{|z-\alpha|^2} d\sigma + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1})$, $|z| = 1$, $\boldsymbol{m} \in \mathbb{C}$ and $|\alpha| < 1$, then we get the Geronimus transformation.

Notice that the linear functional associated with $\tilde{\sigma}$ can be normalized ($\tilde{c}_0 = 1$) if it is quasi-definite. In terms of the Caratéodory functions, all the above perturbations correspond to transformations of the form

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{D(z)},$$

where $A(z)$, $B(z)$, and $D(z)$ are polynomials, and $\widetilde{F}(z)$ is the Carathéodory function associated with the perturbed measure $\tilde{\sigma}$. Hence, these perturbations yield linear spectral transformations in the corresponding Carathéodory functions.

In this work, we analyze these transformations from the point of view of the behavior of the families of Verblunsky parameters. We get explicit expressions for the Verblunsky parameters associated with $\tilde{\sigma}$ in terms of the parameters associated with σ. We also study if the measure $\tilde{\sigma}$ is of bounded variation, provided that σ is.

The structure of the manuscript is as follows. In section 2 we analyze the behavior of the Verblunsky parameters when a Christoffel canonical transform of a probability measure supported on the unit circle is considered. Section 3 is focussed on the Uvarov transformation in a more general framework than one analyzed in [22]. Two particular examples of the Uvarov transformation are studied in Section 4 in order to analyze how this perturbation in the probability measure is reflected on the behavior of their Verblunsky parameters.

2. The Christoffel transformation. Let α be a complex number. Consider the Hermitian bilinear functional

$$\langle p,q\rangle_{\mathcal{L}_C} := \langle (z-\alpha)p,(z-\alpha)q\rangle_{\mathcal{L}}\,, \quad p,q\in\mathbb{P}. \tag{11}$$

If $\mathcal{L}$ is quasi-definite, then necessary and sufficient conditions for $\mathcal{L}_C$ to be quasi-definite have been studied in [16].

Proposition 3. *[14],[16]*

(*i*) *$\mathcal{L}_C$ is quasi-definite if and only if $K_n(\alpha,\alpha)\neq 0$ for every $n\in\mathbb{N}$.*

(*ii*) *If $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ denotes the sequence of monic orthogonal polynomials with respect to $\mathcal{L}_C$, then*

$$\widetilde{\Phi}_n(z) = \frac{1}{z-\alpha}\left(\Phi_{n+1}(z) - \frac{\Phi_{n+1}(\alpha)}{K_n(\alpha,\alpha)}K_n(z,\alpha)\right). \tag{12}$$

$\mathcal{L}_C$ is said to be the canonical Christoffel transformation of the linear functional $\mathcal{L}$.

Proposition 4. *Let $\{\Phi_n(0)\}_{n\geqslant 1}$ be the Verblunsky parameters corresponding to the quasi-definite linear functional $\mathcal{L}$. Then, the Verblunsky parameters associated with $\mathcal{L}_C$ are given by*

$$\widetilde{\Phi_n}(0) = \frac{\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}}{\alpha\mathbf{k}_n K_n(\alpha,\alpha)} - \frac{\Phi_{n+1}(0)}{\alpha}, \quad n\geqslant 1. \tag{13}$$

Proof. From (12), the evaluation in $z=0$ yields

$$\widetilde{\Phi}_n(0) = -\alpha^{-1}\left(\Phi_{n+1}(0) - \frac{\Phi_{n+1}(\alpha)}{K_n(\alpha,\alpha)}\sum_{j=0}^{n}\frac{\Phi_j(0)\overline{\Phi_j(\alpha)}}{\mathbf{k}_j}\right).$$

Applying the Christoffel-Darboux formula, we get

$$\widetilde{\Phi}_n(0) = -\alpha^{-1}\left(\Phi_{n+1}(0) - \frac{\Phi_{n+1}(\alpha)}{K_n(\alpha,\alpha)}\frac{\Phi_n^*(0)\overline{\Phi_n^*(\alpha)}}{\mathbf{k}_n}\right) \tag{14}$$

$$= \frac{\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}}{\alpha\mathbf{k}_n K_n(\alpha,\alpha)} - \frac{\Phi_{n+1}(0)}{\alpha} \tag{15}$$

since $\Phi_n^*(0) = 1$. ☐ Another way to express (15) is

$$\begin{aligned}\widetilde{\Phi}_n(0) &= \frac{[\alpha\Phi_n(\alpha) + \Phi_{n+1}(0)\Phi_n^*(\alpha)]\overline{\Phi_n^*(\alpha)}}{\alpha\mathbf{k}_n K_n(\alpha,\alpha)} - \frac{\Phi_{n+1}(0)}{\alpha},\\ &= \left[\frac{|\Phi_n^*(\alpha)|^2}{\mathbf{k}_n K_n(\alpha,\alpha)} - 1\right]\frac{\Phi_{n+1}(0)}{\alpha} + \frac{\Phi_n(\alpha)\Phi_n^*(\alpha)}{\mathbf{k}_n K_n(\alpha,\alpha)},\end{aligned}$$

i.e., there is a linear relation between both families of Verblunsky parameters.

Notice that, if $|\alpha| \neq 1$, from the Christoffel-Darboux formula we deduce

$$K_n(\alpha,\alpha) = \frac{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}{\mathbf{k}_n(1-|\alpha|^2)},$$

and the expression for the Verblunsky parameters $\{\widetilde{\Phi_n}(0)\}_{n\geqslant 1}$ in terms of $\Phi_n(\alpha)$ and $\Phi_n^*(\alpha)$ is therefore given by

$$\begin{aligned}\widetilde{\Phi}_n(0) &= \frac{\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}(1-|\alpha|^2)}{\alpha[|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2]} - \frac{\Phi_{n+1}(0)}{\alpha},\\ &= \frac{[\alpha\Phi_n(\alpha) + \Phi_{n+1}(0)\Phi_n^*(\alpha)]\overline{\Phi_n^*(\alpha)}(1-|\alpha|^2)}{\alpha[|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2]} - \frac{\Phi_{n+1}(0)}{\alpha},\\ &= \frac{1}{\alpha}\frac{(\alpha\Phi_n(\alpha)\overline{\Phi_n^*(\alpha)} + \Phi_{n+1}(0)|\Phi_n^*(\alpha)|^2)(1-|\alpha|^2)}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}\\ &- \frac{1}{\alpha}\frac{\Phi_{n+1}(0)|\Phi_n^*(\alpha)|^2 + |\alpha|^2\Phi_{n+1}(0)|\Phi_n(\alpha)|^2}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}.\end{aligned}$$

Thus,

$$\widetilde{\Phi}_n(0) = \frac{\Phi_n(\alpha)\overline{\Phi_n^*(\alpha)}(1-|\alpha|^2) + \bar{\alpha}\Phi_{n+1}(0)[|\Phi_n(\alpha)|^2 - |\Phi_n^*(\alpha)|^2]}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2},$$

As a conclusion, $\widetilde{\Phi}_n(0)$ can be expressed as

$$\widetilde{\Phi}_n(0) = A(\alpha;n)\Phi_{n+1}(0) + B(\alpha;n),$$

with

$$\begin{aligned}A(\alpha;n) &= \frac{\bar{\alpha}[|\Phi_n(\alpha)|^2 - |\Phi_n^*(\alpha)|^2]}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2},\\ B(\alpha;n) &= \frac{\Phi_n(\alpha)\overline{\Phi_n^*(\alpha)}(1-|\alpha|^2)}{|\Phi_n^*(\alpha)|^2 - |\alpha|^2|\Phi_n(\alpha)|^2}.\end{aligned}$$

On the other hand, if $|\alpha| = 1$, we have

$$\begin{aligned}K_n(z,\alpha) &= \frac{\overline{\Phi_{n+1}^*(\alpha)}\Phi_{n+1}^*(z) - \overline{\Phi_{n+1}(\alpha)}\Phi_{n+1}(z)}{\mathbf{k}_{n+1}(1-\bar{\alpha}z)}\\ &= \frac{\alpha\overline{\Phi_{n+1}(\alpha)}\Phi_{n+1}(z) - \bar{\alpha}^n\Phi_{n+1}(\alpha)\Phi_{n+1}^*(z)}{\mathbf{k}_{n+1}(z-\alpha)},\end{aligned}$$

and applying the L'Hospital's rule, we obtain

$$K_n(\alpha,\alpha)=\lim_{z\to\alpha}K_n(z,\alpha)=\frac{1}{\mathbf{k}_{n+1}}[\alpha\overline{\Phi_{n+1}(\alpha)}\Phi^{'}_{n+1}(z)-\bar{\alpha}^n\Phi_{n+1}(\alpha)\Phi^{*'}_{n+1}(z)],$$
$$=\frac{1}{\mathbf{k}_{n+1}}[\alpha\overline{\Phi_{n+1}(\alpha)}\Phi^{'}_{n+1}(\alpha)-\bar{\alpha}^n\Phi_{n+1}(\alpha)\Phi^{*'}_{n+1}(\alpha)].$$

Therefore

$$\widetilde{\Phi_n}(0)=\frac{\mathbf{k}_{n+1}\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}}{\alpha\mathbf{k}_n[\alpha\overline{\Phi_{n+1}(\alpha)}\Phi^{'}_{n+1}(\alpha)-\bar{\alpha}^n\Phi_{n+1}(\alpha)\Phi^{*'}_{n+1}(\alpha)]}-\frac{\Phi_{n+1}(0)}{\alpha},$$
$$=\frac{\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}(1-|\Phi_{n+1}(0)|^2)}{\alpha[\alpha\overline{\Phi_{n+1}(\alpha)}\Phi^{'}_{n+1}(\alpha)-\bar{\alpha}^n\Phi_{n+1}(\alpha)\Phi^{*'}_{n+1}(\alpha)]}-\frac{\Phi_{n+1}(0)}{\alpha},$$

with $\Phi^{*'}_{n+1}(\alpha)=\alpha^{-1}[(n+1)\Phi^*_{n+1}(\alpha)-(\Phi^{'}_{n+1})^*(\alpha)]$.

For the remaining of this section, we assume that $\mathcal{L}$ is a positive definite linear functional. Then,

Theorem 5 ([17], [18], [22]). *Suppose* $\sum_{n=0}^{\infty}|\Phi_n(0)|^2<\infty$ *and* $\sum_{n=0}^{\infty}|\Phi_{n+1}(0)-\Phi_n(0)|<\infty$. *Then, for any* $\delta>0$,

$$\sup_{n;\delta<\arg(z)<2\pi-\delta}|\Phi_n^*(z)|<\infty$$

and away from $z=1$, *we have that* $\lim_{n\to\infty}\Phi_n^*(z)$ *exists, is continuous, and equal to* $D(0)D(z)^{-1}$. *Furthermore,* $d\mu_s=0$ *or else a pure mass point at* $z=1$.

Proposition 6. *Suppose* $\sum_{n=0}^{\infty}|\Phi_n(0)|^2<\infty$ *and* $\sum_{n=0}^{\infty}|\Phi_{n+1}(0)-\Phi_n(0)|<\infty$. *Then, for* $|\alpha|\leqslant 1$, $\alpha\neq 1$,

(*i*) $\sum_{n=0}^{\infty}|\widetilde{\Phi}_n(0)|^2<\infty$.

(*ii*) $\sum_{n=0}^{\infty}|\widetilde{\Phi}_{n+1}(0)-\widetilde{\Phi}_n(0)|<\infty$.

Proof.

(*i*) We denote

$$t_{n+1}=\frac{\Phi_{n+1}(\alpha)\overline{\Phi_n^*(\alpha)}}{\alpha\mathbf{k}_nK_n(\alpha,\alpha)}.$$

Let us first assume $|\alpha|=1$, $\alpha\neq 1$. Notice that $\overline{\Phi_{n+1}(\alpha)}=\overline{\alpha^{n+1}}\Phi^*_{n+1}(\alpha)$ and, from Theorem 5, $\lim_{n\to\infty}\Phi_n^*(\alpha)=D(0)D(\alpha)^{-1}$, where D is the Szegő function defined in (10). This also implies that $1/K_n(\alpha,\alpha)=O(1/n)$. On the other hand, if $|\alpha|<1$, notice that $\Phi_n(\alpha)$ and $\Phi_n^*(\alpha)$ are $O(\alpha^n)$ and $\lim_{n\to\infty}(1/K_n(\alpha,\alpha))>0$.

Then t_{n+1} is $O(1/n)$. Since $\sum_{n=0}^{\infty}|\Phi_n(0)|^2<\infty$ and t_{n+1} is $O(1/n)$, then $\sum_{n=0}^{\infty}|\widetilde{\Phi}_n(0)|^2<\infty$.

(*ii*) Since $\sum_{n=0}^{\infty}|\Phi_{n+1}(0)-\Phi_n(0)|<\infty$, we only need to prove that

$$\sum_{n=0}^{\infty}|t_{n+1}-t_n|<\infty.$$

Notice that, from the recurrence relation

$$\Phi_{n+1}^*(\alpha) - \Phi_n^*(\alpha) = \overline{\Phi_{n+1}(0)}\alpha\Phi_n(\alpha).$$

Then $|\Phi_{n+1}^*(\alpha) - \Phi_n^*(\alpha)| = O(|\Phi_{n+1}(0)|)$ and therefore

$$\left|\frac{(\Phi_{n+1}^*(\alpha) - \Phi_n^*(\alpha))\Phi_{n+1}^*(\alpha)}{\mathbf{k}_n K_n(\alpha,\alpha)}\right| = O\left(\frac{|\Phi_{n+1}(0)|}{n}\right). \tag{16}$$

On the other hand,

$$\left|\left(\frac{1}{K_{n+1}(\alpha,\alpha)} - \frac{1}{K_n(\alpha,\alpha)}\right)\frac{\Phi_n(\alpha)\overline{\Phi_{n-1}^*(\alpha)}}{\mathbf{k}_n}\right| = O\left(\frac{1}{n^2}\right). \tag{17}$$

Thus, from (16) and (17) we get

$$|t_{n+1} - t_n| = O\left(\frac{|\Phi_{n+1}(0)|}{n}\right) + O\left(\frac{1}{n^2}\right)$$

and, therefore,

$$\sum_{n=0}^{\infty} |t_{n+1} - t_n| < \infty.$$

□

Remark 7. *This approach was used by M. L. Wong in [22] to prove a similar result for the Uvarov transformation. See more details in the following section.*

3. The Uvarov transformation. In this section we are dealing with the two canonical Uvarov transformations defined in the introduction.

3.1. The bilinear functional $\langle p,q\rangle_{\mathcal{L}_U} := \langle p,q\rangle_{\mathcal{L}} + \boldsymbol{m}p(\alpha)\overline{q(\alpha)}$, $|\alpha| = 1$, $\boldsymbol{m} \in \mathbb{R}$. Now consider the bilinear functional

$$\langle p,q\rangle_{\mathcal{L}_U} := \langle p,q\rangle_{\mathcal{L}} + \boldsymbol{m}p(\alpha)\overline{q(\alpha)}, \quad p,q \in \mathbb{P}, \tag{18}$$

with $\boldsymbol{m} \in \mathbb{R}$ and $|\alpha| = 1$. Notice that $\mathcal{L}_U$ is an Hermitian linear functional. If α is fixed, then we get necessary and sufficient conditions about the choices of $\boldsymbol{m} \in \mathbb{R}$ such that the linear functional $\mathcal{L}_U$ is quasi-definite. Indeed,

Proposition 8 ([14]).

(i) *$\mathcal{L}_U$ is quasi-definite if and only if $1 + \boldsymbol{m}K_{n-1}(\alpha,\alpha) \neq 0$ for every $n \geqslant 1$.*

(ii) *If $\{U_n\}_{n\geqslant 0}$ denotes the sequence of monic orthogonal polynomials with respect to $\mathcal{L}_U$, then*

$$U_n(z) = \Phi_n(z) - \frac{\boldsymbol{m}\Phi_n(\alpha)}{1 + \boldsymbol{m}K_{n-1}(\alpha,\alpha)}K_{n-1}(z,\alpha). \tag{19}$$

In other words, for a fixed $\alpha, |\alpha| = 1$, the linear functional $\mathcal{L}_U$ is quasi-definite up to a numerable set of values of $\boldsymbol{m}$.

Thus, as a consequence, we get

PROPOSITION 9. *Let $\{\Phi_n(0)\}_{n\geqslant 1}$ be the Verblunsky parameters with respect to $\mathcal{L}$. Then, the Verblunsky parameters associated with $\mathcal{L}_U$ are given by*

$$U_n(0) = \Phi_n(0) - \frac{\boldsymbol{m}\Phi_n(\alpha)\overline{\Phi^*_{n-1}(\alpha)}}{\mathbf{k}_{n-1}(1+\boldsymbol{m}K_{n-1}(\alpha,\alpha))} \tag{20}$$

Proof. From (19), evaluating at $z=0$, we obtain

$$U_n(0) = \Phi_n(0) - \frac{\boldsymbol{m}\Phi_n(\alpha)}{1+\boldsymbol{m}K_{n-1}(\alpha,\alpha)}K_{n-1}(0,\alpha), \tag{21}$$

$$= \Phi_n(0) - \frac{\boldsymbol{m}\Phi_n(\alpha)}{1+\boldsymbol{m}K_{n-1}(\alpha,\alpha)}\frac{\overline{\Phi^*_{n-1}(\alpha)}\Phi^*_{n-1}(0)}{\mathbf{k}_{n-1}}, \tag{22}$$

$$= \Phi_n(0) - \frac{\boldsymbol{m}\Phi_n(\alpha)\overline{\Phi^*_{n-1}(\alpha)}}{\mathbf{k}_{n-1}(1+\boldsymbol{m}K_{n-1}(\alpha,\alpha))}. \tag{23}$$

□

REMARK 10. *If $\mathcal{L}$ is a positive definite linear functional and using a formula for the Verblunsky parameters associated with $\tilde{\sigma}$ first given by Geronimus ([6]) and also by Simon (see [19] and the references therein), then this result was also proved in [22], as follows,*

THEOREM 11 ([22]). *Suppose σ is a nontrivial probability measure on the unit circle and $0<\gamma<1$. Let $\tilde{\sigma}$ be the nontrivial probability measure resulting of the addition of a mass point $\zeta = e^{i\theta}\in\mathbb{T}$ to σ as follows*

$$d\tilde{\sigma} = (1-\gamma)d\sigma + \gamma\delta_\theta.$$

Then the Verblunsky parameters associated with $\tilde{\sigma}$ are

$$U_n(0) = \Phi_n(0) + \frac{(1-|\Phi_{n+1}(0)|^2)^{1/2}}{(1-\gamma)\gamma^{-1}+K_n(\zeta,\zeta)}\overline{\varphi_{n+1}(\zeta)}\varphi^*_n(\zeta). \tag{24}$$

Notice that in the above perturbation $\tilde{\sigma}$ is again a nontrivial probability measure. Furthermore, $\mathbf{k}_n/\mathbf{k}_{n-1} = 1-|\Phi_n(0)|^2$, so (20) is equivalent to the expression (24), which also appeared on [6]. There is also an analog of Proposition 6 for the Uvarov transformation on [22], which has been proved (see [7]) in a more general case when masses are added in m points of the unit circle.

Notice that (23) also reads

$$U_n(0) = \Phi_n(0) - \frac{\boldsymbol{m}[\alpha\Phi_{n-1}(\alpha)+\Phi_n(0)\Phi^*_{n-1}(\alpha)]\overline{\Phi^*_{n-1}(\alpha)}}{\mathbf{k}_{n-1}(1+\boldsymbol{m}K_{n-1}(\alpha,\alpha))},$$

or, in other words,

$$U_n(0) = A_U(\alpha;n)\Phi_n(0) + B_U(\alpha;n),$$

with

$$A_U(\alpha;n) = 1 - \frac{\boldsymbol{m}|\Phi^*_{n-1}(\alpha)|^2}{\mathbf{k}_{n-1}(1+\boldsymbol{m}K_{n-1}(\alpha,\alpha))},$$

$$B_U(\alpha;n) = -\frac{\boldsymbol{m}\alpha\Phi_{n-1}(\alpha)\overline{\Phi^*_{n-1}(\alpha)}}{\mathbf{k}_{n-1}(1+\boldsymbol{m}K_{n-1}(\alpha,\alpha))}.$$

3.2. The bilinear functional $\langle p,q\rangle_{\mathcal{L}_V} := \langle p,q\rangle_{\mathcal{L}} + \boldsymbol{m}p(\alpha)\overline{q(\bar\alpha^{-1})} + \bar{\boldsymbol{m}}p(\bar\alpha^{-1})\overline{q(\alpha)}$, $|\alpha|<1$, $\boldsymbol{m}\in\mathbb{C}$. Now let us consider the bilinear functional

$$\langle p,q\rangle_{\mathcal{L}_V} := \langle p,q\rangle_{\mathcal{L}} + \boldsymbol{m}p(\alpha)\overline{q(\bar\alpha^{-1})} + \bar{\boldsymbol{m}}p(\bar\alpha^{-1})\overline{q(\alpha)}, \quad p,q\in\mathbb{P}, \tag{25}$$

with $\boldsymbol{m}\in\mathbb{C}$ and $|\alpha|<1$. Notice that $\mathcal{L}_V$ is also hermitian.

PROPOSITION 12 ([4]). *Assume $\mathcal{L}$ is a quasi-definite functional and α, with $|\alpha|<1$, a fixed complex number. The linear functional $\mathcal{L}_V$ is quasi-definite if and only if $\boldsymbol{m}$ satisfies*

$$\Lambda_n := \begin{vmatrix} 1+\boldsymbol{m}K_n(\alpha,\bar\alpha^{-1}) & \bar{\boldsymbol{m}}K_n(\alpha,\alpha) \\ \boldsymbol{m}K_n(\bar\alpha^{-1},\bar\alpha^{-1}) & 1+\bar{\boldsymbol{m}}K_n(\bar\alpha^{-1},\alpha) \end{vmatrix} \neq 0,$$

for all $n\geqslant 0$. In other words, the linear functional $\mathcal{L}_V$ is quasi-definite for every $\boldsymbol{m}\in\mathbb{C}$ up to a numerable set.

Assuming the conditions of the above proposition we get

PROPOSITION 13 ([4]). *The orthogonal polynomial sequence corresponding to $\mathcal{L}_V$, $\{V_n(z)\}_{n\geqslant 0}$, is given by*

$$\begin{aligned} V_n(z) = \Phi_n(z) - \boldsymbol{m}[A_n\Phi_n(\alpha) + B_n\Phi_n(\bar\alpha^{-1})]K_{n-1}(z,\bar\alpha^{-1}) \\ - \bar{\boldsymbol{m}}[C_n\Phi_n(\alpha) + D_n\Phi_n(\bar\alpha^{-1})]K_{n-1}(z,\alpha), \end{aligned} \tag{26}$$

where

$$A_n = \frac{-[1+\bar{\boldsymbol{m}}K_{n-1}(\bar\alpha^{-1},\alpha)]}{\Lambda_{n-1}} \tag{27}$$

$$B_n = \frac{\bar{\boldsymbol{m}}K_{n-1}(\alpha,\alpha)}{\Lambda_{n-1}} \tag{28}$$

$$C_n = \frac{-\boldsymbol{m}K_{n-1}(\bar\alpha^{-1},\bar\alpha^{-1})}{\Lambda_{n-1}} \tag{29}$$

$$D_n = \frac{1+\boldsymbol{m}K_{n-1}(\alpha,\bar\alpha^{-1})}{\Lambda_{n-1}} \tag{30}$$

with $\Lambda_{n-1} = |\boldsymbol{m}|^2K_{n-1}(\bar\alpha^{-1},\bar\alpha^{-1})K_{n-1}(\alpha,\alpha) - |1+\boldsymbol{m}K_{n-1}(\alpha,\bar\alpha^{-1})|^2$.

Notice that $D_n = -\bar A_n$. Then, the Verblunsky parameters $\{V_n(0)\}_{n\geqslant 1}$ are

$$\begin{aligned} V_n(0) = \Phi_n(0) - \boldsymbol{m}[A_n\Phi_n(\alpha) + B_n\Phi_n(\bar\alpha^{-1})]\Phi^*_{n-1}(0)\overline{\Phi^*_{n-1}(\bar\alpha^{-1})}/\mathbf{k}_{n-1} \\ - \bar{\boldsymbol{m}}[C_n\Phi_n(\alpha) + D_n\Phi_n(\bar\alpha^{-1})]\Phi^*_{n-1}(0)\overline{\Phi^*_{n-1}(\alpha)}/\mathbf{k}_{n-1}. \end{aligned} \tag{31}$$

If $\mathcal{L}$ is a positive linear functional and assuming that $\sum_{n=0}^{\infty}|\Phi_n(0)|^2<\infty$, then we will study the behavior of $V_n(0)$ when $n\to\infty$. Since $|\alpha|<1$, $\lim_{n\to\infty}K_n(\alpha,\alpha)<\infty$ and $\lim_{n\to\infty}K_n(\bar\alpha^{-1},\bar\alpha^{-1})=\infty$ (see [19]). On the other hand, taking into account

$$\frac{K_n(\bar\alpha^{-1},\alpha)}{K_n(\bar\alpha^{-1},\bar\alpha^{-1})} = |\alpha|^{2n}\frac{\overline{K_n(\alpha,\bar\alpha^{-1})}}{K_n(\alpha,\alpha)},$$

we get

$$\lim_{n\to\infty}\frac{K_n(\bar\alpha^{-1},\alpha)}{K_n(\bar\alpha^{-1},\bar\alpha^{-1})} = \lim_{n\to\infty}|\alpha|^{2n}\frac{\overline{K_n(\alpha,\bar\alpha^{-1})}}{K_n(\alpha,\alpha)}.$$

Since $K_n(\alpha, \bar{\alpha}^{-1})$ is $O(\bar{\alpha}^{-n})$ and $|\alpha| < 1$, it follows that

$$\lim_{n\to\infty} \frac{K_n(\bar{\alpha}^{-1}, \alpha)}{K_n(\bar{\alpha}^{-1}, \bar{\alpha}^{-1})} = 0.$$

We obtain the same result for $\frac{K_{n-1}(\alpha,\bar{\alpha}^{-1})}{K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1})}$, since $K_n(\bar{\alpha}^{-1}, \alpha) = \overline{K_n(\alpha, \bar{\alpha}^{-1})}$.

Therefore, if we divide by $K_{n-1}(\bar{\alpha}^{-1}, \bar{\alpha}^{-1})$ in the numerator and denominator of A_n, and take the limit when $n \to \infty$, then the numerator becomes 0, and only $|\boldsymbol{m}|^2 K_{n-1}(\alpha, \alpha)$ survives on the denominator. Hence, $A_n = 0$ when $n \to \infty$.

The same fact occurs with B_n and D_n. In a similar way, we obtain that $C_n \sim -\frac{1}{\bar{\boldsymbol{m}} K_\infty(\alpha,\alpha)}$ as $n \to \infty$.

As a conclusion, when $n \to \infty$

$$V_n(0) \sim \Phi_n(0) + \frac{\Phi_n(\alpha)}{K_{n-1}(\alpha, \alpha)} \varphi^*_{n-1}(0)\overline{\varphi^*_{n-1}(\alpha)} \tag{32}$$

$$= \Phi_n(0) + \frac{\Phi_n(\alpha)}{\mathbf{k}_{n-1} K_{n-1}(\alpha, \alpha)} \Phi^*_{n-1}(0)\overline{\Phi^*_{n-1}(\alpha)} \tag{33}$$

$$= \Phi_n(0) + \frac{\Phi_n(\alpha)\overline{\Phi^*_{n-1}(\alpha)}}{\mathbf{k}_{n-1} K_{n-1}(\alpha, \alpha)}. \tag{34}$$

Notice than (34) has the same form as (13). Therefore,

Proposition 14. *Suppose* $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$ *and* $\sum_{n=0}^{\infty} |\Phi_{n+1}(0) - \Phi_n(0)| < \infty$. *Then,*

(i) $\sum_{n=0}^{\infty} |V_n(0)|^2 < \infty$.

(ii) $\sum_{n=0}^{\infty} |V_{n+1}(0) - V_n(0)| < \infty$.

4. Examples. In the next examples we will illustrate the behavior of the Verblunsky parameters when linear spectral transforms of nontrivial probability measures are considered.

4.1. First, we consider the Uvarov transformation (see Section 3.2) for the normalized Lebesgue measure, i.e. we study the measure $\tilde{\sigma}$ defined by

$$d\tilde{\sigma} = \frac{d\theta}{2\pi} + \boldsymbol{m}\delta(z - \alpha) + \bar{\boldsymbol{m}}\delta(z - \bar{\alpha}^{-1}), \quad |\alpha| < 1, \boldsymbol{m} \in \mathbb{C} \setminus \{0\}.$$

For a fixed α with $|\alpha| < 1$, necessary and sufficient conditions on $\boldsymbol{m}$ for the existence of a sequence of monic polynomials orthogonal with respect to $\tilde{\sigma}$ follow from Proposition 12.

Indeed, for $n \in \mathbb{N}$ we get

$$\begin{vmatrix} 1 + \boldsymbol{m}(n+1) & \bar{\boldsymbol{m}} \sum_{k=0}^{n} |\alpha|^{2k} \\ \frac{\boldsymbol{m}}{|\alpha|^{2n}} \sum_{k=0}^{n} |\alpha|^{2k} & 1 + \bar{\boldsymbol{m}}(n+1) \end{vmatrix}$$

$$= 1 + (\boldsymbol{m} + \bar{\boldsymbol{m}})(n+1) + |\boldsymbol{m}|^2 (n+1)^2 - \frac{|\boldsymbol{m}|^2}{|\alpha|^{2n}} \left[\sum_{k=0}^{n} |\alpha|^{2k} \right]^2,$$

with $|\alpha| < 1$. Notice that this expression is $\neq 0$ if and only if

$$\frac{|\boldsymbol{m}|^2}{|\alpha|^{2n}}\left[\sum_{k=0}^{n}|\alpha|^{2k}\right]^2 \neq 1+(\boldsymbol{m}+\bar{\boldsymbol{m}})(n+1)+|\boldsymbol{m}|^2(n+1)^2,$$

i.e.

$$\frac{|\boldsymbol{m}|^2}{|\alpha|^{2n}}\left(\frac{|\alpha|^{2n+2}-1}{|\alpha|^2-1}\right)^2 \neq 1+(\boldsymbol{m}+\bar{\boldsymbol{m}})(n+1)+|\boldsymbol{m}|^2(n+1)^2,$$

$$|\boldsymbol{m}|^2\left[\frac{1}{|\alpha|^{2n}}\left(\frac{|\alpha|^{2n+2}-1}{|\alpha|^2-1}\right)^2-(n+1)^2\right] \neq 1+(\boldsymbol{m}+\bar{\boldsymbol{m}})(n+1).$$

If $\boldsymbol{m} \in \mathbb{R}$, then the above condition becomes

$$[\boldsymbol{m}(n+1)+1]^2 \neq \frac{\boldsymbol{m}^2}{|\alpha|^{2n}}\left(\frac{|\alpha|^{2n+2}-1}{|\alpha|^2-1}\right)^2, \quad \text{for every } n \in \mathbb{N}.$$

Then, for a fixed α with $|\alpha| < 1$, the linear functional associated with $\tilde{\sigma}$ will be quasi-definite for every $\boldsymbol{m} \in \mathbb{C}$ such that

$$\boldsymbol{m}(n+1)+1 \neq \frac{\boldsymbol{m}}{|\alpha|^n}\left(\frac{|\alpha|^{2n+2}-1}{|\alpha|^2-1}\right) \quad \text{holds for every } n \in \mathbb{N}.$$

For instance, if $|\alpha|^2 = \frac{1}{2}$, then the above condition becomes

$$\boldsymbol{m}\left[n+1-2^{n/2}\left(2-\frac{1}{2^n}\right)\right]+1 \neq 0,$$

and therefore $\mathcal{L}_{\tilde{\sigma}}$ is a quasi-definite linear functional except for a numerable set of values of $\boldsymbol{m}$,

$$\boldsymbol{m} \neq \left(2^{n/2}(2-2^{-n})-n-1\right)^{-1}, \quad \forall n \in \mathbb{N}.$$

Under the above assumptions for $\boldsymbol{m}$, we get

PROPOSITION 15. *Let* $d\tilde{\sigma} = \frac{d\theta}{2\pi} + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1})$, *with* $|\alpha| < 1$ *and* $\boldsymbol{m} \in \mathbb{C} \setminus \{0\}$. *Then, the sequence of monic orthogonal polynomials with respect to* $\tilde{\sigma}$ *is given by*

$$V_n(z) = z^n - \boldsymbol{m}[A_n\alpha^n + B_n\bar{\alpha}^{-n}]\left(\frac{1-\alpha^{-n}z^n}{1-\alpha^{-1}z}\right) - \bar{\boldsymbol{m}}[C_n\alpha^n + D_n\bar{\alpha}^{-n}]\left(\frac{1-\bar{\alpha}^n z^n}{1-\bar{\alpha}z}\right) \tag{35}$$

where

$$\begin{aligned}
A_n &= -(1+n\bar{\boldsymbol{m}})/d_n(\alpha),\\
B_n &= \frac{\bar{\boldsymbol{m}}}{d_n(\alpha)}\sum_{k=0}^{n-1}|\alpha|^{2k},\\
C_n &= -|\alpha|^{-2(n-1)}\bar{B}_n,\\
D_n &= -\bar{A}_n,
\end{aligned}$$

and $d_n(\alpha) = |\boldsymbol{m}|^2|\alpha|^{-2(n-1)}\left[\sum_{k=0}^{n-1}|\alpha|^{2k}\right]^2 - |1+n\boldsymbol{m}|^2$.

Proof. It is well known (see [19]) that the sequence of monic orthogonal polynomials with respect to the normalized Lebesgue measure is $\Phi_n(z) = z^n$. Notice that $\varphi_n(z) = z^n$, as well as $\Phi_n(0) = 0$, $n \geqslant 1$. Then, from (26), we get

$$\begin{aligned} V_n(z) &= z^n - \boldsymbol{m}[A_n\alpha^n + B_n\bar{\alpha}^{-n}]K_{n-1}(z,\bar{\alpha}^{-1}) - \bar{\boldsymbol{m}}[C_n\alpha^n + D_n\bar{\alpha}^{-n}]K_{n-1}(z,\alpha) \\ &= z^n - \boldsymbol{m}[A_n\alpha^n + B_n\bar{\alpha}^{-n}]\left(\frac{1-\alpha^{-n}z^n}{1-\alpha^{-1}z}\right) - \bar{\boldsymbol{m}}[C_n\alpha^n + D_n\bar{\alpha}^{-n}]\left(\frac{1-\bar{\alpha}^n z^n}{1-\bar{\alpha}z}\right). \end{aligned}$$

The values of A_n, B_n, C_n, D_n, and $d_n(\alpha)$ follow from (28) - (30) since $K_{n-1}(\alpha,\alpha) = \sum_{k=0}^{n-1}|\alpha|^{2k}$, $K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1}) = \sum_{k=0}^{n-1}|\alpha|^{-2k}$, and $K_{n-1}(\bar{\alpha}^{-1},\alpha) = K_{n-1}(\alpha,\bar{\alpha}^{-1}) = n$. ∎

Corollary 16. *Assuming the conditions for quasi-definiteness of the linear functional $\mathcal{L}_{\tilde{\sigma}}$ are satisfied, the Verblunsky parameters associated with $\tilde{\sigma}$ are*

$$V_n(0) = -(\boldsymbol{m}[A_n\alpha^n + B_n\bar{\alpha}^{-n}] + \bar{\boldsymbol{m}}[C_n\alpha^n + D_n\bar{\alpha}^{-n}]). \tag{36}$$

Proof. It follows immediately from the evaluation of (35) at $z = 0$. ∎

Now we give an estimate for $V_n(0)$ when $n \to \infty$. We have

$$V_n(0) = -(\boldsymbol{m}A_n + \bar{\boldsymbol{m}}C_n)\alpha^n - (\boldsymbol{m}B_n + \bar{\boldsymbol{m}}D_n)\alpha^{-n}.$$

But

$$\begin{aligned} -(\boldsymbol{m}A_n + \bar{\boldsymbol{m}}C_n)\alpha^n &= \frac{\alpha^n}{d_n(\alpha)}\left[\boldsymbol{m} + n|\boldsymbol{m}|^2 + |\boldsymbol{m}|^2|\alpha|^{-2(n-1)}\sum_{k=0}^{n-1}|\alpha|^{2k}\right], \\ &= \alpha^n\frac{|\alpha|^{2n-2}(\boldsymbol{m}+n|\boldsymbol{m}|^2) + |\boldsymbol{m}|^2\sum_{k=0}^{n-1}|\alpha|^{2k}}{|\boldsymbol{m}|^2\sum_{k=0}^{n-1}|\alpha|^{2k} - |\alpha|^{2n-2}|1+n\boldsymbol{m}|^2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} -\bar{\alpha}^{-n}(\boldsymbol{m}B_n + \bar{\boldsymbol{m}}D_n) &= \bar{\alpha}^{-n}\frac{-|\boldsymbol{m}|^2\frac{|\alpha|^{2n}-1}{|\alpha|^2-1} - (\bar{\boldsymbol{m}} + n|\boldsymbol{m}|^2)}{|\boldsymbol{m}|^2\frac{1}{|\alpha|^{2n-2}}\frac{1-|\alpha|^{2n}}{1-|\alpha|^2} - |1+n\boldsymbol{m}|^2}, \\ &= \alpha^n\frac{-|\boldsymbol{m}|^2\frac{|\alpha|^{2n}-1}{|\alpha|^2-1} - (\bar{\boldsymbol{m}} + n|\boldsymbol{m}|^2)}{|\boldsymbol{m}|^2|\alpha|^2\frac{1-|\alpha|^{2n}}{1-|\alpha|^2} - |1+n\boldsymbol{m}|^2|\alpha|^{2n}}. \end{aligned}$$

As a conclusion, when $n \to \infty$,

$$V_n(0) \sim \frac{|\alpha|^2-1}{|\alpha|^2}n\alpha^n.$$

4.2. The second example corresponds to a Geronimus canonical transformation (see [5]) of the Lebesgue measure. Observe that the Christoffel transform of this

Hermitian linear functional is the Lebesgue measure. Thus, we consider a measure $\tilde{\sigma}$ such that

$$d\tilde{\sigma} = \frac{1}{|z-\alpha|^2}\frac{d\theta}{2\pi} + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1}), \quad |\alpha|<1, \boldsymbol{m}\in\mathbb{C}\setminus\{0\}.$$

Notice that this transformation can also be considered as an Uvarov transformation of the measure $\frac{1}{|z-\alpha|^2}\frac{d\theta}{2\pi}$ with two mass points (Section 3.2). Thus, according to Proposition 12, the linear functional $\mathcal{L}_{\tilde{\sigma}}$ is quasi-definite for every $\boldsymbol{m}\in\mathbb{C}$ up to a numerable set. On the other hand, according to Proposition 13, and for every $\boldsymbol{m}$ such that $\mathcal{L}_{\tilde{\sigma}}$ is quasi-definite, we get

Proposition 17. *Let* $d\tilde{\sigma} = \frac{1}{|z-\alpha|^2}\frac{d\theta}{2\pi} + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1})$, *with* $|\alpha|<1$. *Then, the sequence of monic orthogonal polynomials with respect to* $\tilde{\sigma}$ *is given by*

$$V_n(z) = z^n - \alpha z^{n-1} - \frac{\bar{\alpha}^{-n}(1-|\alpha|^2)^2[|\boldsymbol{m}|^2(1-|\alpha|^2)(\alpha^{-1}z)^{n-1} + \bar{\boldsymbol{m}} + |\boldsymbol{m}|^2(1-|\alpha|^2)]}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2}, \tag{37}$$

for $n \geqslant 1$.

Proof. It is well known ([19]) that the sequence of monic orthogonal polynomials with respect to $\frac{1}{|z-\alpha|^2}\frac{d\theta}{2\pi}$ is

$$\Phi_n(z) = z^n - \alpha z^{n-1}, \qquad |\alpha|<1, n\geqslant 1.$$

Notice that $\Phi_n^*(z) = 1-\bar{\alpha}z$. From (26) we have

$$V_n(z) = z^n - \alpha z^{n-1} - \boldsymbol{m}B_n\bar{\alpha}^{-n}(1-|\alpha|^2)K_{n-1}(z,\bar{\alpha}^{-1}) - \bar{\boldsymbol{m}}D_n\bar{\alpha}^{-n}(1-|\alpha|^2)K_{n-1}(z,\alpha), \tag{38}$$

since $\Phi_n(\alpha)=0$, $n\geqslant 1$, and $\Phi_n(\bar{\alpha}^{-1}) = \bar{\alpha}^{-n}(1-|\alpha|^2)$. Notice that in this case $\mathbf{k}_0 = ||\Phi_0||^2 = \frac{1}{1-|\alpha|^2}$, as well as $\mathbf{k}_n = 1$, $n\geqslant 1$.

We also have $K_{n-1}(z,\alpha) = \frac{1}{\mathbf{k}_0}$ and, as a consequence,

$$K_{n-1}(\alpha,\alpha) = K_{n-1}(\alpha,\bar{\alpha}^{-1}) = K_{n-1}(\bar{\alpha}^{-1},\alpha) = \frac{1}{\mathbf{k}_0} = 1-|\alpha|^2.$$

On the other hand, from the Christoffel-Darboux formula we get

$$\begin{aligned} K_{n-1}(z,\bar{\alpha}^{-1}) &= \frac{\overline{\Phi_n^*(\bar{\alpha}^{-1})}\Phi_n^*(z) - \overline{\Phi_n(\bar{\alpha}^{-1})}\Phi_n(z)}{(1-\alpha^{-1}z)}, \\ &= \frac{-(\alpha^{-n} - \bar{\alpha}\alpha^{-n+1})(z^n - \alpha z^{n-1})}{(1-\alpha^{-1}z)}, \\ &= (\alpha^{-1}z)^{n-1}(1-|\alpha|^2). \end{aligned}$$

As a consequence,

$$\begin{aligned} K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1}) &= \frac{\overline{\Phi_n^*(\bar{\alpha}^{-1})}\Phi_n^*(\bar{\alpha}^{-1})) - \overline{\Phi_n(\bar{\alpha}^{-1})}\Phi_n(\bar{\alpha}^{-1}))}{(1-\alpha^{-1}\bar{\alpha}^{-1}))}, \\ &= -\frac{|\alpha|^{-2n}(1-|\alpha|^2)^2}{1-|\alpha|^{-2}}, \\ &= |\alpha|^{-2n+2}(1-|\alpha|^2). \end{aligned}$$

Therefore

$$B_n = \frac{\bar{\boldsymbol{m}}(1-|\alpha|^2)}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2},$$

and

$$D_n = \frac{1+\boldsymbol{m}(1-|\alpha|^2)}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2}.$$

Then, (38) becomes

$$V_n(z) = z^n - \alpha z^{n-1} - \frac{|\boldsymbol{m}|^2(1-|\alpha|^2)^3\bar{\alpha}^{-n}(\alpha^{-1}z)^{n-1} + \bar{\boldsymbol{m}}[1+\boldsymbol{m}(1-|\alpha|^2)]\bar{\alpha}^{-n}(1-|\alpha|^2)^2}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2}$$

which is equivalent to (37). □

For a fixed $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, we can choose $\boldsymbol{m} \in \mathbb{C}$ such that

$$\begin{vmatrix} 1+\boldsymbol{m}(1-|\alpha|^2) & \bar{\boldsymbol{m}}(1-|\alpha|^2) \\ \boldsymbol{m}|\alpha|^{-2n+2}(1-|\alpha|^2) & 1+\bar{\boldsymbol{m}}(1-|\alpha|^2) \end{vmatrix} =$$

$$1 + (\boldsymbol{m}+\bar{\boldsymbol{m}})(1-|\alpha|^2) + |\boldsymbol{m}|^2(1-|\alpha|^2)^2 - |\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 \neq 0, \quad n \geqslant 0.$$

This condition guarantees the existence of $\{V_n(z)\}_{n\geqslant 0}$. In other words, for every $\boldsymbol{m} \in \mathbb{C}$ such that

$$|\alpha|^{-2n+2} \neq \frac{1+(\boldsymbol{m}+\bar{\boldsymbol{m}})(1-|\alpha|^2) + |\boldsymbol{m}|^2(1-|\alpha|^2)^2}{|\boldsymbol{m}|^2(1-|\alpha|^2)^2},$$

$$(-2n+2)\ln|\alpha| \neq \ln\frac{1+(\boldsymbol{m}+\bar{\boldsymbol{m}})(1-|\alpha|^2) + |\boldsymbol{m}|^2(1-|\alpha|^2)^2}{|\boldsymbol{m}|^2(1-|\alpha|^2)^2},$$

$$n \neq 1 - \frac{1}{2}\frac{\ln\frac{1+(\boldsymbol{m}+\bar{\boldsymbol{m}})(1-|\alpha|^2)+|\boldsymbol{m}|^2(1-|\alpha|^2)^2}{|\boldsymbol{m}|^2(1-|\alpha|^2)^2}}{\ln|\alpha|}, \quad \text{for every } n \in \mathbb{N},$$

the linear functional $\mathcal{L}_{\tilde{\sigma}}$ is quasi-definite.

In particular, for a fixed α such that $|\alpha|^2 = \frac{1}{2}$, and taking $\boldsymbol{m} \in \mathbb{R}$, the above condition becomes

$$n \neq 1 + \frac{\ln\left(1+\frac{2}{\boldsymbol{m}}\right)^2}{\ln 2},$$

i.e.

$$\frac{\ln\left(1+\frac{2}{\boldsymbol{m}}\right)^2}{\ln 2} \notin \mathbb{N}.$$

In other words, for $\boldsymbol{m} \in \mathbb{C}$ such that

$$\boldsymbol{m} \neq \frac{2}{2^{n/2}-1}, \quad \text{for every } n \in \mathbb{N},$$

the linear functional is quasi-definite.

COROLLARY 18. *Assuming the conditions for quasi-definiteness are satisfied, the Verblunsky parameters associated with $\tilde{\sigma}$ are*

$$V_1(0) = -\alpha - \frac{\bar{\alpha}^{-1}(1-|\alpha|^2)^2[\bar{\boldsymbol{m}} + 2|\boldsymbol{m}|^2(1-|\alpha|^2)]}{|\boldsymbol{m}|^2(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2}$$
$$V_n(0) = -\frac{\bar{\alpha}^{-n}(1-|\alpha|^2)^2[\bar{\boldsymbol{m}} + |\boldsymbol{m}|^2(1-|\alpha|^2)]}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2}, \quad n \geqslant 2.$$

Proof. It follows immediately from the evaluation of (37) at $z = 0$. ∎

Finally, we obtain an estimate for $V_n(0)$ when $n \to \infty$. From Corollary 18

$$V_n(0) = -\frac{\bar{\boldsymbol{m}}(1+\boldsymbol{m}(1-|\alpha|^2))(1-|\alpha|^2)^2\alpha^n}{|\boldsymbol{m}|^2|\alpha|^{-2n+2}(1-|\alpha|^2)^2 - |1+\boldsymbol{m}(1-|\alpha|^2)|^2}\frac{1}{|\alpha|^{2n}}.$$

In other words,

$$V_n(0) \sim \left[1 - \frac{1}{|\alpha|^2} - \frac{1}{\boldsymbol{m}|\alpha|^2}\right]\alpha^n.$$

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