Methods and Applications of Analysis.
Vol. 16, No. 1, pp. 001–032, March 2009

VISCOUS LIMITS TO PIECEWISE SMOOTH SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS OF ONE-DIMENSIONAL COMPRESSIBLE VISCOUS HEAT-CONDUCTING FLUIDS*

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Abstract. In this paper, we study the zero dissipation limit problem for the Navier-Stokes equations of one-dimensional compressible viscous heat-conducting fluids. We prove that if the solution of the inviscid Euler equations is piecewise smooth with finitely many noninteracting shocks satisfying the entropy condition, then there exist solutions to Navier-Stokes equations which converge to the inviscid solution away from shock discontinuities at a rate of ε^1 as the viscosity ε tend to zero, provided that the heat-conducting coefficient $k = O(\varepsilon)$.

Key words. Compressible Navier-Stokes equations, compressible Euler equations, viscous limit, noninteracting shocks.

AMS subject classifications. 35Q30, 76N15

1. Introduction.

The purpose of this paper is to study the asymptotic equivalence between the solutions of the compressible Navier-stokes equations and the compressible Euler equations. The one-dimensional Navier-stokes equations of compressible viscous heat-conducting fluids in Lagrangian coordinate are expressed as

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \varepsilon(\dfrac{u_x}{v})_x, \\ (e + \dfrac{u^2}{2})_t + (pu)_x = \kappa(\dfrac{\theta_x}{v})_x + \varepsilon(\dfrac{uu_x}{v})_x, \quad x \in R, \quad t > 0, \end{cases} \tag{1.1}$$

and the corresponding Euler system is of the form

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (e + \dfrac{u^2}{2})_t + (pu)_x = 0, \quad x \in R, \quad t > 0, \end{cases} \tag{1.2}$$

where v, u, θ, p and e denote the specific volume, the velocity, the temperature, the pressure, and the internal energy, respectively, and ε, κ are the viscosity and heat-conductivity coefficients, respectively. And x is the Lagrangian coordinate, so that $x = constant$ corresponds to a particle path. Here we study the ideal polytropic gas, so that the pressure p and the internal energy e are related with v and θ by the following equations of state

$$p \equiv p(v, \theta) = R\theta/v, \quad e \equiv e(\theta) = R\theta/(\gamma - 1) + constant, \tag{1.3}$$

where $R > 0$ is the gas constant and $\gamma \in (1, 2]$ is the adiabatic exponent.

In the theory of compressible fluids, the basic physics issue motivating the mathematical problem is the asymptotic equivalence between the viscous flows and the associate inviscid flows in the limit of small viscosity. This problem is particularly

*Received October 29, 2008; accepted for publication November 3, 2008.
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important and of great significance in many physics phenomena and their numerical computations in the presence of shock discontinuities. When the underlying inviscid flow is smooth, this problem can be solved by classical methods. However, in the presence of shock discontinuities, the solutions near shock discontinuities exhibit very singular behavior as the viscosity is small. The rigorous mathematical justification of this asymptotic equivalence poses challenging problems in many important cases. For the viscous conservation laws with positive definite viscosity matrix, Goodman & Xin [2], Yu [11], and Bianchini & Bressan [18] studied the convergence of the solutions for the viscous conservation laws to those for the associated hyperbolic systems. In [2], Goodman and Xin gave a very detailed description of the asymptotic behavior of solutions to the viscous systems as this viscosity tends to zero for the case when the solutions of the associated hyperbolic conservation laws contain a finite number of non-interacting shocks, via a method of matching asymptotics. For the general solutions with the initial data having small total variations, Bianchini and Bressan proved the convergence of the solutions for the viscous systems to those for the associated hyperbolic systems by establishing the uniform (independent of viscosity) total variation estimates. The above results are for the viscous conservation laws with positive definite viscosity matrix. However, the viscosity matrix of the compressible Navier-Stokes equations (1.1) is only semi-positive definite, and thus less dissipative. For this case, when the flow is isentropic, Hoff & Liu [7] and Wang [15] studied the limit process from the solutions of the compressible Navier-Stokes equations to the single shock-wave solution of the corresponding compressible Euler system(so called p-system). In [7], Hoff and Liu investigated the case when the underlying inviscid flow is a single weak shock wave. They show that the solutions to the isentropic Navier-Stokes equations with shock data exist and converge to the inviscid shocks as the viscosity vanishes, uniformly away from the shocks. And then by smooth initial perturbation, Wang [15] obtains the convergence rates. In this paper, we consider the full compressible Navier-Stokes equations. Motivated by [2] and [15], we use the matched asymptotic expansion analysis and energy estimates to establish that the piecewise smooth solutions of (1.2), with finitely many noninteracting shocks satisfying the entropy condition, are strong limits of solutions of (1.1) as the viscosity and heat-conductivity coefficients ε, κ tend to zero. We assume that for some constant $C > 0$,

$$\kappa = O(\varepsilon) \ \text{ as } \ \varepsilon \to 0 \ \text{ and } \ \kappa(\varepsilon)/\varepsilon \geq C > 0. \tag{1.4}$$

Without loss of generality, we set $\kappa = \varepsilon$. From the kinetic theory, the viscosity and heat-conductivity should be in the same order, the assumption (1.4) is reasonable [10].

For simplicity of presentation, we only consider the case in which the piecewise smooth solution (v, u, θ) to (1.2) is a single-shock solution. And we assume that

$$0 < \underline{v} \leq v(x,t) \leq \bar{v}, \ \text{ and } \ 0 < \underline{\theta} \leq \theta(x,t) \leq \bar{\theta}, \tag{1.5}$$

for some constants $\underline{v}, \bar{v}, \underline{\theta}$ and $\bar{\theta}$.

Definition 1.1. *A function $(v(x,t), u(x,t), \theta(x,t))$ is called a single-shock solution of (1.2) up to time T if*

i) $(v(x,t), u(x,t), \theta(x,t))$ is a distributional solution of the hyperbolic system (1.2) in the region $R^1 \times [0,T]$.

ii) There is a smooth curve, the shock, $x = s(t), 0 \leq t \leq T$, so that $(v(x,t), u(x,t), \theta(x,t))$ is sufficiently smooth at any point $x \neq s(t)$.

iii) The limits

$$\partial_x^l(v,u,\theta)(s(t)-0,t)=\lim_{x\to s(t)-}\partial_x^l(v,u,\theta)(x,t),$$

$$\partial_x^l(v,u,\theta)(s(t)+0,t)=\lim_{x\to s(t)+}\partial_x^l(v,u,\theta)(x,t),$$

exist and are finite for $t\le T$ and $l\ge 0$.

iv) The Lax geometrical entropy condition [13] is satisfied at $x=s(t)$, that is,

$$\dot s<\lambda_1(u(s(t)-0,t)) \ \ \textit{and} \ \ \lambda_1(u(s(t)+0,t))<\dot s<\lambda_2(u(s(t)+0,t)) \ \ (\textit{1-shocks}), \quad (1.6)$$

or

$$\lambda_2(u(s(t)-0,t))<\dot s<\lambda_3(u(s(t)-0,t)) \ \ \textit{and} \ \ \lambda_3(u(s(t)+0,t))<\dot s \ \ (\textit{3-shocks}), \quad (1.7)$$

where $\dot s=\frac{d}{dt}s(t)$ and $\lambda_1=-(\gamma R\theta)^{\frac12}/v$, $\lambda_2=0$, $\lambda_3=(\gamma R\theta)^{\frac12}/v$ are characteristic speeds of the hyperbolic system (1.2) with (1.3).

The Lax's shock condition implies that $\dot s<0$ for 1-shocks and $\dot s>0$ for 3-shocks. Here we only consider the 3-shocks. Our main results are as follows:

THEOREM 1.2. *Let $n\ge 3$ be an integer. Suppose that (v,u,θ) is a single-shock solution of system (1.2) up to time $T>0$ with*

$$\sum_{1\le k\le(2n+3)}\int_0^T\int|\partial_x^k(v(x,t),u(x,t),\theta(x,t))|^2dxdt<\infty. \quad (1.8)$$

Then, there exist constants μ_0 and $\varepsilon_0>0$, such that if

$$(\gamma-1)\sup_{0\le t\le T}|v(s(t)+0,t)-v(s(t)-0,t)|\le\mu_0, \quad (1.9)$$

for any $\epsilon\in(0,\epsilon_0]$ there is a smooth solution $(v^\varepsilon,u^\varepsilon,\theta^\varepsilon)$ to (1.1) with the same initial data as the approximate solution $(\bar{v^\varepsilon},\bar{u^\varepsilon},\bar{\theta^\varepsilon})$, constructed by (2.31), which is a small perturbation to (v,u,θ) in $L^\infty([0,T];L^2(R))$. Moreover, it holds that

$$(v^\varepsilon,u^\varepsilon,\theta^\varepsilon)(x,t)=(\bar{v^\varepsilon},\bar{u^\varepsilon},\bar{\theta^\varepsilon})(x,t)+O(\epsilon^{n-\frac74}),(x,t)\in R\times[0,T], \quad (1.10)$$

in $L^\infty([0,T];L^2(R)\cap L^\infty(R))$ and for any given $\eta\in(0,1)$ that

$$\sup_{0\le t\le T}\int|(v^\varepsilon,u^\varepsilon,\theta^\varepsilon)(x,t)-(v,u,\theta)(x,t)|^2dx\le C_\eta\epsilon^\eta, \quad (1.11)$$

and

$$\sup_{0\le t\le T,|x-s(t)|\ge\epsilon^\eta}|(v^\varepsilon,u^\varepsilon,\theta^\varepsilon)(x,t)-(v,u,\theta)(x,t)|\le C_\eta\epsilon, \quad (1.12)$$

where C_η is a positive constant depending only on η.

REMARK 1.3. *i) For the 1-shock, by a similar way, we can obtain the same results.*

ii) The convergence rate in (1.12) is optimal.

iii) The condition (1.9) implies that when γ is close to 1^+, the shock strength can be large.

iv)To prove Theorem 1.2 and to overcome the difficulties induced by nonisentropy of the flow, we shall adapt and modify the arguments in [2, 15, 16]. That is, we will exploit the smoothing property induced by the parabolic parts in (1.1), make best use of the properties of the shock profile and the smallness condition (1.9), and finally carefully compute the terms with different signs to deduce delicate energy estimates and so obtain the theorem.

NOTATION. In this paper, we use $H^l(l \geq 1)$ to denote the usual Sobolev space with the norm $\|\cdot\|_l$ and $\|\cdot\| = \|\cdot\|_0$ denotes the usual L_2-norm. We also use $O(1)$ to denote any positive bounded function which is independent of ϵ. And we set

$$\mu \equiv \sup_{0\leq t\leq T} |v(s(t)+0,t) - v(s(t)-0,t)|. \tag{1.13}$$

2. Construction of the approximate solutions. In this section, we construct the approximate solutions $(\bar{v^\varepsilon}, \bar{u^\varepsilon}, \bar{\theta^\varepsilon})$ through different scaling and asymptotic expansions in the regions near and away from the shock respectively, such that $(\bar{v^\varepsilon}, \bar{u^\varepsilon}, \bar{\theta^\varepsilon})$ approximates the piecewise smooth inviscid solution (v, u, θ) away from the shock and has a sharp change near the shock.

2.1. Outer and inner expansions and the matching conditions. Let $h_i(x,t) = (v_i, u_i, \theta_i)^t(x,t), i = 0, 1, 2, \ldots$. In the region away from the shock, $x = s(t)$, we approximate the solution of (1.1) by truncating the formal series

$$h^\varepsilon(x,t) \sim h_0(x,t) + \varepsilon h_1(x,t) + \varepsilon^2 h_2(x,t) + \cdots \tag{2.1}$$

Substituting this into (1.1) and comparing the coefficients of powers of ε, we get, for $x \neq s(t)$, that

$$O(1): \begin{cases} v_{0t} - u_{0x} = 0, \\ u_{0t} + p(v_0,\theta_0)_x = 0, \\ (e_0 + \dfrac{u_0^2}{2})_t + (p(v_0,\theta_0)u_0)_x = 0, \end{cases} \tag{2.2}$$

$$O(\varepsilon): \begin{cases} v_{1t} - u_{1x} = 0, \\ u_{1t} + (p_v(v_0,\theta_0)v_1 + p_\theta(v_0,\theta_0)\theta_1)_x = (\dfrac{u_{0x}}{v_0})_x, \\ (e_1 + u_0u_1)_t + (p(v_0,\theta_0)u_1 + u_0(p_v(v_0,\theta_0)v_1 + p_\theta(v_0,\theta_0)\theta_1))_x \\ = (\dfrac{\theta_{0x}}{v_0})_x + (\dfrac{u_0u_{0x}}{v_0})_x. \end{cases} \tag{2.3}$$

$$O(\varepsilon^2): \begin{cases} v_{2t} - u_{2x} = 0, \\ u_{2t} + (p_v(v_0,\theta_0)v_2 + p_\theta(v_0,\theta_0)\theta_2)_x = (\dfrac{u_{1x}}{v_0} - \dfrac{u_{0x}v_1}{v_0^2})_x - f_1(v_0,\theta_0;v_1,\theta_1)_x, \\ (e_2 + u_0u_2)_t + (p(v_0,\theta_0)u_2 + u_0(p_v(v_0,\theta_0)v_2 + p_\theta(v_0,\theta_0)\theta_2))_x \\ = (\dfrac{\theta_{1x}}{v_0} - \dfrac{\theta_{0x}v_1}{v_0^2})_x + (\dfrac{u_1u_{0x}}{v_0} + \dfrac{u_0u_{1x}}{v_0} - \dfrac{u_0u_{0x}v_1}{v_0^2})_x \\ \quad -\dfrac{1}{2}(u_1^2)_t - f_2(v_0,u_0,\theta_0;v_1,u_1,\theta_1)_x, \end{cases} \tag{2.4}$$

and etc., where $e_i = e(\theta_i), i = 0, 1, 2, \cdots$, and

$$\begin{aligned} f_1(v_0, \theta_0; v_1, \theta_1) &= \frac{1}{2}\{p_{vv}(v_0, \theta_0)v_1^2 + 2p_{v\theta}(v_0, \theta_0)v_1\theta_1\}, \\ f_2(v_0, u_0, \theta_0; v_1, u_1, \theta_1) &= u_1(p_v(v_0, \theta_0)v_1 + p_\theta(v_0, \theta_0)\theta_1) \\ &\quad + \frac{1}{2}u_0\{p_{vv}(v_0, \theta_0)v_1^2 + 2p_{v\theta}(v_0, \theta_0)v_1\theta_1\}, \end{aligned}$$

and etc. The outer functions $h_0, h_1, ...,$ are generally discontinuous at the shock, $x = s(t)$, but smooth up to the shock. The leading term, h_0, is the single shock solution of (1.2) which is given in Theorem 1.2.

Near the shock, h^ε should be represented by an inner expansion:

$$h^\varepsilon(x,t) \sim H_0(\xi,t) + \varepsilon H_1(\xi,t) + \varepsilon^2 H_2(\xi,t) + \cdots \tag{2.5}$$

where

$$\xi = \frac{x - s(t)}{\varepsilon} + \delta(t, \varepsilon) \tag{2.6}$$

and $\delta(t, \varepsilon)$ is a perturbation of the shock position to be determined later. We assume that $\delta(t, \varepsilon)$ has the form

$$\delta(t,\varepsilon) = \delta_0(t) + \varepsilon\delta_1(t) + \varepsilon^2\delta_2(t) + \cdots \tag{2.7}$$

Substitute (2.5)-(2.7) into (1.1) to obtain

$$O(\frac{1}{\varepsilon}) : \begin{cases} -\dot{s}V_{0\xi} - U_{0\xi} = 0, \\ -\dot{s}U_{0\xi} + p(V_0, \Theta_0)_\xi = (\dfrac{U_{0\xi}}{V_0})_\xi, \\ -\dot{s}(E_0 + \dfrac{U_0^2}{2})_\xi + (p(V_0, \Theta_0)U_0)_\xi = (\dfrac{\Theta_{0\xi}}{V_0})_\xi + (\dfrac{U_0U_{0\xi}}{V_0})_\xi, \end{cases} \tag{2.8}$$

$$O(1) : \begin{cases} -\dot{s}V_{1\xi} - U_{1\xi} = -\dot{\delta}_0 V_{0\xi} - V_{0t}, \\ -\dot{s}U_{1\xi} + (p_v(V_0, \Theta_0)V_1 + p_\theta(V_0, \Theta_0)\Theta_1)_\xi \\ = (\dfrac{U_{1\xi}}{V_0} - \dfrac{U_{0\xi}V_1}{V_0^2})_\xi - \dot{\delta}_0 U_{0\xi} - U_{0t}, \\ -\dot{s}(E_1 + U_0U_1)_\xi + \{p(V_0, \Theta_0)U_1 + (p_v(V_0, \Theta_0)V_1 + p_\theta(V_0, \Theta_0)\Theta_1)U_0\}_\xi \\ = (\dfrac{\Theta_{1\xi}}{V_0} - \dfrac{\Theta_{0\xi}V_1}{V_0^2})_\xi + (\dfrac{U_1U_{0\xi}}{V_0} + \dfrac{U_0U_{1\xi}}{V_0} - \dfrac{U_0U_{0\xi}V_1}{V_0^2})_\xi \\ \quad -\dot{\delta}_0(E_0 + \dfrac{U_0^2}{2})_\xi - (E_0 + \dfrac{U_0^2}{2})_t, \end{cases} \tag{2.9}$$

$$O(\varepsilon):\begin{cases} -\dot{s}V_{2\xi}-U_{2\xi}=-\dot{\delta}_1V_{0\xi}-\dot{\delta}_0V_{1\xi}-V_{1t},\\ -\dot{s}U_{2\xi}+(p_v(V_0,\Theta_0)V_2+p_\theta(V_0,\Theta_0)\Theta_2)_\xi\\ =\{\dfrac{U_{2\xi}}{V_0}-\dfrac{U_{0\xi}V_2}{V_0^2}+\mathcal{B}_1(V_0,U_0;V_1,U_1)\}_\xi-f_1(V_0,\Theta_0;V_1,\Theta_1)_\xi\\ \quad -\dot{\delta}_1U_{0\xi}-\dot{\delta}_0U_{1\xi}-U_{1t},\\ -\dot{s}(E_2+U_0U_2)_\xi+\{p(V_0,\Theta_0)U_2+(p_v(V_0,\Theta_0)V_2+p_\theta(V_0,\Theta_0)\Theta_2)U_0\}_\xi\\ =\{\dfrac{\Theta_{2\xi}}{V_0}-\dfrac{\Theta_{0\xi}V_2}{V_0^2}+\mathcal{B}_1(V_0,\Theta_0;V_1,\Theta_1)\}_\xi\\ \quad +\{\dfrac{U_2U_{0\xi}}{V_0}+\dfrac{U_0U_{2\xi}}{V_0}-\dfrac{U_0U_{0\xi}V_2}{V_0^2}+\mathcal{B}_2(V_0,U_0;V_1,U_1)\}_\xi\\ \quad -\dot{s}(\dfrac{U_1^2}{2})_\xi-f_2(V_0,U_0,\Theta_0;V_1,U_1,\Theta_1)_\xi\\ \quad -\dot{\delta}_1(E_0+\dfrac{U_0^2}{2})_\xi-\dot{\delta}_0(E_1+U_0U_1)_\xi-(E_1+U_0U_1)_t, \end{cases} \tag{2.10}$$

and etc., where $\dot{s}=ds/dt, \dot{\delta}_0=d\delta_0/dt$, etc., and

$$\mathcal{B}_1(V_0,U_0;V_1,U_1)=-\frac{U_{1\xi}V_1}{V_0^2}+\frac{U_{0\xi}V_1^2}{V_0^3},$$
$$\mathcal{B}_2(V_0,U_0;V_1,U_1)=-\frac{U_0U_{1\xi}V_1}{V_0^2}+\frac{U_0U_{0\xi}V_1^2}{V_0^3}+\frac{U_1U_{1\xi}}{V_0}-\frac{U_1V_1U_{0\xi}}{V_0^2},$$

etc. The inner approximation is supposed to be valid in a small zone of size $O(\varepsilon)$ near the shock $x=s(t)$.

In a matching zone, we expect that the outer and the inner expansion agree with each other. Using the Taylor series to express the outer solutions in terms of ξ, we obtain the following "matching conditions" as $\xi\to\pm\infty$:

$$H_0(\xi,t)=h_0(s(t)\pm 0,t)+o(1), \tag{2.11}$$

$$H_1(\xi,t)=h_1(s(t)\pm 0,t)+(\xi-\delta_0)\partial_x h_0(s(t)\pm 0,t)+o(1), \tag{2.12}$$

$$\begin{aligned} H_2(\xi,t)= \quad & h_2(s(t)\pm 0,t)+(\xi-\delta_0)\partial_x h_1(s(t)\pm 0,t)-\delta_1\partial_x h_0(s(t)\pm 0,t)\\ & +\frac{1}{2}(\xi-\delta_0)^2\partial_x^2 h_0(s(t)\pm 0,t)+o(1), \end{aligned} \tag{2.13}$$

and etc.

2.2. The structure of viscous shock profiles. Our construction of the approximate solutions depends on the properties of the forward traveling waves $H=(V,U,\Theta)^t$, which are the solutions of the following ordinary differential equations

$$\begin{cases} -\sigma V'-U'=0,\\ -\sigma U'+p(V,\Theta)'=(\dfrac{U'}{V})',\\ -\sigma(E+\dfrac{U^2}{2})'+(p(V,\Theta)U)'=(\dfrac{\Theta'}{V}+\dfrac{UU'}{V})', \end{cases}$$

with the boundary conditions

$$H(\xi)\to\begin{cases} h_l=(v_l,u_l,\theta_l)^t, & \text{as} \quad \xi\to-\infty,\\ h_r=(v_r,u_r,\theta_r)^t, & \text{as} \quad \xi\to+\infty, \end{cases}$$

and moving with speed σ satisfying

$$\begin{cases} -\sigma(v_l - v_r) - (u_l - u_r) = 0, \\ -\sigma(u_l - u_r) + (p_l - p_r) = 0, \\ -\sigma(e_l + \frac{u_l^2}{2} - (e_r + \frac{u_r^2}{2})) + (p_l u_l - p_r u_r) = 0, \end{cases} \tag{2.14}$$

and the Lax's shock condition $\lambda_{3r} < \sigma < \lambda_{3l}$, where $p_l = p(v_l, \theta_l), e_l = e(\theta_l)$, etc.. Integrate the differential equations to get

$$\begin{cases} -\sigma V - U = a_1, \\ \frac{U'}{V} = -\sigma U + P + a_2, \\ \frac{\Theta'}{V} + \frac{UU'}{V} = -\sigma(E + \frac{U^2}{2}) + PU + a_3, \end{cases}$$

where $P = p(V,\Theta), E = e(\Theta), a_1 = -\sigma v_l - u_l, a_2 = \sigma u_l - p_l$ and $a_3 = \sigma(e_l + \frac{u_l^2}{2}) - p_l u_l$. This system is transformed into

$$\begin{cases} U = -\sigma V - a_1, \\ \frac{\sigma V'}{V} = -\{P + \sigma^2(V - \frac{b_1}{\sigma^2})\}, \\ \frac{\Theta'}{\sigma V} = -\{E - \frac{\sigma^2}{2}(V - \frac{b_1}{\sigma^2})^2 + \frac{b_1^2}{2\sigma^2} - b_2\}, \end{cases} \tag{2.15}$$

where $b_1 = -\sigma a_1 - a_2$ and $b_2 = \dfrac{\sigma a_1^2 + 2a_1 a_2 + 2a_3}{2\sigma}$.

From [17], we know that there exists a shock profile $H = (V, U, \Theta)^t$, which connects the states h_l and h_r. By a direct calculation[16], we can deduce that H satisfies $\sigma V' = -U' > 0, \sigma\Theta' < 0$ and

$$\begin{aligned} &|\partial_\xi (V,U)| \le \bar{c}|v_r - v_l|, |\partial_\xi^2 (V,U,\Theta)| \le \bar{c}|v_r - v_l|, \\ &|\partial_\xi \Theta| \le \bar{c}(\gamma-1)|v_r - v_l| \ \text{ and } \ |\partial_\xi \Theta| \le \bar{c}(\gamma-1)|\partial_\xi V|, \end{aligned} \tag{2.16}$$

where the constant $\bar{c}$ depends only on h_l. Moreover, as $\xi \to -\infty$,

$$\frac{\partial H}{\partial h_l} - I = O(1)e^{-\alpha|\xi|}, \tag{2.17}$$

$$\frac{\partial H}{\partial \sigma} = O(1)e^{-\alpha|\xi|}. \tag{2.18}$$

As $\xi \to +\infty$,

$$\frac{\partial H}{\partial h_l} - \frac{\partial h_r}{\partial h_l} = O(1)e^{-\alpha|\xi|}, \tag{2.19}$$

$$\frac{\partial H}{\partial \sigma} - \frac{\partial h_r}{\partial \sigma} = O(1)e^{-\alpha|\xi|}. \tag{2.20}$$

2.3. Solutions of the outer and inner problems. Now we construct h_j and H_j order by order.

The leading order outer function, h_0, is the single-shock solution in Theorem 1.2. For any fixed t, the leading order inner solution $H_0(\xi,t)$ is exactly the viscous shock profile with $h_l(t) \equiv (v_l(t), u_l(t), \theta_l(t))^t = h(s(t)-0,t), h_r(t) \equiv (v_r(t), u_r(t), \theta_r(t))^t = h(s(t)+0,t)$ and $\sigma = \dot{s}(t)$. So

$$H_0(\xi,t) = H(\xi, h_l(t), \dot{s}(t)). \tag{2.21}$$

Here we take the shift to be zero since it can be absorbed into $\delta_0(t)$.

Next we determine h_1, H_1 and $\delta_0(t)$ together. By the matching condition (2.12), we expect that

$$H_1(\xi,t) = \xi \cdot \partial_x h_0(s(t) \pm 0, t) + O(1) \quad \text{as} \quad \xi \to \pm\infty.$$

So we set

$$H_1(\xi,t) = \chi(\xi,t) + D(\xi,t), \tag{2.22}$$

where $\chi(\xi,t) = (\chi_1,\chi_2,\chi_3)^t$ and $D(\xi,t) = (D_1,D_2,D_3)^t$ is a smooth function satisfying

$$D(\xi,t) = \begin{cases} \xi \cdot \partial_x h_0(s(t)-0,t), & \xi < -1, \\ \xi \cdot \partial_x h_0(s(t)+0,t), & \xi > 1. \end{cases}$$

Then inserting (2.22) into (2.9) and using (2.16)-(2.20), we obtain

$$\begin{cases} \dot{s}\chi_{1\xi} + \chi_{2\xi} = \dot{\delta_0} V_\xi + g_1(\xi,t), \\ \dot{s}\chi_{2\xi} - (p_v(V,\Theta)\chi_1 + p_\theta(V,\Theta)\chi_3)_\xi + (\dfrac{\chi_{2\xi}}{V} - \dfrac{U_\xi \chi_1}{V^2})_\xi = \dot{\delta_0} U_\xi + g_2(\xi,t), \\ \dot{s}(\dfrac{R}{\gamma-1}\chi_3 + U\chi_2)_\xi - \{p(V,\Theta)\chi_2 + (p_v(V,\Theta)\chi_1 + p_\theta(V,\Theta)\chi_3)U\}_\xi \\ \quad +(\dfrac{\chi_{3\xi}}{V} - \dfrac{\Theta_\xi \chi_1}{V^2})_\xi + (\dfrac{U_\xi \chi_2}{V} + \dfrac{U\chi_{2\xi}}{V} - \dfrac{UU_\xi \chi_1}{V^2})_\xi = \dot{\delta_0}(E + \dfrac{U^2}{2})_\xi + g_3(\xi,t), \end{cases}$$

where $|g_i(\xi,t)| \le c \exp\{-\alpha|\xi|\}$ for large $|\xi|, i = 1,2,3$. Define $G_i(\xi,t) = \int_0^\xi g_i(\eta,t)d\eta$. Then we have

$$\begin{cases} \dot{s}\chi_1 + \chi_2 = \dot{\delta_0} V + G_1(\xi,t) + c_1(t), \\ \dot{s}\chi_2 - (p_v(V,\Theta)\chi_1 + p_\theta(V,\Theta)\chi_3) + \dfrac{\chi_{2\xi}}{V} - \dfrac{U_\xi \chi_1}{V^2} = \dot{\delta_0} U + G_2(\xi,t) + c_2(t), \\ \dot{s}(\dfrac{R}{\gamma-1}\chi_3 + U\chi_2) - \{p(V,\Theta)\chi_2 + (p_v(V,\Theta)\chi_1 + p_\theta(V,\Theta)\chi_3)U\} \\ \quad +(\dfrac{\chi_{3\xi}}{V} - \dfrac{\Theta_\xi \chi_1}{V^2}) + (\dfrac{U_\xi \chi_2}{V} + \dfrac{U\chi_{2\xi}}{V} - \dfrac{UU_\xi \chi_1}{V^2}) = \dot{\delta_0}(E + \dfrac{U^2}{2}) + G_3(\xi,t) + c_3(t), \end{cases} \tag{2.23}$$

where $c_i(t) \in R^1, i = 1,2,3$, are integration constants to be determined later. Letting

$\xi \to \pm\infty$ and using the matching condition (2.12), we obtain

$$\begin{cases} ((\gamma-1)p_l - \dot{s}^2 v_l)(\dot{\delta_0} v_l + c_1(t) + G_{1-}) + (\dot{s}v_l - (\gamma-1)u_l)(\dot{\delta_0} u_l + c_2(t) + G_{2-}) \\ \quad +(\gamma-1)(\dot{\delta_0}(\dfrac{R}{\gamma-1}\theta_l + \dfrac{u_l^2}{2}) + c_3(t) + G_{3-}) = \dot{s}(\gamma p_l - \dot{s}^2 v_l)(v_1^l - \delta_0\partial_x v_l), \\ p_l(\dot{\delta_0} v_l + c_1(t) + G_{1-}) + ((\gamma-1)u_l - \dot{s}v_l)(\dot{\delta_0} u_l + c_2(t) + G_{2-}) \\ \quad -(\gamma-1)(\dot{\delta_0}(\dfrac{R}{\gamma-1}\theta_l + \dfrac{u_l^2}{2}) + c_3(t) + G_{3-}) = (\gamma p_l - \dot{s}^2 v_l)(u_1^l - \delta_0\partial_x u_l), \\ (\gamma-1)\{p_l^2(\dot{\delta_0} v_l + c_1(t) + G_{1-}) - (\dot{s}v_l p_l + (p_l - \dot{s}^2 v_l)u_l)(\dot{\delta_0} u_l + c_2(t) + G_{2-}) \\ \quad +(p_l - \dot{s}^2 v_l)(\dot{\delta_0}(\dfrac{R}{\gamma-1}\theta_l + \dfrac{u_l^2}{2}) + c_3(t) + G_{3-})\} = R\dot{s}(\gamma p_l - \dot{s}^2 v_l)(\theta_1^l - \delta_0\partial_x\theta_l), \end{cases} \tag{2.24}$$

and

$$\begin{cases} ((\gamma-1)p_r - \dot{s}^2 v_r)(\dot{\delta_0} v_r + c_1(t) + G_{1+}) + (\dot{s}v_r - (\gamma-1)u_r)(\dot{\delta_0} u_r + c_2(t) + G_{2+}) \\ \quad +(\gamma-1)(\dot{\delta_0}(\dfrac{R}{\gamma-1}\theta_r + \dfrac{u_r^2}{2}) + c_3(t) + G_{3+}) = \dot{s}(\gamma p_r - \dot{s}^2 v_r)(v_1^r - \delta_0\partial_x v_r), \\ p_r(\dot{\delta_0} v_r + c_1(t) + G_{1+}) + ((\gamma-1)u_r - \dot{s}v_r)(\dot{\delta_0} u_r + c_2(t) + G_{2+}) \\ \quad -(\gamma-1)(\dot{\delta_0}(\dfrac{R}{\gamma-1}\theta_r + \dfrac{u_r^2}{2}) + c_3(t) + G_{3+}) = (\gamma p_r - \dot{s}^2 v_r)(u_1^r - \delta_0\partial_x u_r), \\ (\gamma-1)\{p_r^2(\dot{\delta_0} v_r + c_1(t) + G_{1+}) - (\dot{s}v_r p_r + (p_r - \dot{s}^2 v_r)u_r)(\dot{\delta_0} u_r + c_2(t) + G_{2+}) \\ \quad +(p_r - \dot{s}^2 v_r)(\dot{\delta_0}(\dfrac{R}{\gamma-1}\theta_r + \dfrac{u_r^2}{2}) + c_3(t) + G_{3+})\} = R\dot{s}(\gamma p_r - \dot{s}^2 v_r)(\theta_1^r - \delta_0\partial_x\theta_r). \end{cases} \tag{2.25}$$

Write $\beta_{in} \equiv \{v_1^r, u_1^r, \theta_1^r, \theta_1^l\}$ and $\beta_{out} \equiv \{v_1^l, u_1^l\}$. We first consider (2.25). Since the determinant of the Jacobian matrix

$$detJ = -\frac{(\gamma-1)\dot{s}}{v_r}(\gamma v_r p_r - \dot{s}^2 v_r^2) \neq 0,$$

we can solve $\dot{\delta_0} v_r + c_1(t), \dot{\delta_0} u_r + c_2(t)$ and $\dot{\delta_0}(\frac{R}{\gamma-1}\theta_r + \frac{u_r^2}{2}) + c_3(t)$ from (2.25) in terms of the terms on the right-hand side of (2.25). Then substituting the resulting expression into the last equation of (2.24), we arrive at the ordinary differential equation for δ_0 :

$$\dot{\delta_0} + E_1(t)\delta_0 = E_{21}(t)\partial_x v_1^r + E_{22}(t)\partial_x u_1^r + E_{23}(t)\partial_x\theta_1^r + E_{24}(t)\partial_x\theta_1^l + F(t), \tag{2.26}$$

provided that $(\gamma-1)\mu$ is suitably small. Here $E_1(t), E_{2j}(t)$ and $F(t)$ are some known smooth functions, and $E_1(t)$ and $E_{2j}(t)$ remain bounded even as $\mu \to 0, 1 \le j \le 4$. Solving for δ_0 from (2.26) up to a constant, we obtain $c(t)$ uniquely in terms of β_{in}. Then substituting the expression of δ_0 and $c(t)$ into the first two equations of (2.24), we can express v_1^l, u_1^l in terms of β_{in}. Then the theory of linear hyperbolic equations [3, 4] shows that the problem (2.3), (2.26) has a solution smooth up to the shock provided that the initial value, $h_1(x,0)$, is chosen to satisfy the appropriate compatibility conditions at $x = s(0)$. Thus $h_1(x,t)$ is completely determined, which in turn gives δ_0 and $c(t)$ by (2.24)-(2.26), and therefore $H_1(\xi,t)$. Now we summarize the above discussion to achieve

Proposition 2.1. *If $(\gamma-1)\mu$ is suitably small, then $h_1(x,t), H_1(\xi,t)$ and δ_0 can be established such that*

(i) $h_1(x,t)$ and its derivatives are uniformly continuous up to $x = s(t)$, and

$$\sum_{0\le k\le 2n+1} \int_0^T \int |\partial_x^k h_1(x,t)|^2 dxdt < \infty. \tag{2.27}$$

(ii) $H_1(\xi,t)$ and δ_0 are smooth functions, and there is an $\alpha>0$ such that as $\xi\to\pm\infty$,

$$H_1(\xi,t)=h_1(s(t)\pm 0,t)+(\xi-\delta_0)\partial_x h_0(s(t)\pm 0,t)+O(1)\exp\{-\alpha|\xi|\}. \tag{2.28}$$

The above constructions can be carried out to any order. In particular, we can determine $h_2,H_2,\delta_1;\cdots;h_n,H_n$ and δ_{n-1} for $n\geq 3$ and the similar results as in Proposition 2.1 hold for them.

2.4. Approximate solutions. Now we can construct an approximate solution to (1.1) by patching the truncated outer and inner solutions in the previous discussion. For $n\geq 3$, define

$$I(x,t)=\quad H(\frac{x-s(t)}{\varepsilon}+\sum_{j=0}^{n-1}\varepsilon^j\delta_j(t),t)+\sum_{i=1}^{n}\varepsilon^i H_i(\frac{x-s(t)}{\varepsilon}+\sum_{j=0}^{n-1}\varepsilon^j\delta_j(t),t), \tag{2.29}$$

and

$$O(x,t)=\sum_{i=0}^{n}\varepsilon^i h_i(x,t). \tag{2.30}$$

Let $m\in C_0^\infty(R)$ satisfy $0\leq m(y)\leq 1$, and

$$m(y)=\begin{cases}1, & |y|\leq 1,\\ 0, & |y|\geq 2.\end{cases}$$

Set $\nu\in(\frac{1}{2},1)$ to be a constant. Then we define the approximate solution to (1.1) as

$$S^\varepsilon(x,t)=m(\frac{x-s(t)}{\varepsilon^\nu})I(x,t)+(1-m(\frac{x-s(t)}{\varepsilon^\nu}))O(x,t)+d(x,t), \tag{2.31}$$

where $d(x,t)$ is a higher-order correction term to be determined. We use the following notations:

$$S^\varepsilon=(\bar{v^\varepsilon},\bar{u^\varepsilon},\bar{\theta^\varepsilon})^t,\quad I=(I_1,I_2,I_3)^t,\quad O=(O_1,O_2,O_3)^t,\quad d=(d_1,d_2,d_3)^t.$$

Using the structures of the various orders of inner and outer solutions, we compute that

$$\begin{cases}\bar{v^\varepsilon_t}-\bar{u^\varepsilon_x}=d_{1t}-d_{2x}+q_1(x,t),\\ \bar{u^\varepsilon_t}+p(\bar{v^\varepsilon},\bar{\theta^\varepsilon})-\varepsilon(\dfrac{\bar{u^\varepsilon_x}}{\bar{v^\varepsilon}})_x=d_{2t}-\varepsilon(\dfrac{d_{2x}}{B_1})_x+\displaystyle\sum_{i=2}^{4}q_i(x,t)-q_{5x}(x,t)+q_{6x}(x,t),\\ (e(\bar{\theta^\varepsilon})+\dfrac{\bar{u^\varepsilon}^2}{2})_t+(p(\bar{v^\varepsilon},\bar{\theta^\varepsilon})\bar{u^\varepsilon})_x-\kappa(\dfrac{\bar{\theta^\varepsilon_x}}{\bar{v^\varepsilon}})_x-\varepsilon(\dfrac{\bar{u^\varepsilon}\bar{u^\varepsilon_x}}{\bar{v^\varepsilon}})_x\\ =\dfrac{R}{\gamma-1}d_{3t}-\varepsilon(\dfrac{d_{3x}}{B_1})_x+\displaystyle\sum_{j=7}^{10}q_j(x,t)-q_{11x}(x,t)+q_{12x}(x,t).\end{cases} \tag{2.32}$$

Here

$$q_1(x,t)=m_t(I_1-O_1)-m_x(I_2-O_2)+m(I_{1t}-I_{2x}), \tag{2.33}$$

$$q_2(x,t)=m_t(I_2-O_2)+\{p(B_1,B_3)-mp(I_1,I_3)-(1-m)p(O_1,O_3)\}_x$$

$$+ m_x(p(I_1,I_3)-p(O_1,O_3)) - \varepsilon m_x(\frac{I_{2x}}{I_1}-\frac{O_{2x}}{O_1}), \tag{2.34}$$

$$\begin{aligned} q_3(x,t) =& m\{(p(I_1,I_3)-\Gamma(p(I_1,I_3)))_x - \varepsilon(\frac{I_{2x}}{I_1}-\Gamma(\frac{I_{2x}}{I_1}))_x \\ &+ \varepsilon^n U_{nt} + \varepsilon^{n+1}(\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\varepsilon^i\delta_j U_{n+i-j})_x\}, \end{aligned} \tag{2.35}$$

$$q_4(x,t) = (1-m)\{(p(O_1,O_3)-\Gamma(p(O_1,O_3)))_x - \varepsilon(\frac{O_{2x}}{O_1}-\Gamma(\frac{O_{2x}}{O_1}))_x\}, \tag{2.36}$$

$$q_5(x,t) = \varepsilon\{\frac{1}{\bar{v^\varepsilon}}B_{2x} - m\frac{I_{2x}}{I_1} - (1-m)\frac{O_{2x}}{O_1} + (\frac{1}{\bar{v^\varepsilon}}-\frac{1}{B_1})d_{2x}\}, \tag{2.37}$$

$$q_6(x,t) = p(\bar{v^\varepsilon},\bar{\theta^\varepsilon}) - p(B_1,B_3), \tag{2.38}$$

$$\begin{aligned} q_7(x,t) =& \frac{R}{\gamma-1}m_t(I_3-O_3) + mm_t(I_2-O_2)^2 - m(1-m)(I_2-O_2)(I_2-O_2)_t \\ &+ m_tO_2(I_2-O_2) + m_x(p(I_1,I_3)I_2 - p(O_1,O_3)O_2) \\ &+ \{p(B_1,B_3)B_2 - m(p(I_1,I_3)I_2 - (1-m)(p(O_1,O_3)O_2\}_x \\ &- \varepsilon m_x(\frac{I_{3x}}{I_1}-\frac{O_{3x}}{O_1}+\frac{I_2I_{2x}}{I_1}-\frac{O_2O_{2x}}{O_1}), \end{aligned} \tag{2.39}$$

$$\begin{aligned} q_8(x,t) =& m\{[p(I_1,I_3)I_2 - \Gamma(p(I_1,I_3)I_2)]_x - \varepsilon[\frac{I_{3x}}{I_1} - \Gamma(\frac{I_{3x}}{I_1}) + \frac{I_2I_{2x}}{I_1} - \Gamma(\frac{I_2I_{2x}}{I_1})]_x \\ &+ \varepsilon^n(E_{nt} + \frac{1}{2}\sum_{i=0}^{n}\sum_{j=i}^{n}\varepsilon^iU_jU_{n+i-j})_t \\ &+ \varepsilon^{n+1}(\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\varepsilon^i\delta_jE_{n+i-j} + \frac{1}{2}\sum_{i=0}^{n-1}\sum_{j=i}^{n+i}\sum_{k=j-i}^{n}\delta_i\varepsilon^jU_kU_{n-i+j-k} \\ &\quad + \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=0}^{i-1}\sum_{k=0}^{n-i+j}\delta_i\varepsilon^jU_kU_{n-i+j-k} - \frac{\dot{s}}{2}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n}\varepsilon^iU_jE_{n+1+i-j})_x\}, \end{aligned} \tag{2.40}$$

$$\begin{aligned} q_9(x,t) =& (1-m)\{[p(O_1,O_3)O_2 - \Gamma(p(O_1,O_3)O_2)]_x - \varepsilon[\frac{O_{3x}}{O_1} - \Gamma(\frac{O_{3x}}{O_1}) + \frac{O_2O_{2x}}{O_1} \\ &- \Gamma(\frac{O_2O_{2x}}{O_1})]_x + \frac{1}{2}\varepsilon^{n+1}(\sum_{i=0}^{n-1}\sum_{j=i+1}^{n}\varepsilon^iu_ju_{n+1+i-j})_t\}, \end{aligned} \tag{2.41}$$

$$q_{10}(x,t) = d_2\bar{u^\varepsilon_t} + d_{2t}\bar{u^\varepsilon} - \varepsilon(\frac{d_{2x}}{B_1}\bar{u^\varepsilon})_x, \tag{2.42}$$

$$q_{11}(x,t) =\varepsilon\{(\frac{1}{\bar{v^\varepsilon}}B_{3x}-m\frac{I_{3x}}{I_1}-(1-m)\frac{O_{3x}}{O_1})+(\frac{1}{\bar{v^\varepsilon}}-\frac{1}{B_1})d_{3x}+(\frac{\bar{u^\varepsilon}}{\bar{v^\varepsilon}}B_{2x}-m\frac{I_2I_{2x}}{I_1}$$
$$-(1-m)\frac{O_2O_{2x}}{O_1})+(\frac{\bar{u^\varepsilon}}{\bar{v^\varepsilon}}-\frac{\bar{u^\varepsilon}}{B_1})d_{2x}\}, \tag{2.43}$$

$$q_{12}(x,t)=p(\bar{v^\varepsilon},\bar{\theta^\varepsilon})\bar{u^\varepsilon}-p(B_1,B_3)B_2, \tag{2.44}$$

where $B_j = mI_j+(1-m)O_j, j=1,2,3$; $\Gamma(p(I_1,I_3)),\Gamma(\frac{I_{2x}}{I_1}),\Gamma(p(I_1,I_3)I_2),\Gamma(\frac{I_{3x}}{I_1})$ and $\Gamma(\frac{I_2I_{2x}}{I_1})$ denote the truncated Taylor's expansion of $p(I_1,I_3)$, $\frac{I_{2x}}{I_1}, p(I_1,I_3)I_2, \frac{I_{3x}}{I_1}$ and $\frac{I_2I_{2x}}{I_1}$, respectively, at (V,U,Θ), including all the terms of the orders $O(1)\varepsilon^k, 0\le k\le n$; $\Gamma(p(O_1,O_3)),\Gamma(\frac{O_{2x}}{O_1}),\Gamma(p(O_1,O_3)O_2),\Gamma(\frac{O_{3x}}{O_1})$ and $\Gamma(\frac{O_2O_{2x}}{O_1})$ denote the truncated Taylor's expansion of $p(O_1,O_3)$, $\frac{O_{2x}}{O_1}, p(O_1,O_3)O_2, \frac{O_{3x}}{O_1}$ and $\frac{O_2O_{2x}}{O_1}$, respectively, at (v_0,u_0,θ_0), including all the terms of the orders $O(1)\varepsilon^k, 0\le k\le n$.

In view of our construction, we have

i) supp $(q_1,q_3,q_8)\subseteq\{(x,t):|x-s(t)|\le 2\varepsilon^\nu, 0\le t\le T\}$, and

$$\partial_x^l(q_1,q_3,q_8)(x,t)=O(1)\varepsilon^{(n-l)\nu}, \qquad l=0,1,2,3. \tag{2.45}$$

ii) supp$(q_2,q_7)\subseteq\{(x,t):\varepsilon^\nu\le|x-s(t)|\le 2\varepsilon^\nu, 0\le t\le T\}$, and

$$\partial_x^l(q_2,q_7)(x,t)=O(1)\varepsilon^{(n-l)\nu}, \qquad l=0,1,2,3. \tag{2.46}$$

iii) supp $(q_4,q_9)\subseteq\{(x,t):|x-s(t)|\ge\varepsilon^\nu, 0\le t\le T\}$, and

$$\partial_x^l(q_4,q_9)(x,t)=O(1)\varepsilon^{n+1-l\nu}, \qquad (\int_0^T\|(q_4,q_9)(\cdot,t)\|^2dt)^{\frac12}\le O(1)\varepsilon^{n+1},$$
$$(\int_0^T\|\partial_x^l(q_4,q_9)(\cdot,t)\|^2dt)^{\frac12}\le O(1)\varepsilon^{n+1-(l-\frac12)\nu}, \qquad l=1,2,3. \tag{2.47}$$

We now choose $d(x,t)=(d_1(x,t),d_2(x,t),d_3(x,t))$ to be the solution of

$$\begin{cases} d_{1t}-d_{2x}=-q_1(x,t),\\ d_{2t}-\varepsilon(\frac{d_{2x}}{B_1})_x=-\sum_{i=2}^4 q_i(x,t),\\ \frac{R}{\gamma-1}d_{3t}-\varepsilon(\frac{d_{3x}}{B_1})_x=-\sum_{j=7}^{10}q_j(x,t),\\ d_1(x,0)=d_2(x,0)=d_3(x,0)=0, \end{cases} \tag{2.48}$$

so that S^ε satisfies

$$\begin{cases} \bar{v_t^\varepsilon}-\bar{u_x^\varepsilon}=0,\\ \bar{u_t^\varepsilon}+p(\bar{v^\varepsilon},\bar{\theta^\varepsilon})_x=\varepsilon(\frac{\bar{u_x^\varepsilon}}{\bar{v^\varepsilon}})_x-q_{5x}(x,t)+q_{6x}(x,t),\\ (e(\bar{\theta^\varepsilon})+\frac{\bar{u^\varepsilon}^2}{2})_t+(p(\bar{v^\varepsilon},\bar{\theta^\varepsilon})\bar{u^\varepsilon})_x=\kappa(\frac{\bar{\theta_x^\varepsilon}}{\bar{v^\varepsilon}})_x+\varepsilon(\frac{\bar{u^\varepsilon}\bar{u_x^\varepsilon}}{\bar{v^\varepsilon}})_x-q_{11x}(x,t)+q_{12x}(x,t). \end{cases} \tag{2.49}$$

Since $B_1(x,t)>0$ is bounded below and above, and uniformly continuous, by the result of [14], $(2.48)_{2,3}$ admit fundamental solutions $G_2(x,t)$ and $G_3(x,t)$, respectively, which are bounded as follows:

$$|G_2(x,t),G_3(x,t)|\le k_1(\varepsilon t)^{-\frac12}e^{-\frac{k_2}{\varepsilon t}}, \quad \forall t\in[0,T], \tag{2.50}$$

where the constants k_1 and k_2 depend only on the lower and upper bounds of B_1 and T. By the same method of [2] and a direct calculation, and the fact $d_1(x,t) = \int_0^t d_{2x}(x,\tau)d\tau - \int_0^t q_1(x,\tau)d\tau$, we have the following results. Here we omit the proof.

LEMMA 2.2. *Let $d(x,t)$ be the solution of (2.48). The following estimates hold for all $t \in [0,T]$:*

i)
$$\|\partial_x^l d_2(\cdot,t)\|_{L^\infty} \le O(1)\varepsilon^{(n+1-l)\nu-\frac{1}{2}}, \quad \text{for} \quad l = 0,1,2,3,4, \tag{2.51}$$
$$\|d_2(\cdot,t)\| \le O(1)\varepsilon^{(n+1)\nu+\frac{\alpha}{2}-\frac{1}{2}}, \quad \alpha \in (0,\frac{1}{2}), \tag{2.52}$$
$$\|\partial_x^l d_2(\cdot,t)\| \le O(1)\varepsilon^{(n+1-l+\frac{1}{2})\nu-\frac{1}{2}}, \quad l = 1,2,3,4. \tag{2.53}$$
ii)
$$\|\partial_x^l d_1(\cdot,t)\|_{L^\infty} \le O(1)\varepsilon^{(n-l)\nu-\frac{1}{2}}, \quad \text{for} \quad l = 0,1,2,3, \tag{2.54}$$
$$\|\partial_x^l d_1(\cdot,t)\| \le O(1)\varepsilon^{(n-l+\frac{1}{2})\nu-\frac{1}{2}}, \quad l = 0,1,2,3. \tag{2.55}$$
iii)
$$\|\partial_x^l q_{10}(\cdot,t)\|_{L^\infty} \le O(1)\varepsilon^{(n-l)\nu-\frac{1}{2}}, \quad \text{for} \quad l = 0,1,2,3, \tag{2.56}$$
$$\|\partial_x^l q_{10}(\cdot,t)\| \le O(1)\varepsilon^{(n-l+\frac{1}{2})\nu-\frac{1}{2}}, \quad l = 0,1,2,3. \tag{2.57}$$
iv)
$$\|\partial_x^l d_3(\cdot,t)\|_{L^\infty} \le O(1)\varepsilon^{(n-l)\nu-\frac{1}{2}}, \quad \text{for} \quad l = 0,1,2,3, \tag{2.58}$$
$$\|\partial_x^l d_3(\cdot,t)\| \le O(1)\varepsilon^{(n-l+\frac{1}{2})\nu-\frac{1}{2}}, \quad l = 0,1,2,3. \tag{2.59}$$
v)
$$\|\partial_x^l (q_5,q_6,q_{11},q_{12})(\cdot,t)\| \le O(1)\varepsilon^{(n+\frac{1}{2})\nu-l-\frac{1}{2}}, \quad l = 0,1,2,3. \tag{2.60}$$

It follows from our construction that S^ε has the following property.

LEMMA 2.3. *Let S^ε be defined in (2.31), then*

$$S^\varepsilon(x,t) = \begin{cases} h_0(x,t) + O(1)\varepsilon, & \text{for } |x - s(t)| \ge \varepsilon^\nu, \\ H_0(\xi,t) + O(1)\varepsilon^\nu, & \text{for } |x - s(t)| \le 2\varepsilon^\nu. \end{cases} \tag{2.61}$$

Under the following coordinate transformation

$$y = \frac{x - s(t)}{\varepsilon}, \qquad \tau = \frac{t}{\varepsilon},$$

we have

$$\partial_y^l S^\varepsilon = m\partial_y^l H_0 + O(1)\varepsilon, \;\; \partial_\tau S^\varepsilon = O(1)\varepsilon, \;\; 1 \le l \le 3. \tag{2.62}$$

3. Stability analysis. We now show that there exists an exact solution to (1.1) in a neighborhood of the approximate solution $S^\varepsilon(x,t)$, and that the asymptotic behavior of the viscous solution is given by S^ε for small viscosity ε.

Suppose that $h^\varepsilon = (v^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ is the exact solution to (1.1) with the initial data $h^\varepsilon(x,0) = S^\varepsilon(x,0)$. We decompose the solution as

$$\begin{aligned} v^\varepsilon(x,t) &= \bar{v^\varepsilon}(x,t) + \phi(x,t), \\ u^\varepsilon(x,t) &= \bar{u^\varepsilon}(x,t) + \psi(x,t), \\ \theta^\varepsilon(x,t) &= \bar{\theta^\varepsilon}(x,t) + \zeta(x,t), \\ (e(\theta^\varepsilon) + \frac{(u^\varepsilon)^2}{2})(x,t) &= (e(\bar{\theta^\varepsilon}) + \frac{\bar{u^\varepsilon}^2}{2})(x,t) + w(x,t), \end{aligned} \tag{3.1}$$

for $(x,t) \in R \times [0,T]$. Then using the relation (2.49) for S^ε, we obtain that

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + (\bar{p_v}\phi + \bar{p_\theta}\zeta)_x + Q_1(\bar{v^\varepsilon}, \bar{\theta^\varepsilon}; \phi, \psi)_x = \varepsilon(\frac{u^\varepsilon_x}{v^\varepsilon} - \frac{\bar{u^\varepsilon_x}}{\bar{v^\varepsilon}})_x + (q_5 - q_6)_x, \\ w_t + (\bar{u^\varepsilon}\bar{p_v}\phi + \bar{p}\psi + \bar{u^\varepsilon}\bar{p_\theta}\zeta)_x + Q_2(\bar{v^\varepsilon}, \bar{u^\varepsilon}, \bar{\theta^\varepsilon}; \phi, \psi, \zeta)_x \\ = \kappa(\frac{\theta^\varepsilon_x}{v^\varepsilon} - \frac{\bar{\theta^\varepsilon_x}}{\bar{v^\varepsilon}})_x + \varepsilon(\frac{u^\varepsilon u^\varepsilon_x}{v^\varepsilon} - \frac{\bar{u^\varepsilon}\bar{u^\varepsilon_x}}{\bar{v^\varepsilon}})_x + (q_{11} - q_{12})_x, \\ \phi(x,0) = \psi(x,0) = w(x,0) = 0, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} &\bar{p} = p(\bar{v^\varepsilon}, \bar{\theta^\varepsilon}), \bar{p_v} = p_v(\bar{v^\varepsilon}, \bar{\theta^\varepsilon}), \bar{p_\theta} = p_\theta(\bar{v^\varepsilon}, \bar{\theta^\varepsilon}), \\ &Q_1 = p(v^\varepsilon, \theta^\varepsilon) - p(\bar{v^\varepsilon}, \bar{\theta^\varepsilon}) - (\bar{p_v}\phi + \bar{p_\theta}\zeta) \text{ satisfies } |Q_1| \le O(1)(|\phi|^2 + |\zeta|^2), \\ &Q_2 = p(v^\varepsilon, \theta^\varepsilon)u^\varepsilon - \bar{p}\bar{u^\varepsilon} - (\bar{u^\varepsilon}\bar{p_v}\phi + \bar{p}\psi + \bar{u^\varepsilon}\bar{p_\theta}\zeta) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{satisfies } |Q_2| \le O(1)(|\phi|^2 + |\psi|^2 + |\zeta|^2). \end{aligned}$$

To exploit the fact that a shock satisfying the entropy condition is compressive, we need to integrate the system (3.2) once. Thus we set $(\phi, \psi, w)(x,t) = (\bar{\Phi}_x, \bar{\Psi}_x, \tilde{W}_x)(x,t)$ and $\bar{W} = \frac{\gamma-1}{R}(\tilde{W} - \bar{u^\varepsilon}\bar{\Psi})$. Then

$$\zeta = \bar{W}_x + \frac{\gamma-1}{R}(\bar{u^\varepsilon_x}\bar{\Psi} - \frac{1}{2}\bar{\Psi}_x^2) \text{ and } w = \frac{R}{\gamma-1}\bar{W}_x + (\bar{u^\varepsilon}\bar{\Psi})_x. \tag{3.3}$$

Substitute these quantities into (3.2) and integrate the resulting equation with respect to x to obtain

$$\begin{cases} \bar{\Phi}_t - \bar{\Psi}_x = 0, \\ \bar{\Psi}_t + \bar{p_v}\phi + \bar{p_\theta}\zeta + Q_1 = \varepsilon(\frac{u^\varepsilon_x}{v^\varepsilon} - \frac{\bar{u^\varepsilon_x}}{\bar{v^\varepsilon}}) + q_5 - q_6, \\ (\frac{R}{\gamma-1}\bar{W} + \bar{u^\varepsilon}\bar{\Psi})_t + (\bar{u^\varepsilon}\bar{p_v}\phi + \bar{p}\psi + \bar{u^\varepsilon}\bar{p_\theta}\zeta) + Q_2 \\ = \kappa(\frac{\theta^\varepsilon_x}{v^\varepsilon} - \frac{\bar{\theta^\varepsilon_x}}{\bar{v^\varepsilon}}) + \varepsilon(\frac{u^\varepsilon u^\varepsilon_x}{v^\varepsilon} - \frac{\bar{u^\varepsilon}\bar{u^\varepsilon_x}}{\bar{v^\varepsilon}}) + q_{11} - q_{12}, \\ \bar{\Phi}(x,0) = \bar{\Psi}(x,0) = \bar{W}(x,0) = 0. \end{cases} \tag{3.4}$$

This system can be written as

$$\begin{cases} \bar{\Phi}_t - \bar{\Psi}_x = 0, \\ \bar{\Psi}_t + \bar{p_v}\bar{\Phi}_x + \bar{p_\theta}(\bar{W}_x + \frac{\gamma-1}{R}(\bar{u^\varepsilon_x}\bar{\Psi} - \frac{1}{2}\bar{\Psi}_x^2)) = \varepsilon(\frac{u^\varepsilon_x}{v^\varepsilon} - \frac{\bar{u^\varepsilon_x}}{\bar{v^\varepsilon}}) - Q_1 + q_5 - q_6, \\ \frac{R}{\gamma-1}\bar{W}_t + \bar{u^\varepsilon_t}\bar{\Psi} + \bar{p}\bar{\Psi}_x \\ = \kappa(\frac{\theta^\varepsilon_x}{v^\varepsilon} - \frac{\bar{\theta^\varepsilon_x}}{\bar{v^\varepsilon}}) + \varepsilon\frac{u^\varepsilon_x}{v^\varepsilon}\bar{\Psi}_x + \bar{u^\varepsilon}Q_1 - Q_2 - \bar{u^\varepsilon}(q_5 - q_6) + q_{11} - q_{12}, \\ \bar{\Phi}(x,0) = \bar{\Psi}(x,0) = \bar{W}(x,0) = 0. \end{cases} \tag{3.5}$$

By making the following rescalings,

$$(\bar{\Phi}, \bar{\Psi}, \bar{W})(x,t) = \varepsilon(\Phi, \Psi, W)(y, \tau), \quad y = \frac{x - s(t)}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon}, \tag{3.6}$$

we transform (3.5) into

$$
\begin{cases}
\Phi_\tau - \dot{s}(\varepsilon\tau)\Phi_y - \Psi_y = 0, \\
\Psi_\tau - \dot{s}(\varepsilon\tau)\Psi_y - \bar{\varrho}\Phi_y + \bar{p_\theta}\{W_y + \frac{\gamma-1}{R}(\bar{u_y^\varepsilon}\Psi - \frac{1}{2}\Psi_y^2)\} \\
= \frac{\Psi_{yy}}{\bar{v^\varepsilon}} - \frac{1}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y\Psi_{yy} + \frac{\bar{u_y^\varepsilon}}{\bar{v^\varepsilon}^2 v^\varepsilon}\Phi_y^2 - Q_1 + q_5 - q_6, \\
\frac{R}{\gamma-1}(W_\tau - \dot{s}(\varepsilon\tau)W_y) + (\bar{u_\tau^\varepsilon} - \dot{s}(\varepsilon\tau)\bar{u_y^\varepsilon})\Psi + \bar{p}\Psi_y \\
= \frac{W_{yy}}{\bar{v^\varepsilon}} - \frac{1}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y W_{yy} + \frac{\gamma-1}{Rv^\varepsilon}(\bar{u_y^\varepsilon}\Psi - \frac{1}{2}\Psi_y^2)_y - \frac{\bar{\theta_y^\varepsilon}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y + \frac{1}{v^\varepsilon}\Psi_y\Psi_{yy} + \frac{\bar{u_y^\varepsilon}}{v^\varepsilon}\Psi_y \\
\quad + \bar{u^\varepsilon}Q_1 - Q_2 - \bar{u^\varepsilon}(q_5 - q_6) + q_{11} - q_{12}, \\
\Phi(y,0) = \Psi(y,0) = W(y,0) = 0,
\end{cases}
\tag{3.7}
$$

where $\bar{\varrho} = -(\bar{p_v} + \bar{u_y^\varepsilon}/\bar{v^\varepsilon}^2)$, and

$$
\begin{aligned}
Q_1 &= p(\bar{v^\varepsilon} + \Phi_y, \bar{\theta^\varepsilon} + W_y + \frac{\gamma-1}{R}(\bar{u_y^\varepsilon}\Psi - \frac{1}{2}\Psi_y^2)) - p(\bar{v^\varepsilon}, \bar{\theta^\varepsilon}) \\
&\quad - (\bar{p_v}\Phi_y + \bar{p_\theta}(W_y + \frac{\gamma-1}{R}(\bar{u_y^\varepsilon}\Psi - \frac{1}{2}\Psi_y^2))) \\
&\quad \text{satisfies } |Q_1| \le O(1)(\Phi_y^2 + W_y^2 + (\gamma-1)\bar{u_y^\varepsilon}^2\Psi^2 + \Psi_y^4), \\
Q_2 &= p(\bar{v^\varepsilon} + \Phi_y, \bar{\theta^\varepsilon} + W_y + \frac{\gamma-1}{R}(\bar{u_y^\varepsilon}\Psi - \frac{1}{2}\Psi_y^2))(\bar{u^\varepsilon} + \Psi_y) - \bar{p}\bar{u^\varepsilon} \\
&\quad - (\bar{u^\varepsilon}\bar{p_v}\Phi_y + \bar{p}\Psi_y + \bar{u^\varepsilon}\bar{p_\theta}(W_y + \frac{\gamma-1}{R}(\bar{u_y^\varepsilon}\Psi - \frac{1}{2}\Psi_y^2))) \\
&\quad \text{satisfies } |Q_2| \le O(1)(\Phi_y^2 + \Psi_y^2 + W_y^2 + (\gamma-1)\bar{u_y^\varepsilon}^2\Psi^2 + \Psi_y^4).
\end{aligned}
\tag{3.8}
$$

Then we only need to show that for suitably small ε, (3.7) has a unique "small" smooth solution up to T/ε. By the standard existence and uniqueness theory, and the continuous induction argument for hyperbolic-parabolic equations [5], it suffices to close the following a priori estimate

$$
N(\tau) \equiv \|(\Phi, \Psi, \frac{W}{\sqrt{\gamma-1}})(\cdot,\tau)\|_3 \le \delta,
\tag{3.9}
$$

where δ is a positive small constant depending on the initial data and the strength of the shock. In fact, we have the following result.

Proposition 3.1. *Suppose that the Cauchy problem (3.7) has a solution* (Φ, Ψ, W)
$\in C^1([0,\tau_0] : H^3(R^1))$ *for some* $\tau_0 \in (0, T/\varepsilon]$. *Then there exist positive constants* μ_1, ε_1 *and* C, *which are independent of* ε *and* τ_0, *such that if* $0 < \varepsilon < \varepsilon_1$ *and* $\delta + (\gamma-1)\mu \le \mu_1$, *then*

$$
\sup_{0\le\tau\le\tau_0} N(\tau)^2 + \int_0^{\tau_0}(\|\Phi_y(\cdot,\tau)\|_2^2 + \|(\Psi_y, W_y)(\cdot,\tau)\|_3^2)d\tau \le C\varepsilon^{(2n+1)\nu-4},
\tag{3.10}
$$

where ν *is defined in Section 2.4.*

The proof of Proposition 3.1 occupies the rest of this section. We separate it into two parts. In what follows, we use c to denote any positive constant which is independent of ε, y and τ; and $\bar{c}$ to denote any positive constant which is independent of ε and $(\gamma-1)\mu$. And we set $\varepsilon \le 1$.

Lemma 3.2. *Suppose that the conditions in Proposition 3.1 are satisfied. Then*

$$\begin{aligned}&\|(\Phi,\Psi,\frac{W}{\sqrt{\gamma-1}})(\cdot,\tau)\|_1^2+\int_0^\tau(\|\Phi_y(\cdot,\tau)\|^2+\|(\Psi_y,W_y)\|_1^2)d\tau\\&+\int_0^\tau\int mV_y(\Psi^2+\frac{W^2}{\gamma-1})dyd\tau\le c\varepsilon^{(2n+1)\nu-4},\end{aligned}\tag{3.11}$$

for all $\tau\in[0,\tau_0]$, *where the constant* c *is independent of* τ_0 *and* ε.

Proof. **Step 1** Multiplying $(3.7)_1$, $(3.7)_2$ and $(3.7)_3$ by Φ, $\frac{\Psi}{\bar{\varrho}}$ and $\frac{\bar{p_\theta}W}{\bar{\varrho}\bar{p}}$, respectively, then integrating over R^1, and adding the resulting equations, we obtain after integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\tau}\int(\Phi^2+\frac{1}{\bar{\varrho}}\Psi^2+\frac{R\bar{p_\theta}}{(\gamma-1)\bar{\varrho}\bar{p}}W^2)(y,\tau)dy\\&\quad+\int\{(\frac{\dot{s}}{2}(\frac{1}{\bar{\varrho}})_y+\frac{(\gamma-1)\bar{p_\theta}\bar{u^\varepsilon_y}}{R\bar{\varrho}})\Psi^2+(\frac{1}{\bar{v^\varepsilon}\bar{\varrho}})_y\Psi\Psi_y+\frac{1}{\bar{v^\varepsilon}\bar{\varrho}}\Psi_y^2\}dy\\&\quad+\int\{\frac{\dot{s}R}{2(\gamma-1)}(\frac{\bar{p_\theta}}{\bar{\varrho}\bar{p}})_yW^2+\frac{\bar{p_\theta}}{\bar{v^\varepsilon}\bar{\varrho}\bar{p}}W_y^2\}dy\\&=\int\{\frac{1}{2}(\frac{1}{\bar{\varrho}})_\tau\Psi^2-\frac{\bar{p_\theta}}{\bar{\varrho}\bar{p}}\bar{u^\varepsilon_\tau}\Psi W+\frac{R}{2(\gamma-1)}(\frac{\bar{p_\theta}}{\bar{\varrho}\bar{p}})_\tau W^2\}dy\\&\quad+\int\{(\frac{\bar{p_\theta}}{\bar{\varrho}})_y+\frac{\dot{s}\bar{p_\theta}\bar{u^\varepsilon_y}}{\bar{\varrho}\bar{p}}\}\Psi Wdy-\int\frac{\gamma-1}{R}(\frac{\bar{p_\theta}}{v^\varepsilon\bar{\varrho}\bar{p}})_y\bar{u^\varepsilon_y}\Psi Wdy\\&\quad-\int\{\frac{\bar{p_\theta}\bar{\theta^\varepsilon_y}}{\bar{v^\varepsilon}v^\varepsilon\bar{\varrho}\bar{p}}\Phi_yW-\frac{\bar{p_\theta}\bar{u^\varepsilon_y}}{v^\varepsilon\bar{\varrho}\bar{p}}W\Psi_y+\frac{\gamma-1}{R}\frac{\bar{p_\theta}\bar{u^\varepsilon_y}}{v^\varepsilon\bar{\varrho}\bar{p}}\Psi W_y+(\frac{\bar{p_\theta}}{\bar{v^\varepsilon}\bar{\varrho}\bar{p}})_yWW_y\}dy\\&\quad+\int\{(\frac{\bar{u^\varepsilon_y}}{\bar{v^\varepsilon}^2v^\varepsilon\bar{\varrho}}\Phi_y^2\Psi+\frac{\gamma-1}{2R}(\frac{\bar{p_\theta}}{\bar{\varrho}}\Psi+\frac{\bar{p_\theta}}{v^\varepsilon\bar{\varrho}\bar{p}}W_y+(\frac{\bar{p_\theta}}{v^\varepsilon\bar{\varrho}\bar{p}})_yW)\Psi_y^2\}dy\\&\quad-\int\{\frac{1}{\bar{v^\varepsilon}v^\varepsilon\bar{\varrho}}\Phi_y\Psi\Psi_{yy}-\frac{\bar{p_\theta}}{v^\varepsilon\bar{\varrho}\bar{p}}\Psi_y\Psi_{yy}W+\frac{\bar{p_\theta}}{\bar{v^\varepsilon}v^\varepsilon\bar{\varrho}\bar{p}}\Phi_yWW_{yy}\}dy\\&\quad-\int\{\frac{1}{\bar{\varrho}}Q_1\Psi-\frac{\bar{p_\theta}}{\bar{\varrho}\bar{p}}(\bar{u^\varepsilon}Q_1-Q_2)W\}dy\\&\quad+\int\{\frac{1}{\bar{\varrho}}(q_5-q_6)\Psi-\frac{\bar{p_\theta}}{\bar{\varrho}\bar{p}}(\bar{u^\varepsilon}(q_5-q_6)-(q_{11}-q_{12}))W\}dy.\end{aligned}\tag{3.12}$$

We denote the last two terms on the left by I_1 and I_2 respectively, and the terms on the right hand side above in order by $J_i, 1\le i\le 8$. Now we estimate them separately as follows.

First, Using Lemma 2.3, we have $\bar{v^\varepsilon}\bar{\varrho}=\bar{p}-\frac{\bar{u^\varepsilon_y}}{\bar{v^\varepsilon}}=\bar{p}-\frac{mU_y}{\bar{v^\varepsilon}}+O(1)\varepsilon>0$ for sufficiently small ε. Then it follows from Young's inequality that

$$I_1\ge(1-\eta_1)\int\frac{1}{\bar{v^\varepsilon}\bar{\varrho}}\Psi_y^2dy+\int\{\frac{\dot{s}}{2}(\frac{1}{\bar{\varrho}})_y+\frac{(\gamma-1)\bar{p_\theta}\bar{u^\varepsilon_y}}{R\bar{\varrho}}-\frac{1}{4\eta_1}\bar{v^\varepsilon}\bar{\varrho}((\frac{1}{\bar{v^\varepsilon}\bar{\varrho}})_y)^2\}\Psi^2dy$$

for some $\eta_1\in(0,1)$. Denote the second term by $\int z(S^\varepsilon)\Psi^2dy$. Then Due to (2.15), Lemma 2.3 and the fact

$$|\partial_y(V,U,\Theta)|=O(1)\varepsilon,\ \text{ on }\ |y|\ge\varepsilon^{\gamma-1},\tag{3.13}$$

we get

$$
\begin{aligned}
z(S^\varepsilon) &= \frac{\dot{s}}{2}(\frac{1}{\bar{\varrho}})_y + \frac{(\gamma-1)\bar{p_\theta}\bar{u_y^\varepsilon}}{R\bar{\varrho}} - \frac{1}{4\eta_1}\bar{v^\varepsilon}\bar{\varrho}((\frac{1}{\bar{v^\varepsilon}\bar{\varrho}})_y)^2 \\
&= -\frac{\dot{s}}{2}(\frac{\bar{v^\varepsilon}}{\bar{u_y^\varepsilon}/\bar{v^\varepsilon}-\bar{p}})_y - \frac{(\gamma-1)\bar{u_y^\varepsilon}}{\bar{u_y^\varepsilon}/\bar{v^\varepsilon}-\bar{p}} + \frac{1}{4\eta_1}(\bar{u_y^\varepsilon}/\bar{v^\varepsilon}-\bar{p})((\frac{1}{\bar{u_y^\varepsilon}/\bar{v^\varepsilon}-\bar{p}})_y)^2 \\
&= -\frac{\dot{s}}{2}m(\frac{V}{U_y/V-P})_y - \frac{(\gamma-1)mU_y}{U_y/V-P} + \frac{1}{4\eta_1}m(U_y/V-P)((\frac{1}{U_y/V-P})_y)^2 \\
&\quad + O(1)\varepsilon^\gamma mV_y + O(1)\varepsilon \\
&= m\{\frac{\dot{s}}{2}(\frac{V}{b_1-\dot{s}^2V})_y - \frac{(\gamma-1)\dot{s}V_y}{b_1-\dot{s}^2V} - \frac{1}{4\eta_1}(b_1-\dot{s}^2V)((\frac{1}{b_1-\dot{s}^2V})_y)^2\} \\
&\quad + O(1)\varepsilon^\gamma mV_y + O(1)\varepsilon \\
&= mV_y\frac{\dot{s}}{4(b_1-\dot{s}^2V)^2}\{2b_1 - 4(\gamma-1)(b_1-\dot{s}^2V) - \frac{1}{\eta_1}\frac{\dot{s}^3V_y}{b_1-\dot{s}^2V}\} \\
&\quad + O(1)\varepsilon^\gamma mV_y + O(1)\varepsilon \\
&= mV_y\frac{\dot{s}}{4(b_1-\dot{s}^2V)^2}\{[b_1 + \frac{\dot{s}^2R\Theta}{b_1-\dot{s}^2V} + (b_1-\dot{s}^2V) - 4(\gamma-1)(b_1-\dot{s}^2V)] \\
&\quad + [(\frac{1}{\eta_1}-1)(\frac{\dot{s}^2R\Theta}{b_1-\dot{s}^2V} - \dot{s}^2V)]\} + O(1)\varepsilon^\gamma mV_y + O(1)\varepsilon \\
&\equiv: mV_y\frac{\dot{s}}{4(b_1-\dot{s}^2V)^2}\{z_1(H)+z_2(H)\} + O(1)\varepsilon^\gamma mV_y + O(1)\varepsilon.
\end{aligned}
$$

As in [16], using the fact

$$
\dot{s}^2 \geq \frac{\gamma p_l}{v_r} - c(\gamma-1)(v_r-v_l) \geq \frac{\gamma p_l}{v_r} - \bar{c}(\gamma-1)\mu, \tag{3.14}
$$

which follows from the Rankine-Hugniot condition, we can obtain

$$
z_1(H) \geq (p_l + \dot{s}^2 v_l) + \frac{\dot{s}^2R\theta_r}{p_l} - 4(\gamma-1)p_l \geq \frac{(5-2\gamma)R\theta_l}{v_r} - c(\gamma-1)\mu.
$$

On the other hand,

$$
z_2(H) \geq (\frac{1}{\eta_1}-1)(\frac{\dot{s}^2R\theta_r}{p_l} - \dot{s}^2v_r) = -(\frac{1}{\eta_1}-1)\frac{\dot{s}^4v_r}{p_l}(v_r-v_l) \geq -(\gamma-1)\mu,
$$

if we choose $\dfrac{\dot{s}^4v_r}{\dot{s}^4v_r+(\gamma-1)p_l} \leq \eta_1 < 1$. So there is a constant $\underline{c} > 0$, such that

$$
I_1 \geq \underline{c}\int mV_y\Psi^2dy + \underline{c}\int \Psi_y^2dy - (\bar{c}(\gamma-1)\mu + c\varepsilon^\gamma)\int mV_y\Psi^2dy - c\varepsilon\|\Psi(\cdot,\tau)\|^2.
$$

Next we estimate I_2. Denote the first term of I_2 as $I_2^{(1)}$. Then we have

$$
\begin{aligned}
I_2^{(1)} = &-\frac{\dot{s}R}{2(\gamma-1)}\int m(\frac{V}{\Theta(U_y/V-P)})_yW^2dy \\
&+ O(1)\varepsilon^\nu\int mV_y\frac{W^2}{\gamma-1}dy + O(1)\varepsilon\|\frac{W}{\sqrt{\gamma-1}}(\cdot,\tau)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\dot{s}R}{2(\gamma-1)}\int m\frac{b_1\Theta V_y+(\dot{s}^2V-b_l)V\Theta_y}{\Theta^2(\dot{s}^2V-b_1)^2}W^2dy\\
&\quad + O(1)\varepsilon^{\nu}\int mV_y\frac{W^2}{\gamma-1}dy+O(1)\varepsilon\|\frac{W}{\sqrt{\gamma-1}}(\cdot,\tau)\|^2\\
&\geq \frac{\dot{s}R}{2(\gamma-1)}\int m\frac{b_1\Theta V_y+\dot{s}^2(V-v_l)V\Theta_y}{\Theta^2(\dot{s}^2V-b_1)^2}W^2dy\\
&\quad - c\varepsilon^{\nu}\int mV_y\frac{W^2}{\gamma-1}dy-c\varepsilon\|\frac{W}{\sqrt{\gamma-1}}(\cdot,\tau)\|^2\\
&\geq \int\frac{\dot{s}Rb_1}{2(\gamma-1)\Theta(\dot{s}^2V-b_1)^2}mV_yW^2dy\\
&\quad -(\bar{c}(\gamma-1)\mu+c\varepsilon^{\nu})\int mV_y\frac{W^2}{\gamma-1}dy+c\varepsilon\|\frac{W}{\sqrt{\gamma-1}}(\cdot,\tau)\|^2\\
&\geq \underline{c}\int mV_y\frac{W^2}{\gamma-1}dy,
\end{aligned}
$$

for some constant $\underline{c}>0$, provided that $(\gamma-1)\mu$ and ε are sufficiently small, where we have used (2.15)-(2.16) and Lemma 2.3. So

$$
I_2\geq \underline{c}\int mV_y\frac{W^2}{\gamma-1}dy+\underline{c}\int W_y^2dy,
$$

for some constant $\underline{c}>0$.

Now we estimate the terms $J_i, 1\leq i\leq 8$. First Lemma 2.3 gives

$$
J_1\leq c\varepsilon(\|\Psi(\cdot,\tau)\|^2+\|\frac{W}{\sqrt{\gamma-1}}(\cdot,\tau)\|^2).
$$

and

$$
\begin{aligned}
J_2&\leq\int m|R(\frac{1}{U_y/V-P})_y+\frac{\dot{s}VU_y}{\Theta(U_y/V-P)}||\Psi||W|dy\\
&\quad + c\varepsilon^{\nu}\int mV_y(\Psi^2+W^2)dy+c\varepsilon(\|\Psi(\cdot,\tau)\|^2+\|W(\cdot,\tau)\|^2)\\
&=\int\frac{m\dot{s}^2VV_y}{\Theta(\dot{s}^2V-b_1)^2}|(P-p_l)+\dot{s}^2(V-v_l)||\Psi||W|dy\\
&\quad + c\varepsilon^{\nu}\int mV_y(\Psi^2+W^2)dy+c\varepsilon(\|\Psi(\cdot,\tau)\|^2+\|W(\cdot,\tau)\|^2)\\
&\leq(\bar{c}\eta^{-1}(\gamma-1)\mu+c\varepsilon^{\nu})\int mV_y\Psi^2dy+(\eta+c\varepsilon^{\nu})\int mV_y\frac{W^2}{\gamma-1}dy\\
&\quad + c\varepsilon(\|\Psi(\cdot,\tau)\|^2+\|W(\cdot,\tau)\|^2),
\end{aligned}
$$

where $\eta>0$ is a constant to be determined later, which is different from the one in Theorem 1.2 and be temporarily used in this subsection. Using (2.15)-(2.16) and Lemma 2.3 again, one finds

$$
|\partial_y^lS^{\varepsilon}|^2\leq\bar{c}\mu mV_y+O(1)\varepsilon\ \ ,1\leq l\leq 2. \tag{3.15}
$$

Thus,

$$
J_3=-\frac{\gamma-1}{R}\int(\frac{\bar{p_\theta}}{\bar{v^\varepsilon}\,\bar{\varrho}\bar{p}})_yu_y^{\bar{\varepsilon}}\Psi Wdy-\frac{\gamma-1}{R}\int\frac{\bar{p_\theta}}{\bar{v^\varepsilon}v^\varepsilon\,\bar{\varrho}\bar{p}}\Phi_y(u_{yy}^{\bar{\varepsilon}}\Psi W+u_y^{\bar{\varepsilon}}(\Psi_yW+\Psi W_y))dy
$$

$$
\begin{aligned}
&\le c(\gamma-1)\int(\bar{v^\varepsilon_y}^2+\bar{u^\varepsilon_y}^2+\bar{u^\varepsilon_{yy}}^2+\bar{\theta^\varepsilon_y}^2)(\Psi^2+W^2)dy\\
&\quad+c(\gamma-1)(\|\Psi\|_{L^\infty}+\|W\|_{L^\infty})\|\Phi_y\|^2+c\|W\|_{L^\infty}\|\Psi_y\|^2+c\|\Psi\|_{L^\infty}\|W_y\|^2\\
&\le \bar{c}(\gamma-1)\mu\int mV_y(\Psi^2+\frac{W^2}{\gamma-1})dy+c\delta(\|\Phi_y(\cdot,\tau)\|^2+\|\Psi_y(\cdot,\tau)\|^2+\|W_y(\cdot,\tau)\|^2)\\
&\quad+c\varepsilon(\|\Psi(\cdot,\tau)\|^2+\|W(\cdot,\tau)\|^2).
\end{aligned}
$$

Noticing that $\bar{\varrho}>0$ and $\bar{p_v}<0$ for sufficiently small ε, J_4 can be estimated as

$$
\begin{aligned}
J_4 \le& -\eta\int\bar{p_v}\Phi_y^2dy+\frac{1}{4}\underline{c}\int(\Psi_y^2+W_y^2)dy\\
&+c(\gamma-1)\int\bar{u^\varepsilon_y}^2\Psi^2dy+c\int(\eta^{-1}\bar{\theta^\varepsilon_y}^2+\bar{u^\varepsilon_y}^2+\bar{v^\varepsilon_y}^2+\bar{u^\varepsilon_{yy}}^2+\bar{\theta^\varepsilon_y}^2)W^2)dy\\
\le& -\eta\int\bar{p_v}\Phi_y^2dy+\frac{1}{4}\underline{c}\int(\Psi_y^2+W_y^2)dy\\
&+\bar{c}(\gamma-1)\mu\{\int mV_y\Psi^2dy+(\eta^{-1}+1)\int mV_y\frac{W^2}{\gamma-1}dy\}\\
&+c\varepsilon(\|\Psi(\cdot,\tau)\|^2+(\eta^{-1}+1)\|W(\cdot,\tau)\|^2),
\end{aligned}
$$

Continuing, we compute that

$$
\begin{aligned}
J_5 =& \int\frac{\bar{u^\varepsilon_y}}{\bar{v^\varepsilon}^2v^\varepsilon\bar{\varrho}}\Psi\Phi_y^2dy+\frac{\gamma-1}{2R}\int\frac{\bar{p_\theta}}{\bar{\varrho}}\Psi\Psi_y^2dy-\frac{\gamma-1}{R}\int\frac{\bar{p_\theta}}{v^\varepsilon\bar{\varrho}\bar{p}}W\Psi_y\Psi_{yy}dy\\
\le& c\|\Psi\|_{L^\infty}\|\Phi_y\|^2+c(\|\Psi\|_{L^\infty}+\|W\|_{L^\infty})\|\Psi_y\|^2+c\|W\|_{L^\infty}\|\Psi_{yy}\|^2\\
\le& c\delta(\|\Phi_y\|^2+\|\Psi_y\|^2+\|\Psi_{yy}\|^2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J_6 \le& c(\|\Psi\|_{L^\infty}+\|W\|_{L^\infty})(\|\Phi_y\|^2+\|\Psi_{yy}\|^2)+c\|W\|_{L^\infty}(\|\Psi_y\|^2+\|W_{yy}\|^2)\\
\le& c\delta(\|\Phi_y\|^2+\|\Psi_y\|^2+\|\Psi_{yy}\|^2+\|W_{yy}\|^2).
\end{aligned}
$$

In view of (3.8), we get

$$
\begin{aligned}
J_7 \le& c\int(\Phi_y^2+(\gamma-1)\bar{u^\varepsilon_y}^2\Psi^2+\Psi_y^4+W_y^2)|\Psi|dy\\
&+(\Phi_y^2+\Psi_y^2+(\gamma-1)\bar{u^\varepsilon_y}^2\Psi^2+\Psi_y^4+W_y^2)|W|dy\\
\le& c(\|\Psi\|_{L^\infty}+\|W\|_{L^\infty})\{\|\Phi_y\|^2+(1+\|\Psi_y\|_{L^\infty}^2)\|\Psi_y\|^2+\|W_y\|^2\}\\
&+c\int(\gamma-1)|\bar{u^\varepsilon_y}|^2\Psi^2dy\\
\le& c\delta(\|\Phi_y\|^2+\|\Psi_y\|^2+\|W_y\|^2)+\bar{c}(\gamma-1)\mu\int mV_y\Psi^2dy+c\varepsilon\|\Psi(\cdot,\tau)\|^2,
\end{aligned}
$$

provided that $\|\Psi(\cdot,\tau)\|_2$ is bounded. Finally, Young's inequality and Lemma 2.2 lead to

$$
\begin{aligned}
J_8 \le& c\varepsilon(\|\Psi\|^2+\|W\|^2)+c\varepsilon^{-1}\int(q_5^2+q_6^2+q_{11}^2+q_{12}^2)dy\\
\le& c\varepsilon(\|\Psi\|^2+\|W\|^2)+c\varepsilon^{(2n+1)\nu-3}.
\end{aligned}
$$

Collecting all the estimates we have obtained, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\int(\Phi^2+\frac{1}{\bar{\varrho}}\Psi^2+\frac{R\bar{p_\theta}}{(\gamma-1)\bar{\varrho}\bar{p}}W^2)(y,\tau)dy+\underline{c}\int(\Psi_y^2+W_y^2)dy\\
&\le-\eta\int\bar{p_v}\Phi_y^2dy+c\delta\|\Phi_y\|^2+\frac{1}{4}\underline{c}\int(\Psi_y^2+W_y^2)dy+c\delta(\|\Psi_y\|^2+\|W_y\|^2)\\
&\quad+(-\underline{c}+\bar{c}(\eta^{-1}+1)(\gamma-1)\mu+c\varepsilon^\nu)\int mV_y(\Psi^2+\frac{W^2}{\gamma-1})dy+\eta\int mV_y\frac{W^2}{\gamma-1}dy\\
&\quad+c\delta(\|\Psi_{yy}\|^2+\|W_{yy}\|^2)+c(\eta^{-1}+1)\varepsilon(\|\Psi(\cdot,\tau)\|^2+\|\frac{W}{\sqrt{\gamma-1}}(\cdot,\tau)\|^2)\\
&\quad+c\varepsilon^{(2n+1)\nu-3}.
\end{aligned}$$

By choosing δ sufficiently small, we conclude that

$$\begin{aligned}
&\frac{d}{d\tau}\int(\Phi^2+\frac{1}{\bar{\varrho}}\Psi^2+\frac{R\bar{p_\theta}}{(\gamma-1)\bar{\varrho}\bar{p}}W^2)(y,\tau)dy+\underline{c}\int(\Psi_y^2+W_y^2)dy\\
&+\eta\int\bar{p_v}\Phi_y^2dy+2\underline{c}\int mV_y(\Psi^2+\frac{W^2}{\gamma-1})dy\\
&\le(\bar{c}(\eta^{-1}+1)(\gamma-1)\mu+c\varepsilon^\nu)\int mV_y(\Psi^2+\frac{W^2}{\gamma-1})dy+\eta\int mV_y\frac{W^2}{\gamma-1}dy\\
&\quad+c\delta(\|\Psi_{yy}\|^2+\|W_{yy}\|^2)+c(\eta^{-1}+1)\varepsilon(\|\Psi(\cdot,\tau)\|^2+\|\frac{W}{\sqrt{\gamma-1}}(\cdot,\tau)\|^2)\\
&\quad+c\varepsilon^{(2n+1)\nu-3}.
\end{aligned}\tag{3.16}$$

Step 2 We first rewrite (3.7) as

$$\begin{cases}
\Phi_\tau-\dot{s}(\varepsilon\tau)\Phi_y-\Psi_y=0,\\
\Psi_\tau-\dot{s}(\varepsilon\tau)\Psi_y+p(v^\varepsilon,\theta^\varepsilon)-p(\bar{v^\varepsilon},\bar{\theta^\varepsilon})\\
=\frac{\Psi_{yy}}{\bar{v^\varepsilon}}-\frac{1}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y\Psi_{yy}-\frac{\bar{u_y^\varepsilon}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y+q_5-q_6,\\
\frac{R}{\gamma-1}(W_\tau-\dot{s}(\varepsilon\tau)W_y)+(\bar{u_\tau^\varepsilon}-\dot{s}(\varepsilon\tau)\bar{u_y^\varepsilon})\Psi+p(v^\varepsilon,\theta^\varepsilon)\Psi_y\\
=\frac{W_{yy}}{\bar{v^\varepsilon}}-\frac{1}{\bar{v^\varepsilon}v^\varepsilon}\Phi_yW_{yy}+\frac{\gamma-1}{Rv^\varepsilon}(\bar{u_y^\varepsilon}\Psi-\frac{1}{2}\Psi_y^2)_y-\frac{\bar{\theta_y^\varepsilon}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y+\frac{1}{v^\varepsilon}\Psi_y\Psi_{yy}+\frac{\bar{u_y^\varepsilon}}{v^\varepsilon}\Psi_y\\
\quad-\bar{u^\varepsilon}(q_5-q_6)+q_{11}-q_{12},\\
\Phi(y,0)=\Psi(y,0)=W(y,0)=0.
\end{cases}\tag{3.17}$$

Then differentiating $(3.17)_{2,3}$ with respect to y, multiplying both sides of the resulting equations by Ψ_y, W_y, respectively, then summing them up, and integrating over R^1, we obtain after integration by parts and using Young's inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}(\|\Psi_y\|^2+\frac{R}{\gamma-1}\|W_y\|^2)+\int\frac{1}{\bar{v^\varepsilon}}(\Psi_{yy}^2+W_{yy}^2)dy\\
&=\int\frac{\Phi_y}{\bar{v^\varepsilon}v^\varepsilon}(\Psi_{yy}^2+W_{yy}^2)dy\\
&\quad+\int\Psi_{yy}\{p(v^\varepsilon,\theta^\varepsilon)-p(\bar{v^\varepsilon},\bar{\theta^\varepsilon})+\frac{\bar{u_y^\varepsilon}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y-q_5+q_6\}dy\\
&\quad+\int W_{yy}\{(\bar{u_\tau^\varepsilon}-\dot{s}(\varepsilon\tau)\bar{u_y^\varepsilon})\Psi+p(v^\varepsilon,\theta^\varepsilon)\Psi_y-\frac{\gamma-1}{Rv^\varepsilon}(\bar{u_{yy}^\varepsilon}\Psi+\bar{u_y^\varepsilon}\Psi_y-\Psi_y\Psi_{yy})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\bar{\theta}^\varepsilon_y}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y-\frac{1}{v^\varepsilon}\Psi_y\Psi_{yy}-\frac{\bar{u}^\varepsilon_y}{v^\varepsilon}\Psi_y+\bar{u^\varepsilon}(q_5-q_6)-(q_{11}-q_{12})\}dy\\
\le{}& c(\|\Phi_y\|_{L^\infty}+\|\Psi_y\|_{L^\infty})\int\frac{1}{\bar{v^\varepsilon}}(\Psi_{yy}^2+W_{yy}^2)dy+\frac{1}{4}\int\frac{1}{\bar{v^\varepsilon}}(\Psi_{yy}^2+W_{yy}^2)dy\\
&+c\{\|\Phi_y\|^2+(1+\|\Psi_y\|^2_{L^\infty})\|\Psi_y\|^2+\|W_y\|^2\}+c\varepsilon\|\Psi(\cdot,\tau)\|^2\\
&+c\int(\bar{u^\varepsilon_y}^2+\bar{u^\varepsilon_{yy}}^2)\Psi^2dy+\int(q_5^2+q_6^2+q_{11}^2+q_{12}^2)dy\\
\le{}&(c\delta+\frac{1}{4})\int\frac{1}{\bar{v^\varepsilon}}(\Psi_{yy}^2+W_{yy}^2)dy+c(\|\Phi_y\|^2+\int mV_y\Psi^2dy+\|\Psi_y\|^2+\|W_y\|^2)\\
&+c\varepsilon\|\Psi(\cdot,\tau)\|^2+c\varepsilon^{(2n+1)\nu-3},
\end{aligned}
$$

provided that $\|(\Phi_y,\Psi_y,W_y)\|_{L^\infty}$ is bounded, where we have used (3.1), (3.3), (3.6), (3.15) and (2.60). By taking δ sufficiently small, we arrive at

$$
\begin{aligned}
&\frac{d}{d\tau}(\|\Psi_y\|^2+\frac{R}{\gamma-1}\|W_y\|^2)+\int\frac{1}{\bar{v^\varepsilon}}(\Psi_{yy}^2+W_{yy}^2)dy\\
&-c(\|\Phi_y\|^2+\int mV_y\Psi^2dy+\|\Psi_y\|^2+\|W_y\|^2)\\
&\le c\varepsilon\|\Psi(\cdot,\tau)\|^2+c\varepsilon^{(2n+1)\nu-3}.
\end{aligned}\tag{3.18}
$$

We denote the constant c on the left by c_1.

Step 3 Noting that $\dfrac{d}{d\tau}\|\Phi_y\|^2$ is not included in (3.18), we need to estimate it separately. Multiply both sides of $(3.7)_2$ by Φ_y and integrate over R^1 to obtain

$$
\begin{aligned}
&\int\{\Phi_y\Psi_\tau-\dot{s}\Phi_y\Psi_y+\bar{p_v}\Phi_y^2+\bar{p_\theta}\Phi_y(W_y+\frac{\gamma-1}{R}(\bar{u^\varepsilon_y}\Psi-\frac{1}{2}\Psi_y^2))\}dy\\
={}&\int\frac{\Phi_y\Psi_{yy}}{\bar{v^\varepsilon}}dy-\int\frac{1}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y^2\Psi_{yy}dy-\int\frac{\bar{u^\varepsilon_y}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y^2dy+\int\Phi_y(-Q_1+q_5-q_6)dy.
\end{aligned}\tag{3.19}
$$

The first term on the left can be written as

$$
\begin{aligned}
\int\Phi_y\Psi_\tau dy&=\frac{d}{d\tau}\int\Phi_y\Psi dy-\Phi_{y\tau}\Psi dy\\
&=-\frac{d}{d\tau}\int\Phi\Psi_ydy+\int\Psi_y^2dy+\dot{s}\int\Phi_y\Psi_ydy,
\end{aligned}
$$

and the first term on the right reads

$$
\begin{aligned}
\int\frac{\Phi_y\Psi_{yy}}{\bar{v^\varepsilon}}dy&=\int\frac{1}{\bar{v^\varepsilon}}\Phi_y(\Phi_{y\tau}-\dot{s}\Phi_{yy})dy\\
&=\frac{1}{2}\frac{d}{d\tau}\int\frac{\Phi_y^2}{\bar{v^\varepsilon}}dy+\frac{1}{2}\int\frac{\Phi_y^2}{\bar{v^\varepsilon}^2}(\bar{v^\varepsilon_\tau}-\dot{s}\bar{v^\varepsilon_y})dy\\
&=\frac{1}{2}\frac{d}{d\tau}\int\frac{\Phi_y^2}{\bar{v^\varepsilon}}dy+\frac{1}{2}\int\frac{\bar{u^\varepsilon_y}}{\bar{v^\varepsilon}^2}\Phi_y^2dy,
\end{aligned}
$$

where we have used $(3.7)_1$ and $(2.49)_1$. Substituting them into (3.19), we get

$$
\frac{d}{d\tau}\int(\frac{1}{2}\frac{\Phi_y^2}{\bar{v^\varepsilon}}+\Phi\Psi_y)dy=\int\bar{p_v}\Phi_y^2dy+\int\bar{p_\theta}\Phi_y(W_y+\frac{\gamma-1}{R}(\bar{u^\varepsilon_y}\Psi-\frac{1}{2}\Psi_y^2))dy
$$

$$+\int \Psi_y^2 dy + \int \frac{1}{\bar{v^\varepsilon} v^\varepsilon}\Phi_y^2\Psi_{yy}dy + \int (\frac{1}{\bar{v^\varepsilon} v^\varepsilon} - \frac{1}{2\bar{v^\varepsilon}^2})\bar{u^\varepsilon_y}\Phi_y^2 dy$$
$$+\int \Phi_y Q_1 dy - \int \Phi_y(q_5 - q_6)dy.$$

We denote the second term on the right by ω. By Young's inequality, one finds

$$\begin{aligned}\omega &\le -\frac{1}{8}\int \bar{p_v}\Phi_y^2 dy + c(\|W_y\|^2 + (\gamma-1)\int \bar{u^\varepsilon_y}^2\Psi^2 dy + \|\Psi_y\|^2_{L^\infty}\|\Psi_y\|^2)\\ &\le -\frac{1}{8}\int \bar{p_v}\Phi_y^2 dy + c(\|\Psi_y\|^2 + \|W_y\|^2) + \bar{c}(\gamma-1)\mu\int mV_y\Psi^2 dy + c\varepsilon\|\Psi(\cdot,\tau)\|^2,\end{aligned}$$

provided that $\|\Psi_y\|_{L^\infty}$ is bounded. Similarly,

$$\begin{aligned}\int \frac{1}{\bar{v^\varepsilon} v^\varepsilon}\Phi_y^2\Psi_{yy}dy &\le -\frac{1}{8}\int \bar{p_v}\Phi_y^2 dy + c\|\Phi_y\|^2_{L^\infty}\|\Psi_{yy}\|^2\\ &\le -\frac{1}{8}\int \bar{p_v}\Phi_y^2 dy + c\delta\|\Psi_{yy}\|^2.\end{aligned}$$

Noting that $\bar{u^\varepsilon_y} = mU_y + O(1)\varepsilon$ and $U_y < 0$, we get

$$\int (\frac{1}{\bar{v^\varepsilon} v^\varepsilon} - \frac{1}{2\bar{v^\varepsilon}^2})\bar{u^\varepsilon_y}\Phi_y^2 dy = \int \frac{\bar{u^\varepsilon_y}}{2\bar{v^\varepsilon}^2}\Phi_y^2 dy - \int \frac{\Phi_y^3}{\bar{v^\varepsilon}^2 v^\varepsilon}dy \le C(\varepsilon+\delta)\|\Phi_y\|^2.$$

Due to (3.8),

$$\begin{aligned}\int \Phi_y Q_1 dy &\le c\int |\Phi_y|(\Phi_y^2 + (\gamma-1)\bar{u^\varepsilon_y}^2\Psi^2 + \Psi_y^4 + W_y^2)dy\\ &\le c\delta(\|\Phi_y\|^2 + \|\Psi_y\|^2 + \|W_y\|^2) + \bar{c}(\gamma-1)\mu\int mV_y\Psi^2 dy + c\varepsilon\|\Psi(\cdot,\tau)\|^2.\end{aligned}$$

Lemma 2.2 and Young's inequality yield

$$-\int \Phi_y(q_5 - q_6)dy \le \varepsilon\|\Phi_y\|^2 + \frac{1}{4\varepsilon}\int (q_5^2 + q_6^2)dy \le \varepsilon\|\Phi_y\|^2 + c\varepsilon^{(2n+1)\nu-3}.$$

Collecting all the estimates we have obtained and taking δ to be sufficiently small, we have

$$\begin{aligned}&\frac{d}{d\tau}\int (\frac{\Phi_y^2}{\bar{v^\varepsilon}} + 2\Phi\Psi_y)dy - \int \bar{p_v}\Phi_y^2 dy - c(\|\Psi_y\|^2 + \|W_y\|^2)\\ &\le c\varepsilon\|\Phi_y\|^2 + c\delta\|\Psi_{yy}\|^2 + \bar{c}(\gamma-1)\mu\int mV_y\Psi^2 dy + c\varepsilon\|\Psi(\cdot,\tau)\|^2 + c\varepsilon^{(2n+1)\nu-3}.\end{aligned} \tag{3.20}$$

We denote the constant c on the left by c_2.
Step 4 Choosing suitable constants $\beta_1, \beta_2 > 0$ and adding up the three inequalities (3.16), β_1(3.18), β_2(3.20), we obtain the following inequality

$$\begin{aligned}&\frac{d}{d\tau}\int \{\Phi^2 + \frac{1}{\bar{\varrho}}\Psi^2 + \beta_2\frac{\Phi_y^2}{\bar{v^\varepsilon}} + 2\beta_2\Phi\Psi_y + \beta_1\Psi_y^2 + \frac{R}{\gamma-1}(\frac{\bar{p_\theta}}{\bar{\varrho}\bar{p}}W^2 + \beta_1 W_y^2)\}dy\\ &-\int ((\beta_2-\eta)\bar{p_v} + c_1\beta_1)\Phi_y^2 dy + \int (\underline{c} - c_1\beta_1 - c_2\beta_2)(\Psi_y^2 + W_y^2)dy\end{aligned}$$

$$
\begin{aligned}
&+\beta_1\int\frac{1}{\bar{v^\varepsilon}}(\Psi_{yy}^2+W_{yy}^2)dy+(2\underline{c}-c_1\beta_1)\int mV_y(\Psi^2+\frac{W^2}{\gamma-1})dy\\
&\le c\beta_1\varepsilon\|\Phi_y\|^2+c\delta(\beta_2+1)\|\Psi_{yy}\|^2+c\delta\|W_{yy}\|^2\\
&\quad+(\bar{c}(\eta^{-1}+\beta_2+1)(\gamma-1)\mu+c\varepsilon^\nu)\int mV_y(\Psi^2+\frac{W^2}{\gamma-1})dy+\eta\int mV_y\frac{W^2}{\gamma-1}dy\\
&\quad+c(\eta^{-1}+\beta_1+\beta_2+1)\varepsilon(\|\Psi(\cdot,\tau)\|^2+\|\frac{W}{\sqrt{\gamma-1}}(\cdot,\tau)\|^2)+c\varepsilon^{(2n+1)\nu-3}.
\end{aligned}
$$

To get desired signs, we first choose $\beta_1=2\beta_2^2,\eta=\frac{\beta_2}{4}$, such that

$$\Phi^2+2\beta_2\Phi\Psi_y+\beta_1\Psi_y^2\ge C_{\beta_2}(\Phi^2+\Psi_y^2),$$

and then choose β_2 satisfying

$$c_1\beta_1+c_2\beta_2\le\frac{1}{2}\underline{c}\ \text{ and }\ \beta_2\le\min\{-\frac{\bar{p_v}}{4c_1},\underline{c}\}.$$

Finally, we choose $\delta,(\gamma-1)\mu$ and ε so small that

$$c\delta(\beta_2+1)\le\frac{\beta_1}{2\bar{v^\varepsilon}}\ \text{ and }\ \bar{c}(\eta^{-1}+\beta_2+1)(\gamma-1)\mu+c\varepsilon^\nu\le\frac{1}{2}\underline{c}.$$

With these constants at hand, it follows from Gronwall type inequality that

$$
\begin{aligned}
&\|(\Phi,\Psi,\frac{W}{\sqrt{\gamma-1}})(\cdot,\tau)\|_1^2+\int_0^\tau(\|\Phi_y(\cdot,\tau)\|^2+\|(\Psi_y,W_y)\|_1^2)d\tau\\
&+\int_0^\tau\int mV_y(\Psi^2+\frac{W^2}{\gamma-1})dyd\tau\le c\varepsilon^{(2n+1)\nu-4}.
\end{aligned}
$$

This completes the proof of Lemma 3.2.

To finish the proof of Proposition 3.1, we need to establish the estimates on the higher derivatives of (Φ,Ψ,W). This is given by the following Lemma.

Lemma 3.3. *Suppose the conditions in Proposition 3.1 are satisfied. Then*

$$\|(\partial_y^2\Phi,\partial_y^2\Psi,\frac{\partial_y^2W}{\gamma-1})(\cdot,\tau)\|_1^2+\int_0^\tau(\|\partial_y^2\Phi(\cdot,\tau)\|_1^2+\|(\partial_y^3\Psi,\partial_y^3W)\|_1^2)d\tau\le c\varepsilon^{(2n+1)\nu-4}. \tag{3.21}$$

with some constant c independent of τ_0 and ε.

Proof. **Step 1** First we rewrite $(3.7)_2$ as

$$\Psi_\tau-\dot{s}(\varepsilon\tau)\Psi_y+\chi_1\Phi_y+\bar{p_\theta}\{W_y+\frac{\gamma-1}{R}(\bar{u_y^\varepsilon}\Psi-\frac{1}{2}\Psi_y^2)\}=\frac{\Psi_{yy}}{v^\varepsilon}-Q_1+q_5-q_6, \tag{3.22}$$

where $\chi_1=\bar{p_v}+\frac{\bar{u_y^\varepsilon}}{\bar{v^\varepsilon}v^\varepsilon}$. Applying ∂_y^l to (3.22), multiplying both sides of the resulting equation by $\partial_y^{l+1}\Phi$, and integrating over $R^1\times[0,\tau]$, we obtain

$$
\begin{aligned}
&\int_0^\tau\int(\partial_y^{l+1}\Phi\partial_y^l\Psi_\tau-\dot{s}\partial_y^{l+1}\Phi\partial_y^{l+1}\Psi)dyd\tau\\
&+\int_0^\tau\int\partial_y^{l+1}\Phi\partial_y^l\{\chi_1\Phi_y+\bar{p_\theta}(W_y+\frac{\gamma-1}{R}(\bar{u_y^\varepsilon}\Psi-\frac{1}{2}\Psi_y^2))\}dyd\tau
\end{aligned}
$$

$$= \int_0^\tau \int \partial_y^{l+1}\Phi \partial_y^l (\frac{\Psi_{yy}}{v^\varepsilon}) dy d\tau - \int_0^\tau \int \partial_y^{l+1}\Phi \partial_y^l (Q_1 - q_5 + q_6) dy d\tau. \tag{3.23}$$

Similar to the methods we have used, with the help of $(3.7)_1$ and $(2.49)_1$, we have

$$\int_0^\tau \int \partial_y^{l+1}\Phi \partial_y^l \Psi_\tau dy d\tau = \int \partial_y^{l+1}\Phi \partial_y^l \Psi dy + \int_0^\tau \int (\partial_y^{l+1}\Psi)^2 dy d\tau + \int_0^\tau \int \dot{s} \partial_y^{l+1}\Phi \partial_y^{l+1}\Psi dy d\tau,$$

and

$$\begin{aligned}
&\int_0^\tau \int \int \partial_y^{l+1}\Phi \partial_y^l (\frac{\Psi_{yy}}{v^\varepsilon}) dy d\tau \\
&= \int_0^\tau \int \frac{1}{v^\varepsilon} \partial_y^{l+1}\Phi \partial_y^{l+2}\Psi dy d\tau + \int_0^\tau \int \partial_y^{l+1}\Phi [\partial_y^l, \frac{1}{v^\varepsilon}] \Psi_{yy} dy d\tau \\
&= \int_0^\tau \int \frac{1}{v^\varepsilon} \partial_y^{l+1}\Phi (\partial_y^{l+1}\Phi_\tau - \dot{s}\partial_y^{l+2}\Phi) dy d\tau + \int_0^\tau \int \partial_y^{l+1}\Phi [\partial_y^l, \frac{1}{v^\varepsilon}] \Psi_{yy} dy d\tau \\
&= \frac{1}{2} \int \frac{(\partial_y^{l+1}\Phi)^2}{v^\varepsilon} dy + \frac{1}{2} \int_0^\tau \int \frac{v_\tau^\varepsilon - \dot{s} v_y^\varepsilon}{(v^\varepsilon)^2} (\partial_y^{l+1}\Phi)^2 dy d\tau \\
&\quad + \int_0^\tau \int \partial_y^{l+1}\Phi [\partial_y^l, \frac{1}{v^\varepsilon}] \Psi_{yy} dy d\tau \\
&= \frac{1}{2} \int \frac{(\partial_y^{l+1}\Phi)^2}{v^\varepsilon} dy + \frac{1}{2} \int_0^\tau \int \frac{\Psi_{yy}}{(v^\varepsilon)^2} (\partial_y^{l+1}\Phi)^2 dy d\tau + \frac{1}{2} \int_0^\tau \int \frac{\bar{u_y^\varepsilon}}{(v^\varepsilon)^2} (\partial_y^{l+1}\Phi)^2 dy d\tau \\
&\quad + \int_0^\tau \int \partial_y^{l+1}\Phi [\partial_y^l, \frac{1}{v^\varepsilon}] \Psi_{yy} dy d\tau,
\end{aligned}$$

where $[\cdot,\cdot]\cdot$ denotes the commutator. Insert them into (3.23) to get

$$\begin{aligned}
\frac{1}{2} \int \frac{(\partial_y^{l+1}\Phi)^2}{v^\varepsilon} dy &= \int \partial_y^{l+1}\Phi \partial_y^l \Psi dy + \int_0^\tau \int (\partial_y^{l+1}\Psi)^2 dy d\tau \\
&\quad + \int_0^\tau \int \partial_y^{l+1}\Phi \partial_y^l (\chi_1 \Phi_y) dy d\tau - \frac{1}{2} \int_0^\tau \int \frac{\bar{u_y^\varepsilon}}{(v^\varepsilon)^2} (\partial_y^{l+1}\Phi)^2 dy d\tau \\
&\quad - \frac{1}{2} \int_0^\tau \int \frac{\Psi_{yy}}{(v^\varepsilon)^2} (\partial_y^{l+1}\Phi)^2 dy d\tau - \int_0^\tau \int \partial_y^{l+1}\Phi [\partial_y^l, \frac{1}{v^\varepsilon}] \Psi_{yy} dy d\tau \\
&\quad + \int_0^\tau \int \partial_y^{l+1}\Phi \partial_y^l \{\bar{p_\theta} (W_y + \frac{\gamma-1}{R} (\bar{u_y^\varepsilon}\Psi - \frac{1}{2}\Psi_y^2))\} dy d\tau \\
&\quad + \int_0^\tau \int \partial_y^{l+1}\Phi \partial_y^l (Q_1 - q_5 + q_6) dy d\tau.
\end{aligned} \tag{3.24}$$

In the case $l = 1$, (3.24) reads

$$\begin{aligned}
\frac{1}{2} \int \frac{\Phi_{yy}^2}{v^\varepsilon} dy &= \int \Phi_{yy} \Psi_y dy + \int_0^\tau \int \Psi_{yy}^2 dy d\tau \\
&\quad + \int_0^\tau \int (\chi_1 - \frac{\bar{u_y^\varepsilon}}{2(v^\varepsilon)^2}) \Phi_{yy}^2 dy d\tau + \int_0^\tau \int \chi_{1y} \Phi_{yy} \Phi_y dy d\tau \\
&\quad + \frac{1}{2} \int_0^\tau \int \frac{1}{(v^\varepsilon)^2} \Psi_{yy} \Phi_{yy}^2 dy d\tau + \int_0^\tau \int \frac{\bar{v_y^\varepsilon}}{(v^\varepsilon)^2} \Phi_{yy} \Psi_{yy} dy d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\tau\int \Phi_{yy}\{\bar{p_\theta}(W_y+\frac{\gamma-1}{R}(\bar{u^\varepsilon_y}\Psi-\frac{1}{2}\Psi_y^2))\}_y dyd\tau\\
&+\int \Phi_{yy}(Q_{1y}-q_{5y}+q_{6y})dyd\tau. \qquad (3.25)
\end{aligned}
$$

Next we use the result we have obtained in Lemma 3.2 to estimate each term on the right hand side of (3.25). First, by Young's inequality,

$$
\int \Phi_{yy}\Psi_y dy \le \frac{1}{4}\int \frac{\Phi_{yy}^2}{v^\varepsilon}dy + c\|\Psi_y\|^2 \le \frac{1}{4}\int \frac{\Phi_{yy}^2}{v^\varepsilon}dy + c\varepsilon^{(2n+1)\nu-4}.
$$

By the definition of χ_1, we have

$$
\chi_{1y}=O(1)(1+\Phi_{yy}),\ \ \chi_{1yy}=O(1)(1+\Phi_{yy}+\Phi_{yy}^2+\Phi_{yyy}). \qquad (3.26)
$$

It follows from this and the facts $\bar{u^\varepsilon_y}=mU_y+O(1)\varepsilon$ and $U_y<0$ that

$$
\begin{aligned}
&\int_0^\tau\int(\chi_1-\frac{\bar{u^\varepsilon_y}}{2(v^\varepsilon)^2})\Phi_{yy}^2 dyd\tau\\
&=\int_0^\tau\int \bar{p_v}\Phi_{yy}^2 dyd\tau+\int_0^\tau\int(\frac{\bar{u^\varepsilon_y}\Phi_y}{\bar{v^\varepsilon}(v^\varepsilon)^2}+\frac{\bar{u^\varepsilon_y}}{2(v^\varepsilon)^2})\Phi_{yy}^2 dyd\tau\\
&\le\int_0^\tau\int \bar{p_v}\Phi_{yy}^2 dyd\tau + c(\delta+\varepsilon)\int_0^\tau\|\Phi_{yy}\|^2 d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^\tau\int \chi_{1y}\Phi_y\Phi_{yy}dyd\tau &\le c\delta\int_0^\tau\|\Phi_{yy}\|^2 d\tau-\frac{1}{8}\int_0^\tau\int\bar{p_v}\Phi_{yy}^2 dyd\tau + c\int_0^\tau\|\Phi_y\|^2 d\tau\\
&\le c\delta\int_0^\tau\|\Phi_{yy}\|^2 d\tau-\frac{1}{8}\int_0^\tau\int\bar{p_v}\Phi_{yy}^2 dyd\tau + c\varepsilon^{(2n+1)\nu-4}.
\end{aligned}
$$

By Sobolev's inequality and Young's inequality, the remaining terms on the right are estimated as follows.

$$
\frac{1}{2}\int_0^\tau\int\frac{1}{(v^\varepsilon)^2}\Psi_{yy}\Phi_{yy}^2 dyd\tau \le c\delta\int_0^\tau\|\Phi_{yy}\|^2 d\tau,
$$

and

$$
\begin{aligned}
\int_0^\tau\int\frac{\bar{v^\varepsilon_y}}{(v^\varepsilon)^2}\Phi_{yy}\Psi_{yy}dyd\tau &\le -\frac{1}{8}\int_0^\tau\int\bar{p_v}\Phi_{yy}^2 dyd\tau + c\int_0^\tau\|\Psi_{yy}\|^2 d\tau\\
&\le -\frac{1}{8}\int_0^\tau\int\bar{p_v}\Phi_{yy}^2 dyd\tau + c\varepsilon^{(2n+1)\nu-4}.
\end{aligned}
$$

Continuing, using Lemma 2.3 and Lemma 3.2, we obtain

$$
\begin{aligned}
&\int_0^\tau\int \Phi_{yy}\{\bar{p_\theta}(W_y+\frac{\gamma-1}{R}(\bar{u^\varepsilon_y}\Psi-\frac{1}{2}\Psi_y^2))\}_y dyd\tau\\
&\le-\frac{1}{8}\int_0^\tau\int\bar{p_v}\Phi_{yy}^2 dyd\tau\\
&\quad + c\int_0^\tau\int(W_y^2+W_{yy}^2+(\bar{u^\varepsilon_y}^2+\bar{u^\varepsilon_{yy}}^2)\Psi^2+\Psi_y^2+\Psi_y^4+\Psi_y^2\Psi_{yy}^2)dyd\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le -\frac{1}{8}\int_0^\tau\int \bar{p_v}\Phi_{yy}^2dyd\tau + c\int_0^\tau(\|\Psi_y\|^2+\|\Psi_y\|_{L^\infty}^2\|\Psi_y\|_1^2+\|W_y\|_1^2)d\tau\\
&\quad + c\int_0^\tau\int mV_y\Psi^2dyd\tau + c\varepsilon\tau\sup_{0\le\tau\le\tau_0}\|\Psi(\cdot,\tau)\|^2\\
&\le -\frac{1}{8}\int_0^\tau\int \bar{p_v}\Phi_{yy}^2dyd\tau + c\varepsilon^{(2n+1)\nu-4},
\end{aligned}
$$

provided that $\|\Psi_y\|_{L^\infty}$ is bounded. By the definition of Q_1, we have

$$
\begin{aligned}
\int_0^\tau\int Q_{1y}^2dyd\tau &\le c\int_0^\tau\int\{\Phi_y^4+W_y^4+\bar{u_y^\varepsilon}^4\Psi^4+\Psi_y^8+(\Phi_y^2+W_y^2+\bar{u_y^\varepsilon}^2\Psi^2\\
&\qquad +\Psi_y^4)(\Phi_{yy}^2+W_{yy}^2+\bar{u_{yy}^\varepsilon}^2\Psi^2+\Psi_y^2+\Psi_y^2\Psi_{yy}^2)\}dyd\tau\\
&\le c\delta\int_0^\tau\int\Phi_{yy}^2dyd\tau + c\int_0^\tau(\|\Phi_y\|^2+\|\Psi_y\|_1^2+\|W_y\|_1^2)d\tau\\
&\qquad + c\int_0^\tau\int mV_y\Psi^2dyd\tau + c\varepsilon\tau\sup_{0\le\tau\le\tau_0}\|\Psi(\cdot,\tau)\|^2\\
&\le c\delta\int_0^\tau\int\Phi_{yy}^2dyd\tau + c\varepsilon^{(2n+1)\nu-4},
\end{aligned}
\tag{3.27}
$$

provided that $\|(\Phi_y,\Psi,\Psi_y,W_y)\|_{L^\infty}$ is bounded. Then combining Lemma 2.2, one finds

$$
\begin{aligned}
&\int\Phi_{yy}(Q_{1y}-q_{5y}+q_{6y})dyd\tau\\
&\le -\frac{1}{8}\int_0^\tau\int\bar{p_v}\Phi_{yy}^2dyd\tau + c\int_0^\tau\int(Q_{1y}^2+q_{5y}^2+q_{6y}^2)dyd\tau\\
&\le -\frac{1}{8}\int_0^\tau\int\bar{p_v}\Phi_{yy}^2dyd\tau + c\delta\int_0^\tau\int\Phi_{yy}^2dyd\tau + c\varepsilon^{(2n+1)\nu-4}.
\end{aligned}
$$

Collecting all the estimates we have obtained and taking δ and ε to be sufficiently small, we get

$$
\int\frac{\Phi_{yy}^2}{v^\varepsilon} - \int_0^\tau\int\bar{p_v}\Phi_{yy}^2dyd\tau \le c\varepsilon^{(2n+1)\nu-4}.
$$

This implies

$$
\|\Phi_{yy}(\cdot,\tau)\|^2 + \int_0^\tau\|\Phi_{yy}(\cdot,\tau)\|^2d\tau \le c\varepsilon^{(2n+1)\nu-4}. \tag{3.28}
$$

Step 2 We rewrite (3.17) as

$$
\begin{cases}
\Phi_\tau - \dot{s}(\varepsilon\tau)\Phi_y - \Psi_y = 0,\\
\Psi_\tau - \dot{s}(\varepsilon\tau)\Psi_y + p(v^\varepsilon,\theta^\varepsilon) - p(\bar{v^\varepsilon},\bar{\theta^\varepsilon}) = \dfrac{\Psi_{yy}}{v^\varepsilon} - \dfrac{\bar{u_y^\varepsilon}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y + q_5 - q_6,\\
\dfrac{R}{\gamma-1}(W_\tau - \dot{s}(\varepsilon\tau)W_y) + (\bar{u_\tau^\varepsilon} - \dot{s}(\varepsilon\tau)\bar{u_y^\varepsilon})\Psi + \chi_2\Psi_y\\
= \dfrac{W_{yy}}{v^\varepsilon} + \dfrac{\gamma-1}{Rv^\varepsilon}(\bar{u_y^\varepsilon}\Psi - \dfrac{1}{2}\Psi_y^2)_y - \dfrac{\bar{\theta_y^\varepsilon}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y + \dfrac{1}{v^\varepsilon}\Psi_y\Psi_{yy}\\
\quad -\bar{u^\varepsilon}(q_5-q_6) + q_{11} - q_{12},\\
\Phi(y,0) = \Psi(y,0) = W(y,0) = 0,
\end{cases}
\tag{3.29}
$$

where $\chi_2 = p(v^\varepsilon, \theta^\varepsilon) - \dfrac{\bar{u^\varepsilon_y}}{v^\varepsilon}$. Applying ∂_y^k to $(3.29)_2$, multiplying both sides of the resulting equation by $\partial_y^k\Psi$ and integrating on $R^1 \times [0,T]$, we obtain

$$\frac{1}{2}\|\partial_y^k\Psi(\cdot,\tau)\|^2 + \int_0^\tau\int\frac{(\partial_y^{k+1}\Psi)^2}{v^\varepsilon}dyd\tau = \int_0^\tau\int\partial_y^{k+1}\Psi\partial_y^{k-1}(\frac{\bar{u^\varepsilon_y}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y)dyd\tau$$
$$+\int_0^\tau\int\partial_y^{k+1}\Psi\partial_y^{k-1}\{p(v^\varepsilon,\theta^\varepsilon) - p(\bar{v^\varepsilon},\bar{\theta^\varepsilon})\}dyd\tau$$
$$-\int_0^\tau\int\partial_y^{k+1}\Psi[\partial_y^{k-1},\frac{1}{v^\varepsilon}]\Psi_{yy}dyd\tau - \int_0^\tau\int\partial_y^{k+1}\Psi\partial_y^{k-1}(q_5 - q_6)dyd\tau. \tag{3.30}$$

In the case $k = 2$, by the Young's inequality and Sobolev's inequality, we have

$$\frac{1}{2}\|\partial_y^2\Psi(\cdot,\tau)\|^2 + \int_0^\tau\int\frac{(\partial_y^3\Psi)^2}{v^\varepsilon}dyd\tau$$
$$= \int_0^\tau\int\partial_y^3\Psi(\frac{\bar{u^\varepsilon_y}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_{yy} + (\frac{\bar{u^\varepsilon_y}}{\bar{v^\varepsilon}v^\varepsilon})_y\Phi_y)dyd\tau + \int_0^\tau\int\partial_y^3\Psi\{p(v^\varepsilon,\theta^\varepsilon) - p(\bar{v^\varepsilon},\bar{\theta^\varepsilon})\}_y dyd\tau$$
$$-\int_0^\tau\int(\frac{1}{v^\varepsilon})_y\partial_y^3\Psi\partial_y^2\Psi dyd\tau - \int_0^\tau\int\partial_y^3\Psi(q_{5y} - q_{6y})dyd\tau$$
$$\le \frac{1}{2}\int_0^\tau\int\frac{(\partial_y^3\Psi)^2}{v^\varepsilon}dyd\tau + c\int_0^\tau(\|\Phi_y\|_1^2 + \|\Psi_y\|_1^2 + \|W_y\|_1^2)d\tau$$
$$+c\int_0^\tau\int mV_y\Psi^2dyd\tau + c\varepsilon\tau\sup_{0\le\tau\le\tau_0}\|\Psi(\cdot,\tau)\|^2 + \int_0^\tau\int(q_{5y}^2 + q_{6y}^2)dyd\tau$$
$$\le \frac{1}{2}\int_0^\tau\int\frac{(\partial_y^3\Psi)^2}{v^\varepsilon}dyd\tau + c\varepsilon^{(2n+1)\nu-4},$$

provided that $\|\Phi\|_2, \|\Psi\|_3$ and $\|W\|_2$ are bounded, where we have used (3.11) and (3.28). This implies

$$\|\partial_y^2\Psi(\cdot,\tau)\|^2 + \int_0^\tau\|\partial_y^3\Psi(\cdot,\tau)\|^2d\tau \le c\varepsilon^{(2n+1)\nu-4}. \tag{3.31}$$

Applying ∂_y^k to $(3.29)_3$, multiplying both sides of the resulting equation by ∂_y^kW and integrating on $R^1 \times [0,T]$, we get

$$\frac{R}{2(\gamma-1)}\|\partial_y^kW(\cdot,\tau)\|^2 + \int_0^\tau\int\frac{(\partial_y^{k+1}W)^2}{v^\varepsilon}dyd\tau$$
$$= \int_0^\tau\int\partial_y^{k+1}W\partial_y^{k-1}\{(\bar{u^\varepsilon_\tau} - \dot{s}(\varepsilon\tau)\bar{u^\varepsilon_y})\Psi + \chi_2\Psi_y\}dyd\tau$$
$$-\int_0^\tau\int\partial_y^{k+1}W[\partial_y^{k-1},\frac{1}{v^\varepsilon}]W_{yy}dyd\tau$$
$$+\int_0^\tau\int\partial_y^{k+1}W\partial_y^{k-1}\{\frac{\bar{\theta^\varepsilon_y}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y - \frac{\gamma-1}{Rv^\varepsilon}(\bar{u^\varepsilon_y}\Psi - \frac{1}{2}\Psi_y^2)_y - \frac{1}{v^\varepsilon}\Psi_y\Psi_{yy}\}dyd\tau$$
$$+\int_0^\tau\int\partial_y^{k+1}W\partial_y^{k-1}\{\bar{u^\varepsilon}(q_5 - q_6) - (q_{11} - q_{12})\}dyd\tau. \tag{3.32}$$

In the case $k = 2$, one has

$$\frac{R}{2(\gamma-1)}\|\partial_y^2W(\cdot,\tau)\|^2 + \int_0^\tau\int\frac{(\partial_y^3W)^2}{v^\varepsilon}dyd\tau$$

$$
\begin{aligned}
&= \int_0^\tau \int \partial_y^3 W\{(\bar{u^\varepsilon_\tau} - \dot{s}(\varepsilon\tau)\bar{u^\varepsilon_y})\Psi + \chi_2\Psi_y\}_y dyd\tau - \int_0^\tau \int (\frac{1}{v^\varepsilon})_y \partial_y^3 W \partial_y^2 W dyd\tau \\
&\quad + \int_0^\tau \int \partial_y^3 W\{\frac{\bar{\theta^\varepsilon_y}}{\bar{v^\varepsilon} v^\varepsilon}\Phi_y - \frac{\gamma-1}{Rv^\varepsilon}(\bar{u^\varepsilon_y}\Psi - \frac{1}{2}\Psi_y^2)_y - \frac{1}{v^\varepsilon}\Psi_y\Psi_{yy}\}_y dyd\tau \\
&\quad + \int_0^\tau \int \partial_y^3 W\{(\bar{u^\varepsilon}(q_5 - q_6)_y - (q_{11y} - q_{12y})\} dyd\tau \\
&\le \frac{1}{2}\int_0^\tau \int \frac{(\partial_y^3 W)^2}{v^\varepsilon} dyd\tau + c\int_0^\tau (\|\Phi_y\|_1^2 + \|\Psi_y\|_2^2 + \|W_y\|_1^2) d\tau \\
&\quad + c\int_0^\tau \int mV_y\Psi^2 dyd\tau + c\varepsilon\tau \sup_{0\le\tau\le\tau_0}\|\Psi(\cdot,\tau)\|^2 \\
&\quad + \int_0^\tau \int (q_5^2 + q_{5y}^2 + q_6^2 + q_{6y}^2 + q_{11y}^2 + q_{12y}^2) dyd\tau \\
&\le \frac{1}{2}\int_0^\tau \int \frac{(\partial_y^3 W)^2}{v^\varepsilon} dyd\tau + c\varepsilon^{(2n+1)\nu-4},
\end{aligned}
$$

provided that $\|\Phi\|_2, \|\Psi\|_3$ and $\|W\|_3$ are bounded, where we have used (3.11), (3.28), (3.31) and Lemma 2.2-2.3. This gives

$$
\|\frac{\partial_y^2 W}{\sqrt{\gamma-1}}(\cdot,\tau)\|^2 + \int_0^\tau \|\partial_y^3 W(\cdot,\tau)\|^2 d\tau \le c\varepsilon^{(2n+1)\nu-4}. \tag{3.33}
$$

Step 3 Similar to Step 1, for $l = 2$, due to (3.8), (3.11), (3.28), (3.31) and (3.33), we have

$$
\begin{aligned}
&\frac{1}{2}\int \frac{(\partial_y^3\Phi)^2}{v^\varepsilon} dy - \int_0^\tau \int \bar{p_v}(\partial_y^3\Phi)^2 dyd\tau \\
&= \int \partial_y^3\Phi\partial_y^2\Psi dy + \int_0^\tau \int (\partial_y^3\Psi)^2 dyd\tau + \int_0^\tau \int (\frac{\bar{u^\varepsilon_y}\Phi_y}{\bar{v^\varepsilon}(v^\varepsilon)^2} + \frac{\bar{u^\varepsilon_y}}{2(v^\varepsilon)^2})(\partial_y^3\Phi)^2 dyd\tau \\
&\quad + \int_0^\tau \int (2\chi_{1y}\Phi_{yy} + \chi_{1yy}\Phi_y)\partial_y^3\Phi dyd\tau + \frac{1}{2}\int_0^\tau \int \frac{\Psi_{yy}}{(v^\varepsilon)^2}(\partial_y^3\Phi)^2 dyd\tau \\
&\quad + \int_0^\tau \int \partial_y^3\Phi\{\frac{2(\Phi_{yy} + \bar{v^\varepsilon_y})}{(v^\varepsilon)^2}\partial_y^3\Psi + (\frac{\bar{v^\varepsilon_{yy}}}{(v^\varepsilon)^2} - \frac{2(\Phi_{yy} + \bar{v^\varepsilon_y})^2}{(v^\varepsilon)^3})\partial_y^2\Psi\} dyd\tau \\
&\quad + \int_0^\tau \int \partial_y^3\Phi\partial_y^2\{\bar{p_\theta}(W_y + \frac{\gamma-1}{R}(\bar{u^\varepsilon_y}\Psi - \frac{1}{2}\Psi_y^2))\} dyd\tau \\
&\quad + \int_0^\tau \int \partial_y^3\Phi(\partial_y^2 Q_1 - \partial_y^2 q_5 + \partial_y^2 q_6) dyd\tau \\
&\le \frac{1}{4}\int \frac{(\partial_y^3\Phi)^2}{v^\varepsilon} dy + c\|\partial_y^2\Psi(\cdot,\tau)\|^2 + c(\delta+\varepsilon)\int_0^\tau \|\partial_y^3\Phi\|^2 d\tau \\
&\quad - \frac{1}{2}\int_0^\tau \int \bar{p_v}(\partial_y^3\Phi)^2 dyd\tau + c\int_0^\tau (\|\Phi_y\|_1^2 + \|\Psi_y\|_2^2 + \|W_y\|_2^2) d\tau \\
&\quad + c\int_0^\tau \int mV_y\Psi^2 dyd\tau + c\varepsilon\tau \sup_{0\le\tau\le\tau_0}\|\Psi(\cdot,\tau)\|^2 \\
&\quad + c\int_0^\tau \int \{(\partial_y^2 Q_1)^2 + (\partial_y^2 q_5)^2 + (\partial_y^2 q_6)^2\} dyd\tau \\
&\le \frac{1}{4}\int \frac{(\partial_y^3\Phi)^2}{v^\varepsilon} dy - \frac{1}{2}\int_0^\tau \int \bar{p_v}(\partial_y^3\Phi)^2 dyd\tau + c(\delta+\varepsilon)\int_0^\tau \|\partial_y^3\Phi\|^2 d\tau + c\varepsilon^{(2n+1)\nu-4},
\end{aligned}
$$

provided that $\|(\Phi,\Psi,W)\|_3$ is bounded. By taking δ and ε to be sufficiently small, we arrive at

$$\int \frac{(\partial_y^3\Phi)^2}{v^\varepsilon}dy-\int_0^\tau\int \bar{p_v}(\partial_y^3\Phi)^2dyd\tau\le c\varepsilon^{(2n+1)\nu-4},$$

which implies

$$\|\partial_y^3\Phi(\cdot,\tau)\|^2+\int_0^\tau\|\partial_y^3\Phi(\cdot,\tau)\|^2d\tau\le c\varepsilon^{(2n+1)\nu-4}. \tag{3.34}$$

Step 4 Similarly, when $k=3$, we have

$$\begin{aligned}
&\frac{1}{2}\|\partial_y^3\Psi(\cdot,\tau)\|^2+\int_0^\tau\int\frac{(\partial_y^4\Psi)^2}{v^\varepsilon}dyd\tau\\
&=\int_0^\tau\int\partial_y^4\Psi\partial_y^2(\frac{\bar{u_y^\varepsilon}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y)dyd\tau+\int_0^\tau\int\partial_y^4\Psi\partial_y^2\{p(v^\varepsilon,\theta^\varepsilon)-p(\bar{v^\varepsilon},\bar{\theta^\varepsilon})\}dyd\tau\\
&\quad-\int_0^\tau\int\partial_y^4\Psi\{\partial_y^2(\frac{1}{v^\varepsilon})\partial_y^2\Psi+2(\frac{1}{v^\varepsilon})_y\partial_y^3\Psi\}dyd\tau-\int_0^\tau\int\partial_y^4\Psi\partial_y^2(q_5-q_6)dyd\tau\\
&\le\frac{1}{2}\int_0^\tau\int\frac{(\partial_y^4\Psi)^2}{v^\varepsilon}dyd\tau+c\int_0^\tau(\|\Phi_y\|_2^2+\|\Psi_y\|_2^2+\|W_y\|_2^2)dyd\tau\\
&\quad+c\int_0^\tau\int mV_y\Psi^2dyd\tau+c\varepsilon\tau\sup_{0\le\tau\le\tau_0}\|\Psi(\cdot,\tau)\|^2+\int_0^\tau\int(((\partial_y^2q_5)^2+(\partial_y^2q_6)^2)dyd\tau\\
&\le\frac{1}{2}\int_0^\tau\int\frac{(\partial_y^4\Psi)^2}{v^\varepsilon}dyd\tau+c\varepsilon^{(2n+1)\nu-4},
\end{aligned}$$

provided that $\|(\Phi,\Psi,W)\|_3$ is bounded. This implies

$$\|\partial_y^3\Psi(\cdot,\tau)\|^2+\int_0^\tau\|\partial_y^4\Psi(\cdot,\tau)\|^2d\tau\le c\varepsilon^{(2n+1)\nu-4}. \tag{3.35}$$

And for W, we have

$$\begin{aligned}
&\frac{R}{2(\gamma-1)}\|\partial_y^3W(\cdot,\tau)\|^2+\int_0^\tau\int\frac{(\partial_y^4W)^2}{v^\varepsilon}dyd\tau\\
&=\int_0^\tau\int\partial_y^4W\partial_y^2\{(\bar{u_\tau^\varepsilon}-\dot{s}(\varepsilon\tau)\bar{u_y^\varepsilon})\Psi+\chi_2\Psi_y\}dyd\tau\\
&\quad-\int_0^\tau\int\partial_y^4W\{\partial_y^2(\frac{1}{v^\varepsilon})\partial_y^2W+2(\frac{1}{v^\varepsilon})_y\partial_y^3W\}dyd\tau\\
&\quad+\int_0^\tau\int\partial_y^4W\partial_y^2\{\frac{\bar{\theta_y^\varepsilon}}{\bar{v^\varepsilon}v^\varepsilon}\Phi_y-\frac{\gamma-1}{Rv^\varepsilon}(\bar{u_y^\varepsilon}\Psi-\frac{1}{2}\Psi_y^2)_y-\frac{1}{v^\varepsilon}\Psi_y\Psi_{yy}\}dyd\tau\\
&\quad+\int_0^\tau\int\partial_y^4W\partial_y^2\{(\bar{u^\varepsilon}(q_5-q_6))-(q_{11}-q_{12})\}dyd\tau\\
&\le\frac{1}{2}\int_0^\tau\int\frac{(\partial_y^4W)^2}{v^\varepsilon}dyd\tau+c\int_0^\tau(\|\Phi_y\|_2^2+\|\Psi_y\|_3^2+\|W_y\|_2^2)dyd\tau\\
&\quad+c\int_0^\tau\int mV_y\Psi^2dyd\tau+c\varepsilon\tau\sup_{0\le\tau\le\tau_0}\|\Psi(\cdot,\tau)\|^2\\
&\quad+\int_0^\tau\int(q_5^2+q_6^2+q_{5y}^2+q_{6y}^2+q_{5yy}^2+q_{6yy}^2+q_{11yy}^2+q_{12yy}^2)dyd\tau
\end{aligned}$$

$$\leq \frac{1}{2}\int_0^\tau \int \frac{(\partial_y^4 W)^2}{v^\varepsilon} dy d\tau + c\varepsilon^{(2n+1)\nu-4},$$

provided that $\|(\Phi, \Psi, W)\|_3$ is bounded. This implies

$$\|\frac{\partial_y^3 W}{\sqrt{\gamma-1}}(\cdot,\tau)\|^2 + \int_0^\tau \|\partial_y^4 W(\cdot,\tau)\|^2 d\tau \leq c\varepsilon^{(2n+1)\nu-4}. \tag{3.36}$$

So far we finish the proof of Lemma 3.3.

Combining Lemma 3.2 and Lemma 3.3 together, we complete the proof of Proposition 3.1.

4. Proof of Theorem 1.2. Using the Proposition 3.1 and the standard continuous induction argument, we conclude that

PROPOSITION 4.1. *There exist positive constants ε_0, μ_0 and C, which are independent of ε such that if $0 < \varepsilon < \varepsilon_0$ and $0 \leq (\gamma-1)\mu \leq \mu_0$, then the Cauchy problem (3.7) has a unique solution $(\Phi, \Psi, W) \in C^1([0, T/\varepsilon] : H^3(R^1))$. Furthermore, the following inequality holds*

$$\begin{aligned}&\sup_{0\leq\tau\leq T/\varepsilon} \|(\Phi, \Psi, \frac{W}{\sqrt{\gamma-1}})(\cdot,\tau)\|_3^2 + \int_0^{T/\varepsilon} (\|\Phi_y(\cdot,\tau)\|_2^2 + \|(\Psi_y, W_y)(\cdot,\tau)\|_3^2) d\tau \\ &\leq C\varepsilon^{(2n+1)\nu-4}.\end{aligned} \tag{4.1}$$

Proof of Theorem 1.2. Now we choose $\nu \in (\eta, 1) \cap (\frac{4n+1}{4n+2}, 1)$. In view of (4.1) and Sobolev inequality, we have

$$\begin{aligned}
&\sup_{0\leq t\leq T} \|(v^\varepsilon - \bar{v^\varepsilon}, u^\varepsilon - \bar{u^\varepsilon}, \theta^\varepsilon - \bar{\theta^\varepsilon})(\cdot,t)\|^2 \\
&= \sup_{0\leq t\leq T} \|(\bar{\Phi}_x, \bar{\Psi}_x, \bar{W}_x + \frac{\gamma-1}{R}(\bar{u^\varepsilon_x}\bar{\Psi} - \frac{1}{2}\bar{\Psi}_x^2))(\cdot,t)\|^2 \\
&= \varepsilon \sup_{0\leq\tau\leq T/\varepsilon} \|(\Phi_y, \Psi_y, W_y + \frac{\gamma-1}{R}(\bar{u^\varepsilon_y}\Psi - \frac{1}{2}\Psi_y^2))(\cdot,\tau)\|^2 \\
&\leq c\varepsilon \sup_{0\leq\tau\leq T/\varepsilon} (\|\Phi_y(\cdot,\tau)\|^2 + \|\Psi(\cdot,\tau)\|^2 + (1 + \|\Psi_y(\cdot,\tau)\|^4)\|\Psi_y(\cdot,\tau)\|^2 \\
&\qquad + \|\Psi_{yy}(\cdot,\tau)\|^2 + \|W_y(\cdot,\tau)\|^2) \\
&\leq c\varepsilon \sup_{0\leq\tau\leq T/\varepsilon} (\|\Phi_y(\cdot,\tau)\|^2 + \|\Psi(\cdot,\tau)\|_2^2 + \|W_y(\cdot,\tau)\|^2) \\
&\leq C\varepsilon^{(2n+1)\nu-3} \leq C\varepsilon^{2n-\frac{5}{2}}.
\end{aligned}$$

On the other hand, it follows from Lemma 2.3 that

$$\sup_{0\leq t\leq T} \|(\bar{v^\varepsilon} - v_0, \bar{u^\varepsilon} - u_0, \bar{\theta^\varepsilon} - \theta_0)(\cdot,t)\|^2 \leq C\varepsilon^\nu \leq C\varepsilon^\eta.$$

Consequently,

$$\begin{aligned}
&\sup_{0\leq t\leq T} \|(v^\varepsilon - v_0, u^\varepsilon - u_0, \theta^\varepsilon - \theta_0)(\cdot,t)\|^2 \\
&\leq \sup_{0\leq t\leq T} \|(v^\varepsilon - \bar{v^\varepsilon}, u^\varepsilon - \bar{u^\varepsilon}, \theta^\varepsilon - \bar{\theta^\varepsilon})(\cdot,t)\|^2 + \sup_{0\leq t\leq T} \|(\bar{v^\varepsilon} - v_0, \bar{u^\varepsilon} - u_0, \bar{\theta^\varepsilon} - \theta_0)(\cdot,t)\|^2
\end{aligned}$$

$$\leq c\varepsilon^{\eta},$$

which gives (1.11). Finally,

$$\begin{aligned}
&\|(v^\varepsilon-\bar{v^\varepsilon},u^\varepsilon-\bar{u^\varepsilon},\theta^\varepsilon-\bar{\theta^\varepsilon})(\cdot,t)\|_{L^\infty}=\|(\Phi_y,\Psi_y,W_y+\frac{\gamma-1}{R}(\bar{u^\varepsilon_y}\Psi-\frac{1}{2}\Psi_y^2))(\cdot,t)\|_{L^\infty}\\
&\leq c\|(\Phi_y,\Psi,\Psi_y,W_y)(\cdot,t)\|^{\frac{1}{2}}\|(\Phi_{yy},\Psi,\Psi_y,\Psi_{yy},W_{yy})(\cdot,t)\|^{\frac{1}{2}}\\
&\leq c\varepsilon^{((2n+1)\nu-4)/2}\leq c\varepsilon^{n-\frac{7}{4}}.
\end{aligned}$$

This gives (1.10). By using Lemma 2.3 again, we obtain (1.12).

We completes the proof of the Theorem 1.2.

Acknowledgement. This paper is part of my Ph.D thesis. I express my hearty thanks to my supervisor, Professor Zhouping Xin, for his encouragement and guidance. I also thank Professor Tao Luo for his helpful advice on this paper.

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