METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 15, No. 4, pp. 469–476, December 2008

004

A GENERAL STABILITY CONDITION FOR MULTI-STAGE VORTICITY BOUNDARY CONDITIONS IN INCOMPRESSIBLE FLUIDS*

CHENG WANG†

Abstract. A stability condition is provided for a class of vorticity boundary formulas used with the second order finite-difference numerical scheme for the vorticity-stream function formulation of the unsteady incompressible Navier-Stokes equations. These local vorticity boundary formulas are derived using the no-slip boundary condition for the velocity. A new form of these long-stencil formulas is needed to classify the stability property, in which local terms are controlled by global quantities via discrete elliptic regularity for the stream functions.

Key words. Vorticity-stream function formulation, local vorticity boundary condition, stability condition.

AMS subject classifications. 65M06, 76M20

1. Introduction.

The vorticity-stream function formulation of the 2-D Navier-Stokes equations is given by

$$\begin{cases} \partial_t \omega + \nabla\cdot(\boldsymbol{u}\omega) = \nu\Delta\omega\,, \\ \Delta\psi = \omega\,, \\ u = -\partial_y\psi\,, \qquad v = \partial_x\psi\,, \end{cases} \tag{1.1}$$

where $\boldsymbol{u} = (u, v)$, $\omega = \nabla \times \boldsymbol{u} = -\partial_y u + \partial_x v$ and ψ denote the velocity vector, the vorticity and the stream function, respectively. The no-penetration, no-slip boundary condition can be rewritten as

$$\psi = 0\,, \quad \frac{\partial\psi}{\partial\mathbf{n}} = 0\,, \qquad \text{on} \quad \partial\Omega\,, \tag{1.2}$$

in a simply-connected domain.

When solving these equations numerically, the absence of an explicit formula for the boundary values of the vorticity makes it difficult to enforce a correct boundary condition. The study of the vorticity boundary condition has a long history, going back to the pioneering work of A. Thom [9] in 1933. Local vorticity boundary formulas for the vorticity, such as Thom's formula or Pearson-Wilkes' formula, have obvious advantages in terms of simplicity and numerical efficiency. In these cases, a biharmonic equation is avoided and there is no coupling between the kinematic equation and the vorticity boundary condition. Relevant discussions on this subject can be found in [1, 2, 5, 6, 8].

Thom's formula, which is the simplest of these local boundary formulas, includes only one interior point of stream function, and its stability analysis is straightforward. The corresponding theoretical convergence analysis of the full second order scheme can be found in Hou and Wetton's work [4]. However, it is significantly harder to perform the stability analysis for the long-stencil formulas, such as Wilkes' formula, in which more than one interior point values of stream function are utilized. The main difficulty in this analysis is that the boundary term complicated the process of

*Received December 8, 2007; accepted for publication November 18, 2008.
†Department of Mathematics, University of Massachusetts Dartmouth, North Dartmouth, MA 02747, USA (cwang1@umassd.edu).

energy estimates. A new technique to overcome this difficulty was proposed in [10]: rewrite the vorticity boundary formula in terms of the second derivative of stream function near the boundary, whose L^2 norm can be controlled by that of the vorticity. This approach allowed a stability analysis for Wilkes' formula, which is given in detail in that article. The bounds resulting from this analysis can be viewed as an elliptic regularity at the discrete level.

In this paper, this methodology is extended to a class of local vorticity boundary formulas in terms of one-sided extrapolation for stream function values at interior grid point. This analysis relies on a reformulation of the boundary formula, which makes the verification of stability straightforward. The resulting numerical stability condition is provided in terms of the coefficients for the stream function values at different numerical grid indices.

2. The second order finite difference scheme with the vorticity boundary condition. For simplicity of presentation, the computation domain is chosen to be $\Omega = (0,1) \times (0,1)$ and the grid sizes are $\Delta x = \Delta y = h$. Consequently, the numerical grid points will be denoted as $(x_i\,, y_j)$, where $x_i = i\Delta x$ and $y_j = j\Delta y$ for $0 \le i, j \le N$. A standard centered difference approximation to (1.1) at the interior grid points gives

$$\begin{cases} \partial_t \omega + \widetilde{D}_x(u\omega) + \widetilde{D}_y(v\omega) = \nu \Delta_h \omega \,, \\ \Delta_h \psi = \omega \,, \qquad \psi \,|_{\Gamma} = 0 \,, \\ u = -\widetilde{D}_y \psi \,, \qquad v = \widetilde{D}_x \psi \,, \end{cases} \tag{2.1}$$

with

$$\begin{aligned} &\widetilde{D}_x u_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \,, \quad \widetilde{D}_y u_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2h} \,, \quad \Delta_h = D_x^2 + D_y^2 \,, \\ &D_x^2 u_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \,, \quad D_y^2 u_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} \,. \end{aligned} \tag{2.2}$$

From (1.2), we see that we need two boundary conditions for ψ. The Dirichlet boundary condition $\psi = 0$ on Γ is implemented to solve the stream function via the vorticity as in (2.1). The normal boundary condition, $\frac{\partial \psi}{\partial \boldsymbol{n}} = 0$ cannot be enforced directly, so it is converted into a vorticity boundary condition. Because of the identically zero values for the stream function on the boundary, the vorticity on the boundary on Γ_x, $j = 0$, can be approximated by

$$\omega_{i,0} = D_y^2 \psi_{i,0} = \frac{1}{h^2}(\psi_{i,1} + \psi_{i,-1}) = \frac{2\psi_{i,1}}{h^2} - \frac{2}{h}\frac{\psi_{i,1} - \psi_{i,-1}}{2h} \,, \tag{2.3}$$

where $(i,-1)$ refers to the "ghost" grid point outside of the computational domain. Taking the approximation identity $\dfrac{\psi_{i,1} - \psi_{i,-1}}{2h} = 0$ as a second order normal boundary condition for $(\partial_y \psi)_{i,0} = 0$, we arrive at **Thom's formula**

$$\omega_{i,0} = \frac{2\psi_{i,1}}{h^2} \,, \qquad \psi_{i,-1} = \psi_{i,1} \,. \tag{2.4}$$

Note that Thom's formula is only first order accurate for ω on the boundary, by a Taylor expansion for ψ. To improve the formal accuracy, a third order one-sided

approximation for the normal boundary condition $\frac{\partial\psi}{\partial \boldsymbol{n}}=0$ can be used:
(2.5)

$$(\partial_y\psi)_{i,0}=\frac{-\psi_{i,-1}+3\psi_{i,1}-\frac{1}{2}\psi_{i,2}}{3h}=0\,,\quad \text{which leads to}\quad \psi_{i,-1}=3\psi_{i,1}-\frac{1}{2}\psi_{i,2}\,.$$

Its substitution into the difference equation (2.3) results in **Wilkes-Pearson's formula**

$$\omega_{i,0}=\frac{1}{h^2}\Big(4\psi_{i,1}-\frac{1}{2}\psi_{i,2}\Big)\,,\qquad \psi_{i,-1}=3\psi_{i,1}-\frac{1}{2}\psi_{i,2}\,. \tag{2.6}$$

See [7] for more details.

In general, we represent the vorticity boundary condition as

$$\omega_{i,0}=\frac{1}{h^2}\Big(C_1\psi_{i,1}+C_2\psi_{i,k}+...+C_k\psi_{i,k}\Big)\,, \tag{2.7}$$

by assuming that altogether k interior grid point values of ψ are utilized in the formula. This representation includes Thom's formula, the Wilkes-Pearson formula, and other local vorticity boundary conditions such as:
(2.8)

Fromm (1963) : $\omega_{i,0}=\frac{1}{h^2}\psi_{i,1}\,,\ \psi_{i,-1}=0\,,$

Orszag and Israeli (1974) : $\omega_{i,0}=\frac{1}{h^2}\Big(\frac{10}{3}\psi_{i,1}-\frac{1}{3}\psi_{i,2}\Big)\,,\ \psi_{i,-1}=\frac{7}{3}\psi_{i,1}-\frac{1}{3}\psi_{i,2}\,,$

Orszag and Israeli (1974) : $\omega_{i,0}=\frac{1}{h^2}\Big(\frac{35}{13}\psi_{i,1}-\frac{1}{13}\psi_{i,2}\Big)\,,\ \psi_{i,-1}=\frac{22}{13}\psi_{i,1}-\frac{1}{13}\psi_{i,2}\,,$

Briley (1971) : $\omega_{i,0}=\frac{1}{h^2}\Big(6\psi_{i,1}-\frac{3}{2}\psi_{i,2}+\frac{2}{9}\psi_{i,3}\Big)\,,\ \psi_{i,-1}=5\psi_{i,1}-\frac{3}{2}\psi_{i,2}+\frac{2}{9}\psi_{i,3}\,.$

It should be noted that Briley's formula was originally proposed in [1] as a local vorticity boundary condition for a fourth order difference method, due to its $O(h^3)$ consistency on the boundary. In the next section we will see that this three-stage formula preserves L^2 stability.

3. A stability classification of vorticity boundary formula in terms of the stream function. When the one-sided (in terms of stream function at interior mesh points) formula (2.7) is used to approximate the boundary value of vorticity, the symmetry of the difference operator is broken. As a result, direct calculations and standard local estimates cannot ensure its stability. To overcome this difficulty, we can rewrite the formula (2.7) in terms of the second normal derivative of the stream function near the boundary (at a discrete level). This transformation leads to some local terms around the boundary, such as $D_x^2\psi$ and $D_y^2\psi$. Certainly, the local terms are bounded by the global quantities of $\|D_x^2\psi\|^2$ and $\|D_y^2\psi\|^2$, respectively. Moreover, by an application of elliptic regularity at the discrete level, namely Lemma 3.3 below (its proof can be found in the earlier article [10]), these global quantities can be controlled by the diffusion term in the energy estimate. This process gives the L^2-stability of a general vorticity boundary condition.

For simplicity, let's consider the Stokes equations to illustrate the numerical stability of a general vorticity boundary condition (2.7). The corresponding second order scheme becomes

$$\left\{ \begin{array}{l} \partial_t \omega = \nu \Delta_h \omega \,, \\ \Delta_h \psi = \omega \,, \qquad \psi \mid_\Gamma = 0 \,, \end{array} \right. \tag{3.1}$$

with a boundary formula (2.7).

Using the property that ψ vanishes on the boundary, we rewrite (2.7) in the following form to facilitate the analysis

$$\begin{aligned} & C_1 \psi_{i,1} + C_2 \psi_{i,2} + ... + C_k \psi_{i,k} \\ & = C_1^* \psi_{i,1} + \widetilde{C}_1 h^2 D_y^2 \psi_{i,1} + \widetilde{C}_2 h^2 D_y^2 \psi_{i,2} + ... + \widetilde{C}_{k-1} h^2 D_y^2 \psi_{i,k-1} \,. \end{aligned} \tag{3.2}$$

Note that such a transformation is always possible by taking recursive formulas:

$$\begin{aligned} & \widetilde{C}_{k-1} = C_k \,, \quad \widetilde{C}_{k-2} = C_{k-1} + 2\widetilde{C}_{k-1} \,, \quad \widetilde{C}_{k-3} = C_{k-2} - \widetilde{C}_{k-1} + 2\widetilde{C}_{k-2} \,, ... \,, \\ & \widetilde{C}_1 = C_2 - \widetilde{C}_3 + 2\widetilde{C}_2 \,, \quad C_1^* = C_1 - \widetilde{C}_2 + 2\widetilde{C}_1 \,. \end{aligned} \tag{3.3}$$

3.1. The stability condition and its derivation. The stability condition for a general formula (2.7) is given in the following theorem.

THEOREM 3.1. *The second order scheme (3.1) along with the vorticity boundary condition (2.7) is L^2-stable if the reformulated coefficients satisfy the following criterion:*

$$C_1^* - \frac{1}{4}\Big(\widetilde{C}_1^2 + \widetilde{C}_2^2 + ... + \widetilde{C}_{k-1}^2 \Big) \equiv B^* > 0 \,. \tag{3.4}$$

To facilitate the proof of this theorem, we first define the discrete L^2-norm and L^2-inner product.

DEFINITION 3.2. *The discrete L^2-norm and L^2-inner product are defined as*

$$\|u\| = \langle u \,, u \rangle^{1/2} \,, \qquad \langle u \,, v \rangle = \sum_{1 \le i,j \le N-1} u_{i,j} \, v_{i,j} \, h^2 \,. \tag{3.5}$$

For $u \mid_\Gamma = 0$, the notation $\|\nabla_h u\|$ is introduced by

$$\begin{aligned} & \|\nabla_h u\|^2 = \sum_{j=1}^{N-1} \sum_{i=0}^{N-1} (D_x^+ u_{i,j})^2 h^2 + \sum_{i=1}^{N-1} \sum_{j=0}^{N-1} (D_y^+ u_{i,j})^2 h^2 \,, \qquad \textit{with} \\ & D_x^+ u_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h} \,, \quad D_y^+ u_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{h} \,. \end{aligned} \tag{3.6}$$

Proof of Theorem 3.1. Taking the inner product of the momentum equation in (3.1) with $-\psi$ gives $-\langle \psi, \partial_t \omega \rangle + \langle \psi, \Delta_h \omega \rangle = 0$. Because of the homogeneous Dirichlet boundary condition for ψ, integration by parts at a discrete level shows that the first term becomes

$$-\langle \psi, \partial_t \omega \rangle = -\langle \psi, \partial_t \Delta_h \psi \rangle = \frac{1}{2} \frac{d}{dt} \|\nabla_h \psi\|^2 \,. \tag{3.7}$$

Similarly, summation by parts for the second term gives

$$\langle \psi, \Delta_h \omega \rangle = \langle \Delta_h \psi, \omega \rangle + \mathcal{B} = \|\omega\|^2 + \mathcal{B} \,, \qquad \text{with} \tag{3.8a}$$

(3.8b)

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4 \,,$$

$$\mathcal{B}_1 = \sum_{i=1}^{N-1} \psi_{i,1}\omega_{i,0} \,, \mathcal{B}_2 = \sum_{i=1}^{N-1} \psi_{i,N-1}\omega_{i,N} \,, \mathcal{B}_3 = \sum_{j=1}^{N-1} \psi_{1,j}\omega_{0,j} \,, \mathcal{B}_4 = \sum_{j=1}^{N-1} \psi_{N-1,j}\omega_{N,j} \,,$$

where we used the fact that $\Delta_h \psi = \omega$.

Obviously, to demonstrate numerical stability we need to control the boundary term $\mathcal{B}$. For simplicity of presentation, we only consider $\mathcal{B}_1$ here. Boundary condition (2.7) is applied to recover $\mathcal{B}_1$

$$\mathcal{B}_1 = \sum_{i=1}^{N-1} \frac{\psi_{i,1}}{h^2} \Big(C_1\psi_{i,1} + C_2\psi_{i,2} + ... + C_k\psi_{i,k} \Big) \,. \tag{3.9}$$

It can be seen that a direct calculation cannot control $\mathcal{B}_1$, since many interior points of stream function: $\psi_{i,1}$, $\psi_{i,2}$, ..., $\psi_{i,k}$, are involved in the formula. At this point, the reformulation (3.2) allows us to carry out the stability analysis, and its substitution into (3.9) yields

$$\mathcal{B}_1 = \sum_{i=1}^{N-1} \Big[C_1^* \frac{\psi_{i,1}^2}{h^2} + \psi_{i,1} \Big(\widetilde{C}_1 D_y^2 \psi_{i,1} + \widetilde{C}_2 D_y^2 \psi_{i,2} + ... + \widetilde{C}_{k-1} D_y^2 \psi_{i,k-1} \Big) \Big] \,. \tag{3.10}$$

This transformation lets us control local terms by global terms; it will later become clear how this is useful. In addition, we define a constant $0 < \widetilde{C}^* < 1$ as

$$\widetilde{C}^* = \frac{C_1^* - B^*}{C_1^*} \,, \qquad \text{in which } 0 < B^* < C_1^* \text{ is given in (3.4)} \,. \tag{3.11}$$

Applying the Cauchy inequality to $\psi_{i,1} \cdot D_y^2 \psi_{i,j}$, $1 \le j \le k-1$, shows that

$$\begin{aligned} &\widetilde{C}_1 \psi_{i,1} D_y^2 \psi_{i,1} \ge -\frac{\widetilde{C}_1^2}{4\widetilde{C}^*} \cdot \frac{\psi_{i,1}^2}{h^2} - \widetilde{C}^* |D_y^2 \psi_{i,1}|^2 h^2 \,, \\ &\widetilde{C}_2 \psi_{i,1} D_y^2 \psi_{i,2} \ge -\frac{\widetilde{C}_2^2}{4\widetilde{C}^*} \cdot \frac{\psi_{i,1}^2}{h^2} - \widetilde{C}^* |D_y^2 \psi_{i,2}|^2 h^2 \,, \\ &\quad ... \quad \widetilde{C}_{k-1} \psi_{i,1} D_y^2 \psi_{i,k-1} \ge -\frac{\widetilde{C}_{k-1}^2}{4\widetilde{C}^*} \cdot \frac{\psi_{i,1}^2}{h^2} - \widetilde{C}^* |D_y^2 \psi_{i,k-1}|^2 h^2 \,. \end{aligned} \tag{3.12}$$

Going back to (3.10), we arrive at

(3.13)

$$\begin{aligned} \mathcal{B}_1 \ge\ & \sum_{i=1}^{N-1} \Big[\frac{\psi_{i,1}^2}{h^2} \Big(C_1^* - \frac{\widetilde{C}_1^2}{4\widetilde{C}^*} - \frac{\widetilde{C}_2^2}{4\widetilde{C}^*} ... - \frac{\widetilde{C}_{k-1}^2}{4\widetilde{C}^*} \Big) - \widetilde{C}^* h^2 \Big(|D_y^2 \psi_{i,1}|^2 + ... + |D_y^2 \psi_{i,k-1}|^2 \Big) \Big] \\ \ge\ & \sum_{i=1}^{N-1} \Big[\frac{\psi_{i,1}^2}{h^2} \Big(C_1^* - \frac{4(C_1^* - B^*)}{4\widetilde{C}^*} \Big) - \widetilde{C}^* h^2 \Big(|D_y^2 \psi_{i,1}|^2 + ... + |D_y^2 \psi_{i,k-1}|^2 \Big) \Big] \\ \ge\ & -\widetilde{C}^* h^2 \sum_{i=1}^{N-1} \Big(|D_y^2 \psi_{i,1}|^2 + ... + |D_y^2 \psi_{i,k-1}|^2 \Big) \,, \end{aligned}$$

where in the second step we used the identity $\widetilde{C}_1^2 + \widetilde{C}_2^2 + ... + \widetilde{C}_{k-1}^2 = 4(C_1^* - B^*)$ indicated by (3.4), and in the last step we used (3.11). If we apply the same argument to $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$, we get

(3.14)

$$\begin{aligned}
\mathcal{B} \geq \quad & -\widetilde{C}^* h^2 \sum_{i=1}^{N-1} \Big(|D_y^2\psi_{i,1}|^2 + ... + |D_y^2\psi_{i,k-1}|^2 + |D_y^2\psi_{i,N-k+1}|^2 + ... + |D_y^2\psi_{i,N-1}|^2 \Big) \\
& -\widetilde{C}^* h^2 \sum_{j=1}^{N-1} \Big(|D_x^2\psi_{1,j}|^2 + ... + |D_x^2\psi_{k-1,j}|^2 + |D_x^2\psi_{N-k+1,j}|^2 + ... + |D_x^2\psi_{N-1,j}|^2 \Big) \\
\geq \quad & -\widetilde{C}^* \|D_x^2\psi\|^2 - \widetilde{C}^* \|D_y^2\psi\|^2 .
\end{aligned}$$

Note again that the transformation (3.2) leads to a bound of the boundary term, which is a local term, by global terms $\|D_x^2\psi\|^2$ and $\|D_y^2\psi\|^2$.

To control the terms $\|D_x^2\psi\|^2$, $\|D_y^2\psi\|^2$ by the diffusion term $\|\omega\|^2$, we use the following Lemma, which was proved in the author's earlier article [10].

LEMMA 3.3. *For any ψ such that $\psi\,|_\Gamma = 0$, we have*

$$\|D_x^2\psi\|^2 + \|D_y^2\psi\|^2 \leq \|(D_x^2 + D_y^2)\psi\|^2 = \|\omega\|^2 . \tag{3.15}$$

Applying Lemma 3.3 to inequality (3.14) implies

$$\mathcal{B} \geq -\widetilde{C}^* \|\omega\|^2 . \tag{3.16}$$

Substituting back into (3.8), along with (3.7), we arrive at the stability estimate of the second order scheme with vorticity boundary formula (2.7):

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h\psi\|^2 + (1 - \widetilde{C}^*)\nu\|\omega\|^2 \leq 0 . \tag{3.17}$$

3.2. Verification of the stability condition for some commonly used vorticity boundary condition. The second order scheme (3.1) is L^2 stable when coupled with any of the local boundary formulas cited in Section 2, including Thom's formula (2.4), Wilkes' formula (2.6) or the ones given in (2.8), because all these formulas satisfy the criterion (3.4). It is particularly interesting that the three-stage Briley's formula is also stable by the condition given by Theorem 3.1. This stability property has also been verified by various numerical experiments.

(3.18)

$$\begin{aligned}
&\textbf{Thom} \quad (1933): \quad C_1^* = 2 , \quad \widetilde{C}_1 = 0 , \qquad B^* = 2 > 0 , \\
&\textbf{Fromm} \quad (1963): \quad C_1^* = 1 , \quad \widetilde{C}_1 = 0 , \qquad B^* = 1 > 0 , \\
&\textbf{Pearson - Wilkes} \quad (1965): \quad C_1^* = 3 , \quad \widetilde{C}_1 = -\frac{1}{2} , \qquad B^* = \frac{47}{16} > 0 , \\
&\textbf{Orszag and Israeli} \quad (1974): \quad C_1^* = \frac{8}{3} , \quad \widetilde{C}_1 = -\frac{1}{3} , \qquad B^* = \frac{95}{36} > 0 , \\
&\textbf{Orszag and Israeli} \quad (1974): \quad C_1^* = \frac{33}{13} , \quad \widetilde{C}_1 = -\frac{1}{13} , \qquad B^* = \frac{1715}{676} > 0 , \\
&\textbf{Briley} \quad (1971): \quad C_1^* = \frac{11}{3} , \quad \widetilde{C}_2 = \frac{2}{9} , \qquad \widetilde{C}_1 = -\frac{19}{18} , \qquad B^* = \frac{4375}{1296} > 0 .
\end{aligned}$$

REFERENCES

[1] W.R. Briley, *A Numerical study of laminar separation bubbles using the Navier-Stokes equations*, Journal of Fluid Mechanics, 47 (1971), pp. 713–736.

[2] W. E and J.-G. Liu, *Vorticity boundary condition for finite difference schemes*, Journal of Computational Physics, 124 (1996), pp. 368-382.

[3] R. Glowinski and O. Pironneau, *Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem*, SIAM Review, 21 (1979), pp. 167–212.

[4] T.Y. Hou and B. Wetton, *Convergence of a finite difference scheme for the Navier-Stokes equations using vorticity boundary conditions*, SIAM Journal of Numerical Analysis, 29 (1992), pp. 615–639.

[5] S. Orszag and M. Israeli, *Numerical simulation of viscous incompressible flows*, Annual Review of Fluid Mechanics, 6 (1974), pp. 281–318.

[6] S. Orszag, M. Israeli and M. Deville, *Boundary conditions for incompressible flows*, Journal of Scientific Computing, 1 (1986), pp. 75–111.

[7] R. Peyret and T. Taylor, *Computational Methods for Fluid Flow,* Springer-Verlag, New York/Berlin, 1983.

[8] L. Quartapelle, *Vorticity conditioning in the computation of two-dimensional Viscous Flows*, Journal of Computational Physics, 40 (1981), pp. 453–477.

[9] A. Thom, *The Flow past circular cylinders at low speeds*, Proceedings Royal Society, London Sect. A., 141 (1933), pp. 651–669.

[10] C. Wang and J.-G. Liu, *Analysis of finite difference schemes for unsteady Navier-Stokes equations in vorticity formulation*, Numerische Mathematik, 91 (2002), pp. 543–576.