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BOUNDS FOR ZEROS OF THE CHARLIER POLYNOMIALS *

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Abstract. We use the method of positive quadratic forms and discrete analogues of the Laguerre inequality recently obtained by the author, to give bounds on the zeros of the Charlier polynomials, which are uniform in all parameters involved.

1. Introduction. The Charlier polynomials are a family of classical discrete orthogonal polynomials, which can be defined either by the recurrence (see e.g. [14, 18]),

$$aC_{k+1}^{(a)}(x) = (k+a-x)C_k^{(a)}(x) - kC_{k-1}^{(a)}(x),\ C_{-1}^{(a)}(x) = 0,\ C_0^{(a)}(x) = 1, \tag{1}$$

or, for $a \neq 0$, by the generating function

$$e^z \left(1 - \frac{z}{a}\right)^x = \sum_{k=0}^{\infty} \frac{C_k^{(a)}(x)}{k!} z^k . \tag{2}$$

They are orthogonal on the set $\{0, 1, ...\}$ with respect to the weight $e^{-a}a^x/x!$ for $a > 0$, and also satisfy the following difference equation

$$aC_k^{(a)}(x+1) = (a-k+x)C_k^{(a)}(x) - xC_k^{(a)}(x-1). \tag{3}$$

The uniform asymptotic behaviour of the Charlier polynomials for all values of x was only recently found by T.M. Dunster [2]. Earlier B. Rui and R. Wong provided asymptotic results for restricted values of x, [17]. Today tackling problems concerning asymptotics of discrete orthogonal polynomials is a very difficult and technically involved business. One of the approaches is based on the steepest descend method applied to a generating function and contains many ad hoc arguments. This program was realized (yet under certain restrictions on the corresponding parameters) for Krawtchouk, Meixner and in part for Charlier polynomials, see [1, 2, 3, 8, 9]. Methods based on different ideas for bounding zeros of orthogonal polynomials can be found in [5, 6, 12, 13]. In particular, applying Theorem 2 in [5] with $a_n = \frac{1}{4}$, one finds that the largest zero x_k of $C_k^{(a)}(x)$ satisfies

$$x_k < k + a - \frac{1}{2} + \sqrt{4ak + \frac{1}{4}}. \tag{4}$$

It is also known that the jth smallest zero converges to $j-1$, for a fixed a, [6]. The aim of this paper is to establish sharper explicit bounds, uniform in k and a, on the extreme zeros of $C_k^{(a)}(x)$, provided $a > 0$. We shall use the approach developed in [4, 10] and the difference analogues of Laguerre inequality found in [11]. Namely, we prove the following results.

THEOREM 1. *Let x_1 and x_k be the least and the largest zeros of the Charlier polynomial $C_k^{(a)}(x)$. Then x_1 and x_k are confined between the only two real roots of the following equation in x, $y = x - a - k$,*

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$$F(x) = (y^2 - 4ak + 2a)^3 - 2y^4 - 2ay^3 + (12ak - 3a^2 - 6a + 1)y^2 +$$

$$2a(6ak - 3a + 1)y - a^2(12k^2 - 16ak - 12k + 8a - 1) = 0\,, \tag{5}$$

provided $k \geq 3$ and $a \geq (\sqrt{k}+1)^2$. Moreover, for $a \geq k \geq 2$, x_k still does not exceed the largest root of (5).

To state the next two theorems it will be convenient to set $\beta = \sqrt{\frac{a}{k}}$, the substitution which also will be used in the sequel.

THEOREM 2. *For $\frac{5}{4} < a < k$, $k \geq 3$, x_k does not exceed the largest root in x of the following equation*

$$\Phi(x) = (1+\beta)(1+\beta+\beta^2)y^4 - ((1+2\beta+2\beta^2)k - 2(1+\beta)(1+\beta+\beta^2))y^3 -$$

$$((1+4\beta+9\beta^2+8\beta^3+2\beta^4-2\beta^5)k - (1+\beta)(1+\beta+\beta^2))y^2 -$$

$$\beta^2 k(1-2\beta^2-2\beta^3)y + \beta^3(1+\beta)^4 k^2 = 0\,, \tag{6}$$

where $y = k(1+\beta)^2 - x$.

The explicit bounds corresponding to (5) and (6) are given by

THEOREM 3. *For $a \geq (\sqrt{k}+1)^2$, $k \geq 3$,*

$$x_1 > k(\beta-1)^2 + 2^{-4/3}k^{1/3}\beta^{1/3}(\beta-1)^{2/3} + \frac{\beta}{2}\,. \tag{7}$$

For $a \geq k$,

$$x_k < k(\beta+1)^2 - 2^{-4/3}k^{1/3}\beta^{1/3}(\beta+1)^{2/3} - \frac{\beta}{2}\,. \tag{8}$$

For $\frac{5}{4} < a < k$

$$x_k < k(\beta+1)^2 - \frac{\beta(1+\beta)^{4/3}k^{2/3}}{\left(k(1+2\beta+2\beta^2)^2 - 2(1+2\beta+2\beta^2+\beta^3)\right)^{1/3}}\,. \tag{9}$$

For completeness we shall present the following two claims, the first giving very simple unconditional (and weaker) bounds for all $a > 0$, and the second providing some information about the location of the least zero for $a \leq (\sqrt{k}+1)^2$.

THEOREM 4. *For $a \geq k$, the zeros of Charlier polynomial $C_k^{(a)}(x)$ are confined in the interval*

$$\Big((\sqrt{k} - \sqrt{a}\,)^2 + 1, (\sqrt{k} + \sqrt{a}\,)^2 \Big)\,,$$

whenever for $0 < a < k$ they are in the interval $\Big(0,\, (\sqrt{k} + \sqrt{a}\,)^2\Big)$.

It is easy to compare the results of Theorems 3 and 4 since $(\sqrt{k}\pm\sqrt{a}\,)^2 = k(\beta\pm1)^2$. Notice also that (4) gives slightly better bound for the largest zero than Theorem 4 as

$$k+a-\frac{1}{2}+\sqrt{4ak+\frac{1}{4}} = (\sqrt{k}+\sqrt{a}\,)^2-\frac{1}{2}+O(\frac{1}{\sqrt{ak}}).$$

Theorem 5. *For* $k \geq 2$,
$x_1 < 1$ *if* $a \leq k$,
$x_1 < 2$ *if* $a \leq k+\sqrt{k}$,
$x_1 < 3$ *if* $a \leq k+\delta\sqrt{k}$,
$x_1 < 4$, *if* $a \leq (\sqrt{k}+1)^2$,
where δ *is the positive zero of* $\delta^3-3\delta-\frac{2}{\sqrt{k}}=0$, $\delta=\sqrt{3}+O(\frac{1}{\sqrt{k}})$.

The paper is organized as follows. In Section 2 we present the inequalities being our main tool in the sequel and shortly describe our approach. In Section 3 we prove Theorems 4 and 5. In fact, the proof of Theorem 4 contains the main idea of the method. It can also serve as a good illustration for the rest of the results without tiresome calculations which are necessary for establishing (5) and (6). The next three sections are devoted to proving Theorems 1,2 and 3. Application of the Laguerre type inequalities reduces the problem to investigation of graphs of certain polynomials in 3 variables x, k and a. This requires a substantial amount of symbolic calculations and we used Mathematica to perform them. In particularly we need to calculate a great deal of resultants of polynomials. The resultant in x of polynomials $f(x) = a_n x^n + ...$, and $g(x) = b_k x^k + ...$, $a_n, b_k \neq 0$, is defined by

$$Result_x(f,g) = a_n^k b_k^n \prod_{i=1}^{n}\prod_{j=1}^{k}(x_i-y_j),$$

where $x_1, x_2, ...$, and $y_1, y_2, ..$, are the zeros of f and g respectively. Since all we have to know about resultants is whether they vanish in a certain region, we will ignore their sign and omit multiplicative numerical constants.

2. Inequalities. We denote by $\mathcal{RP}$ the set of all hyperbolic polynomials (that is real polynomials with only real zeros). Let $x_1 \leq x_2 \leq ... \leq x_k$, be the zeros of $f = f(x) \in \mathcal{RP}$, the mesh $M(f)$ of f is defined by

$$M(f) = \min_{1\leq i\leq k-1}(x_{i+1}-x_i).$$

First of all we need the following simple observation (see [11] for a proof).

Theorem 6. *Let* $p(x)$ *be a discrete orthogonal polynomial corresponding to an orthogonality measure supported on a subset of integers. Suppose that* $p(x)$ *satisfies*

$$p(x+1) = b(x)p(x) - c(x)p(x-1) \tag{10}$$

and has all its zeros in the open interval I. *Then* $M(p) > 1$ *provided* $c(x) > 0$ *for* $x \in I$. *If in addition* $b(x) > 0$ *on* I, *then* $M(p) \geq 2$.

It is well known and readily follows from the orthogonality that all the zeros of the Charlier polynomials are positive, provided $a > 0$. Thus, applying the above Theorem to (3) we get

Corollary 1. *$M(C_k^{(a)}(x)) > 1$ for $a > 0$ and $M(C_k^{(a)}(x)) \geq 2$ for $a \geq k$.*

Our main tools are the following inequalities established in [11], which are the discrete analogue of the generalization of the Laguerre inequality given by Jensen [7] and rediscovered by Patrick [15], [16].

Theorem 7. *Let $p = p(x)$ be a hyperbolic polynomial, then*
(i) for $M(p) \geq \sqrt{4 - \frac{6}{m+2}}$,

$$V_m(p) = \sum_{j=-m}^{m} (-1)^j \frac{p(x-j)p(x+j)}{(m-j)!(m+j)!} \geq 0, \quad m = 1, 2, ... \tag{11}$$

In particular

$$V_1(p) = p^2(x) - p(x-1)p(x+1) \geq 0, \tag{12}$$

for $M(p) \geq \sqrt{2}$, and

$$12V_2(p) = 3p^2(x) - 4p(x-1)p(x+1) + p(x-2)p(x+2) \geq 0, \tag{13}$$

for $M(p) \geq \sqrt{5/2}$.
(ii) If $M(p) \geq 1$ then for any $\mu(x) \geq 0$,

$$U_\mu(p) = p^2(x) - p(x-1)p(x+1) + \frac{1}{4}\left(p(x+1) - \mu(x)p(x) + p(x-1)\right)^2 \geq 0. \tag{14}$$

As (11) does not contain any additional parameters and leads to easier calculations, we split our investigation into two cases. For $a \geq k$ we shall apply (12) and (13), and use (14) for $0 < a < k$.
We will need two more technical statements.

Corollary 2. *Each branch of the function $t_k^a(x) = \frac{C_k^{(a)}(x+1)}{C_k^{(a)}(x)}$ is a decreasing function in x, (i.e. $\frac{d}{dx} t_k^a(x) < 0$), and $t_k^a(0) = 1 - \frac{k}{a}$, and $\lim_{x\to\infty} t_k^a(x) = 1$.*

Proof. Let $x_1 < ... < x_k$, be the zeros of $C_k^{(a)}(x)$. By the previous corollary the zeros of $C_k^{(a)}(x)$ and $C_k^{(a)}(x+1)$ interlace. Hence, all the coefficients a_i in the partial fraction decomposition $t_k^a(x) = \sum_{i=1}^k \frac{a_i}{x - x_i}$ are positive. This implies the first claim. The second one follows from (2) and the last is obvious. □
Inequality (14) being applied directly to an orthogonal polynomial $p(x)$ yields relatively weak bounds. To obtain a sharper result we perturb the zeros of $p(x)$ by considering instead $p(x) - \gamma p(x-1)$, for an appropriate constant γ. As easily can be checked, if $M(p) \geq 1$, this new polynomial has only real zeros for any $\gamma \geq 0$. Moreover, a quick inspection of the intersection of the straight line $y = \frac{1}{\gamma}$ with the graph of the functions $\frac{p(x-1)}{p(x)}$ yields

Lemma 1. *Let $p(x)$ be a hyperbolic polynomial with $M(p) > 1$. Then the mesh of the hyperbolic polynomial $q(x) = p(x) - \gamma p(x-1)$ satisfies $M(q) > 1$, for any $\gamma \geq 0$.*

Let us now explain how the inequalities of Theorem 7 can be used for bounding zeros of discrete orthogonal polynomials. Let $p(x)$ be such a polynomial with zeros $x_1 < ... < x_k$. First of all notice that by (10) one can express $p(x+j)$ as a linear combination of $p(x+1)$ and $p(x)$ with coefficients depending on x. Choosing either (11) or (14), depending on the mesh, and putting $t = t(x) = p(x+1)/p(x)$, one obtains a nonnegative expression

$$W(x) = A(x)t^2 + B(x)t + C(x) \geq 0. \tag{15}$$

By Corollary 2 the graph of $t(x)$ consists of decreasing, cotangent-shaped branches in the middle and two hyperbolic branches at the ends. Suppose for simplicity that A, B and C are continuous functions, changing relatively slowly in comparison with the rapidly oscillating $p(x)$. Since in the oscillatory region $[x_1, x_n]$, $t(x)$ attains all real values from $-\infty$ to ∞, one should have here $A(x) > 0$, by (15). Moreover, consider the minimum of W viewed as a quadratic in t. It is equal to $\frac{4AC-B^2}{2A}$ and attains for $t = \bar{\mathbf{t}}(x) = -\frac{B}{2A}$. If $\bar{\mathbf{t}}$ intersects all the branches of t, then at such an intersection point one has $\frac{4AC-B^2}{2A} \geq 0$. Thus, the discriminant of (15) in t is negative on the interval confining the roots of the polynomial. Therefore, the solution of the last inequality provides bounds on the extreme zeros of $p(x)$. In fact, as the coefficients of W depend on parameters, such as k and a in the case of Charlier polynomials, to justify these arguments one needs rather involved investigation of the inequality $\frac{4AC-B^2}{2A} \geq 0$. Of course, if one is looking for asymptotics rather than explicit bounds, the calculation can be substantially simplified.

Yet, it seems difficult to formulate a general theorem capturing those arguments. In fact, even a continuous function (and $\bar{\mathbf{t}}(x)$ is not necessary continuous) may not intersect the first and the last hyperbolic branches of $t(x)$. For instance, this happens if one applies (14) to the discrete Chebyshev polynomials. Therefore, in general the zeros of $4AC - B^2 = 0$, may not embrace the extreme roots of the polynomial. Fortunately, in our case this yields only some weak restrictions (as $a > \frac{5}{4}$ in Theorem 2) on the parameters.

3. First order bounds. In this section we prove Theorems 4 and 5. It will be convenient to use the following substitutions: $v = C_k^{(a)}(x)$, $u = C_k^{(a)}(x+1)$.

Proof of Theorem 4.

We will consider two cases $a \geq k$ and $0 < a < k$.

Case 1. $a \geq k$

Consider $V_1 = V_1(v)$ and set $t = t(x) = u/v$. As all the zeros $x_1 < ... < x_k$, of $C_k^{(a)}(x)$ are positive we assume $x > 0$. By (12) and Corollary 1 $V_1 \geq 0$, and on excluding $C_k^{(a)}(x-1)$ by (3) we get

$$\frac{xV_1}{v^2} = at^2 - (a + x - k)t + x \geq 0.$$

The minimum of this expression in t is attained for $\bar{\mathbf{t}}(x) = \frac{a+x-k}{2a}$ and is equal to

$$r(x) = -x^2 + 2kx + 2ax - k^2 + 2ak - a^2.$$

Comparing the graphs of the functions $t(x)$ and $\bar{\mathbf{t}}(x)$ we see that they are intersected for some $x_M > x_k$. Moreover, by $C_k^{(a)}(0) = 1 > \frac{a-k}{2a} > 0$, the functions are intersected also at some x_m, $0 < x_m < x_1 - 1$. Therefore $r(x_m) \geq 0$, and $r(x_M) \geq 0$. Since $r(x) \geq 0$ for $(\sqrt{a} - \sqrt{k})^2 \leq x \leq (\sqrt{a} + \sqrt{k})^2$, this yields

$$1 + (\sqrt{a} - \sqrt{k})^2 \leq x_1 < x_k \leq (\sqrt{a} + \sqrt{k})^2.$$

Case 2. $0 < a \le k$
Put $s = \sqrt{k} + \sqrt{a}$, $q = \sqrt{k} - \sqrt{a}$, and choose $\mu = \frac{2s}{s-q} + \frac{s-q}{2s}$ in (14). Similarly to the previous case we have

$$\frac{4x^2}{v^2}U_\mu = (x+a)^2t^2 - 2t((a+x)^2 + \mu x^2 - a\mu x - ak - kx) +$$

$$(x+a-k)^2 - x(2a\mu - 2k\mu - 4x + 2\mu x - \mu^2 x).$$

The minimum value of this expression in t is

$$r(x) = \frac{4x^2(s^2-x)(8sx(3s^2+q^2) + (s-q)^2(s^3+7qs^2-q^2s+q^3))}{s^2(s-q)(x+a)^2},$$

and attains at

$$\bar{\mathfrak{t}}(x) = 1 + \frac{\mu x^2 - a\mu x - kx - ak}{(a+x)^2}.$$

Since $\lim_{x\to\infty} \bar{\mathfrak{t}}(x) = \frac{7}{2} + \frac{q(3s+q)}{2s(s-q)} > 1$, whenever $\lim_{x\to\infty} t(x) = 1$, the graphs of t and $\bar{\mathfrak{t}}(x)$ intersects for some $x_M > x_k$. Thus, $r(x_M) \ge 0$, what implies $x_k < s^2$. $\square$

Proof of Theorem 5.
It is enough to check that $C_k^{(a)}(x)$ changes sign on each of the corresponding intervals. We have $C_k^{(a)}(0) = 1$, and $C_k^{(a)}(1) = 1 - \frac{k}{a}$, giving the first claim. Assuming $k \le a < k + \sqrt{k}$, we get $C_k^{(a)}(2) = \frac{(a-k)^2-k}{a^2} < 0$, proving the second. The third claim is similar. To prove the last it is enough to show that $C_k^{(a)}(4) < 0$, for $k + \sqrt{k} \le a \le (\sqrt{k}+1)^2$. For $2 \le k \le 8$ the claim can be checked directly. Assuming $k \ge 9$ we have

$$C_k^{(a)}(4) = \frac{(a-k)^4 - 6k(a-k)^2 + k(11k - 8a - 6)}{a^4}.$$

Now, $(\sqrt{k}+1)^2 \le k + \frac{7}{3}\sqrt{k}$, for $k \ge 9$, and it is enough to show that $C_k^{(a)}(4) < 0$, for $k + \sqrt{k} \le a \le k + \frac{7}{3}\sqrt{k}$. Putting $a = k + \epsilon\sqrt{k},\ 1 \le \epsilon \le \frac{7}{3}$, we obtain

$$C_k^{(a)}(4) = k^2(\epsilon^4 - 6\epsilon^2 + 3) - k(8\epsilon\sqrt{k} + 6) < k^2(\epsilon^4 - 6\epsilon^2 + 3) < 0.$$

This completes the proof. $\square$

4. Second Order Bounds, $a \ge k$**.** The aim of this section is to prove Theorem 1. In view of Theorem 4 we may assume $x > 1$. On using the substitution $x = a+k+y$, we get

$$F(t,x) = \frac{ax(x-1)}{v^2}V_2(v) = aA(x)t^2 + B(x)t + C(x) \ge 0,$$

where

$$A(x) = -y^2 - 4a + 4ak + 1,$$

$$B(x) = y^3 + 2ay^2 - (4ak - 3a + 1)y - a(8ak - 8a + 2k + 3),$$

$$C(x) = -y^3 - (k+a)y^2 + (4ak - 3a + k + 1)y + 2a(2k^2 + 2ak - 2a + 1).$$

Since $V_2 \geq 0$, at $x = x_1, x_k$, it follows $A(x_1), A(x_k) \geq 0$. This gives $c_1 < x_1 < x_k < c_2$, where

$$c_1 = k + a - \sqrt{4a(k-1)+1}, \quad c_2 = k + a + \sqrt{4a(k - 1a + 1}.$$

The minimum of $F(t,x)$ in t is attained at $t = \bar{\mathbf{t}}(x) = -\frac{B(x)}{2aA(x)}$, $F(\bar{\mathbf{t}}, x) = \frac{4aA(x)C(x)-B^2(x)}{4aA(x)}$. We shall show that the roots of $C_k^{(a)}(x)$ lay between the only two real roots of the equation $A(x)C(x) - B^2(x) = 0$. The last one is precisely (5). We need the following technical lemmas.

Lemma 2. *$\bar{\mathbf{t}}(x)$ is a continuous function increasing on (c_1, c_2), from $-\infty$ to ∞, provided $k \geq 2$ and $a > \frac{k(k+1)}{k-1}$.*

Proof. Calculating the resultant of $A(x)$ and $B(x)$ in x, we obtain

$$Result_x(A(x), B(x)) = a^2(k^2 - ak + k + a) \neq 0.$$

Thus, $A(x)$ and $B(x)$ have no common zeros for $x \in [c_1, c_2]$. Hence it is enough to prove $\frac{d}{dx}\bar{\mathbf{t}}(x) = \frac{D(x)}{2aA^2(x)} > 0$, where

$$D(x) = y^4 - (8ak - 9a + 2)y^2 + 2a(2k+1)y + (4ak - 3a + 1)(4ak - 4a + 1).$$

The discriminant of $D(x)$ in x, that is $Result_x(D(x), D'(x))$, is

$$Dis_x(D(x)) = -64a^2(k^2 - ak + k + a)((4k-3)(32k-33)^2a^3 + 12(265k^2 - 471k + 225)a^2 +$$

$$48(16k - 15)a + 64).$$

The last factor is positive for $k \geq 2$, and thus the number of real roots of $D(x)$ does not depend on a and k for $k \geq 2$ and $a > \frac{k(k+1)}{k-1}$, since the discriminant does not vanish in this region. Taking $k = 2$, $a = 7$ we get $D(x) = x^4 - 36x^3 + 435x^2 - 1928x + 2844 > 0$, implying $\frac{d}{dx}\bar{\mathbf{t}}(x) > 0$. ∎

Lemma 3. *For $a \geq k \geq 2$, $\bar{\mathbf{t}}(x)$ is a continuous function on $(a+k, c_2)$ and $\lim_{x \to c_2^{(-)}} \bar{\mathbf{t}}(x) = \infty$.*

Proof. The continuity clearly follows from $c_1 < a + k$. We also have

$$A(c_2 - \epsilon) = \epsilon(2\sqrt{4a(k-1)+1} - \epsilon) > 0$$

$$B(c_2) = -a(1 + 2k + \sqrt{4a(k-1)+1}) < 0,$$

hence $\bar{\mathbf{t}}(c_2 - \epsilon) > 0$, for sufficiently small $\epsilon > 0$, hence $\lim_{x \to c_2^{(-)}} \bar{\mathbf{t}}(x) = \infty$. ∎

Lemma 4. *The function $\Delta(x) = 4aA(x)C(x) - B^2(x)$ has precisely two real zeros x_m, x_M, provided $k \geq 3$, $a \geq (\sqrt{k}+1)^2$. Moreover, $c_1 < x_m < x_M < c_2$ and $\Delta(x) > 0$ for $x_m < x < x_M$.*

Proof. The discriminant of $\Delta(x)$ in x is

$$\begin{gathered}Dis_x(\Delta) = a^8(k^2-k)^3((a+3)(108a^3-81a^2+36a-4)^2 - \\ 3(7776a^7+31104a^6-2106a^5+27522a^4-24759a^3+5970a^2-436a-8)k + \\ 3(31104a^6+69984a^5+58626a^4-34101a^3+3360a^2+388a-8))k^2 - \\ 2a(69984a^4+93312a^3-14661a^2-7830a+388)k^3 + \\ 54a^2(1728a^2+1080a-145)k^4-23328a^3k^5-23328a^3k^5).\end{gathered}$$

It is easy to show that this expression does not vanish for $k \geq 2$, $a \geq (\sqrt{k}+1)^2$. Choosing $k=4, a=9$ we obtain $\Delta(x) = 285156-288924x+109813x^2-19422x^3+1651x^4-66x^5+x^6$, having two real roots.
It is left to show that the roots of Δ are in (c_1,c_2). Noticing that $(\sqrt{k}+1)^2 > \frac{k(k+1)}{(k-1)}$, for $k \geq 3$, we have $Result_x(\Delta, A(x)) = a^4(k^2-ak+k+a)^2 > 0$. Therefore, it is enough to check the claim for any suitable a and k, say, $k=4$, $a=9$. We omit the details. $\square$

Lemma 5. *For $a \geq k \geq 2$, the function $\Delta(x)$ has the only zero x_M in the interval*

$$(a+k, c_2).$$

Proof. By the previous lemma it suffices to show $\Delta(c_1) \leq 0$, $\Delta(c_2) < 0$, and $\Delta(a+k) > 0$. Calculations yield

$$\begin{aligned}&\Delta(a+k \pm \sqrt{4a(k-1)+1}) \\ &= -2a^2\left(2k(k+a+1)-2a+1 \pm (2k+1)\sqrt{4a(k-1)+1}\right).\end{aligned}$$

Thus $\Delta(c_2) < 0$, and by

$$\Delta(c_1)\Delta(c_2) = 16a^4(k^2-ak+k+a)^2 \geq 0,$$

it follows $\Delta(c_1) \leq 0$. Similarly,

$$\Delta(a+k) = a^2(64ak^3-96ak^2+12k^2+32ak-12k-1) > 0.$$

$\square$

Proof of Theorem 1. Recall that equation (5) is merely $\Delta(x) = 0$. Assume first that $k \geq 3$ and $a \geq (\sqrt{k}+1)^2 > \frac{k(k+1)}{k-1}$. By Lemma 2 the function $\bar{\mathfrak{t}}(x)$ intersects all the branches of $t(x)$, consequently all the zeros of $C_k^{(a)}(x)$ are confined between the points P and Q - the first and the last intersection respectively. As $F(t,x) \geq 0$ one should have $F(\bar{\mathfrak{t}}, P) \geq 0$, $F(\bar{\mathfrak{t}}, Q) \geq 0$, or, by the positivity of $A(x)$ on (c_1,c_2), $\Delta(P) > 0$, $\Delta(Q) > 0$. Now the result follows by Lemma 4. Finally, under the weaker condition $a \geq k \geq 3$, we can assume $x_M > a+k$, otherwise there is nothing to prove. Then the function $\bar{\mathfrak{t}}(x)$ still intersects the rightmost branch of $t(x)$ by Lemma 3. Now the result follows by Lemma 5. $\square$

5. Second Order Bounds, $a < k$. The aim of this section is to establish Theorem 2. We will use the substitutions $\alpha = \sqrt{a}$, $\kappa = \sqrt{k}$. Choose in (14)

$$p(x) = u(x) - \gamma u(x-1), \quad \gamma = 1 + \frac{\kappa}{\alpha}, \quad \mu = \gamma + \frac{1}{\gamma}.$$

Notice that the optimal choice of γ and μ is not known. Numerical experiments show that the answer is not too sensitive to the value of μ. Our choice was motivated rather by attempts to simplify the calculations.
Consider $U_\mu(p)$, by Theorem 6, Lemma 1, and (14) we get

$$\frac{4x^2(x-1)^2\alpha^6\gamma^2}{v^2}U_\mu(p) = \alpha^4(\alpha+\kappa)^2A(x)t^2 + 2\alpha^2(\alpha+\kappa)B(x)t + C(x) \geq 0 \qquad (16)$$

where

$$A(x) = y^4 - 2(\kappa^2+\alpha\kappa-1)y^3 + (\kappa^4 - 2\alpha\kappa^3 - 7\alpha^2\kappa^2 - 2\kappa^2 - 8\alpha^3\kappa + 2\alpha^2 + 1)y^2 +$$

$$2\alpha(2\kappa(\alpha+\kappa)^4 - 5\kappa^3 - 8\alpha\kappa^2 - 5\alpha^2\kappa + \kappa + 2\alpha^3 + \alpha)y +$$

$$\alpha(\kappa-\alpha)(4(\kappa+\alpha)^4 - 5\alpha^2 - 7\alpha\kappa - 4\kappa^2);$$

and $B(x), C(x)$ are certain polynomials too complicated to be given here. As before, we will consider $\bar{\mathfrak{t}}(x) = -\frac{B(x)}{\alpha^2(\alpha+\kappa)A(x)}$, yielding the extremal value $\frac{A(x)C(x)-B^2(x)}{A(x)}$ of (16), where

$$A(x)C(x) - B^2(x) = -4\alpha\kappa^2(y+1-(\alpha+\kappa)^2)^2(y-(\alpha+\kappa)^2)^2\Phi(x)\,, \qquad (17)$$

and $\Phi(x)$ is given by (6). We shall show that the zeros of $C_k^{(a)}(x)$ lie between the zeros of the equation $A(x)C(x) - B^2(x) = 0$, or, equivalently $\Phi(x) = 0$.

LEMMA 6. *Let $\frac{5}{4} < a < k$, $k \geq 3$, then $A(x)$ has precisely two real zeros, the smallest is in the interval $\left((\alpha+\kappa)^2, (\alpha+\kappa+1)^2\right)$.*

Calculation yields

$$Dis_x(A(x)) = \alpha^3(\kappa-\alpha)^2(\kappa+\alpha)^4R(\alpha,\kappa),$$

where $R(\alpha,\kappa)$ is a certain polynomial of degree 11 in α and 13 in κ. In turn,

$$Dis_\alpha(R(\alpha,\kappa)) = \kappa^{28}(1-2\kappa+2\kappa^2)^2(1+2\kappa+2\kappa^2)^2S(\kappa)T^3(\kappa),$$

where S and T are polynomials of degree 24 and 40 in κ respectively, positive for $\kappa \geq 1$. Thus, the number of real zeros of $R(\alpha,\kappa)$ does not depend on κ in our region. Choosing $\kappa = \alpha$ we get

$$R(\alpha,\alpha) = \alpha^5(1-4\alpha^2)^2(5-4\alpha^2)(1-8\alpha^2)^2 \neq 0$$

for $\sqrt{\frac{5}{4}} < \alpha \leq \kappa$. Hence to find the number of real zeros of $A(x)$ we can substitute any α and κ such that $\sqrt{\frac{5}{4}} < \alpha < \kappa$. Now $R(2,3) = 4804 + 13728y - 480y^2 - 28y^3 + y^4 = 0$, has precisely two real roots.
Finally, we shall show that the least zero of $A(x)$ lies between $(\alpha+\kappa)^2$ and $(\alpha+\kappa+1)^2$. We have

$$A((\alpha+\kappa)^2) = \alpha(\kappa-\alpha)\left(4(\alpha+\kappa)^4 - 4(\alpha+\kappa)^2 + \alpha(\alpha-\kappa)\right) > 0.$$

by $\alpha + \kappa > 2\sqrt{\frac{5}{4}}$. On the other hand

$$A((\alpha+\kappa+1)^2) = (\alpha+\kappa)^2(4(1-2\alpha)\kappa^4 - 4(6\alpha^2+2\alpha-5)\kappa^3 - (24\alpha^3+32\alpha^2-40\alpha-33)\kappa^2 -$$

$$(8\alpha^4+40\alpha^3-12\alpha^2-58\alpha-20)\kappa-4\alpha^4-8\alpha^3+25\alpha^2+20\alpha+4)\,.$$

It is easy to check that the last factor decreases in α. Substitution $\alpha=\sqrt{\frac{5}{4}}$, shows that the above expression is negative. ∎

LEMMA 7. *Let x_m be the least zero of $A(x)$. Then for $k>a\geq\frac{5}{4}$, $\bar{\mathbf{t}}(x)$ is a continuous function for $x<x_m$, and* $\lim_{x\to x_m^{(-)}}\bar{\mathbf{t}}(x)=\infty$.

Proof. It is enough to show that $B(x)<0$ for $(\alpha+\kappa)^2<x<(\alpha+\kappa+1)^2$. Calculations yield

$$Result_x(A(x),B(x))=\alpha^6\kappa^{18}(\kappa-\alpha)^2(\kappa+\alpha)^8M(\alpha,\kappa)N(\alpha,\kappa),$$

where

$$M(\alpha,\kappa)=\beta^2(1-\beta)^2(1+\beta)^4k^2-2(\beta-\beta^3)((1+\beta)(2\beta^2+\beta+1)k+8\beta^4+12\beta^3+9\beta^2+2\beta+1$$

$$N(\alpha,\kappa)=8\beta^2(1+\beta)^4(2\beta^6+2\beta^5-\beta^3+\beta^2+2\beta+1)k^2-$$

$$(1+\beta)^2(16\beta^{10}+16\beta^9+8\beta^8+28\beta^7+73\beta^6+76\beta^5+45\beta^4+23\beta^3+19\beta^2+12\beta+4)k+$$

$$(2\beta^2+2\beta+1)(4\beta^6+4\beta^5+\beta^4+4\beta^3+12\beta^2+11\beta+1),$$

and $\beta=\frac{\alpha}{\kappa}$. The discriminant of $M(\alpha,\kappa)$, which is a quadratic in k, is $-16(1-\beta^2)^2(\beta+\beta^2)^4<0$, hence $M(\alpha,\kappa)>0$.
Solving now in k the inequality $N(\alpha,\kappa)<0$, we obtain that

$$k<\frac{L(\beta)-K(\beta)\sqrt{4\beta^4-4\beta^3+\beta^2+4\beta+4}}{16\beta^2(1+\beta)^2(2\beta^6+2\beta^5-\beta^3+\beta^2+2\beta+1)}\,,$$

where

$$L(\beta)=16\beta^{10}+16\beta^9+8\beta^8+28\beta^7+73\beta^6+76\beta^5+45\beta^4+23\beta^3+19\beta^2+12\beta+4,$$

$$K(\beta)=8\beta^8+12\beta^7+10\beta^6+15\beta^5+24\beta^4+17\beta^3+\beta^2-5\beta-2.$$

Thus, if $N(\alpha,\kappa)<0$ then

$$a=\beta^2k<\frac{L(\beta)-K(\beta)\sqrt{4\beta^4-4\beta^3+\beta^2+4\beta+4}}{16(1+\beta)^2(2\beta^6+2\beta^5-\beta^3+\beta^2+2\beta+1)}.$$

The maximum of the last expression is $\frac{1}{2}$ and attains for $\beta=0$. Thus in our region $Result_x(A(x),B(x))$ does not vanish. Therefore, one can check the claim, say, for $k=3$, $a=2$. ∎

Now, we change the variable x into $y=k(1+\beta)^2-x$, to transform the equation $\Phi(x)=0$ into the equation $\Phi^*(y)=0$. (in fact (6)).

LEMMA 8. *For $0<a<k$, $k\geq 3$, equation $\Phi^*(y)=0$ has two positive zeros, the least lies in the interval $(0,k-1)$, whenever the largest is greater than $k-1$.*

Proof. It is easy to check that for $k > 2$, and $0 < \beta < 1$, the following sign pattern $(+, -, -, \pm, +)$ holds for the coefficients of $\Phi^*(x)$. Thus, according to the Descartes rule of signs there are at most two positive zeros. Obviously $\Phi^*(0) > 0$, and also, with a little more effort,

$$\Phi^*(k-1) = -bk^2((4\beta^3 + 7\beta^2 + 6\beta + 2)k^2 - (2\beta^4 + 10\beta^3 + 14\beta^2 + 11\beta + 4)k +$$

$$(\beta^2 + 3\beta + 3)(1 + \beta + \beta^2 - \beta^3 - \beta^4)) < 0.$$

Moreover,

$$\Phi^*\left(\frac{(1+2\beta+2\beta^2)^2 k}{1+2\beta+2\beta^2+\beta^3}\right) > 0,$$

since its expansion has only positive terms. Now the result follows by comparing the signs of $\Phi^*(x)$ at those 3 points. □

Proof of Theorem 2. The proof is similar to that of Theorem 1 and readily follows from the above lemmas. We omit the details. □

6. Proof of Theorem 3. To prove (7) it is enough to show that $F(x) > 0$ in (5), for x given by the right hand side of (5) (7). We put $k = n^6,\ a = \delta^6 n^6,\ r = (1+\delta^3)^{1/3}$. Thus $\delta \geq 1$. The substitution into $F(x)$ yields a quadratic in r. It can be checked by the substitution $n := n + 1,\ \delta := \delta + 1$, that the leading coefficient contains only nonnegative terms and therefore is positive. The discriminant of this quadratic is a polynomial in k and is negative for $k \geq 2$. This can be checked by the substitution $k := k + 2,\ \delta := \delta + 1$, in the same way. The proof of (8) is similar.

To demonstrate (9) we notice that in fact the term

$$y^* = \frac{\beta(1+\beta)^{4/3}k^{2/3}}{\left(k(1+2\beta+2\beta^2)^2 - 2(1+2\beta+2\beta^2+\beta^3)\right)^{1/3}},$$

is just the real solution in y of the equation

$$\left(k(1+2\beta+2\beta^2)^2 - 2(1+2\beta+2\beta^2+\beta^3)\right) y^3 - \beta^3(1+\beta)^4 k^2 = 0,$$

which is obtained from equation (6) by omitting the terms corresponding to y, y^2 and y^4. Hence one has to show that the left hand side of (2) is positive for $y = y^*$. It can be routinely done using easy to check inequality $(\frac{4}{5})^{2/3}\beta k^{1/3} \leq y^* \leq 3^{1/3}\beta k^{1/3}$, for $k \geq 3$, and the assumption $\alpha > \sqrt{\frac{5}{4}}$. We omit the details.

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REFERENCES

[1] W. Van Assche and A. B. J. Kuijlaars, *The asymptotic distribution of orthogonal polynomials with varying recurrence coefficients*, J. Approx. Theory, 99 (1999), pp. 167–197.

[2] T.M. Dunster, *Uniform asymptotic expansions for Charlier polynomials*, J. Approx. Theory, 112 (2001), pp. 93–133.

[3] P. D. Dragnev and E. B. Saff, *A problem in potential theory and zero asymptotics of Krawtchouk polynomials*, Journal of Approximation Theory, 102 (1999), pp. 120–140.

[4] W. H Foster and I. Krasikov, *Bounds for the extreme roots of orthogonal polynomials*, Int. J. of Math. Algorithms, 2 (2000), pp. 121–132.

[5] M.E.H. Ismail and X. Li, *Bounds on the extreme zeros of orthogonal polynomials*, Proc. Amer. Math. Soc., 115 (1992), pp. 131–140.

[6] M.E.H. Ismail, M.E. Muldoon, *A discrete approach to monotonicity of zeros of orthogonal polynomials*, Trans. Amer. Math. Soc., 323 (1991), pp. 65–78.

[7] J.L.W.V. Jensen, *Recherches sur la theorie des equations*, Acta Math., 36 (1913), pp. 181–195.

[8] X.-S. Jin, R. Wong, *Uniform asymptotic expansion for Meixner polynomials*, Constr. Approx., 14 (1998), pp. 113–150.

[9] X.-S. Jin, R. Wong, *Asymptotic formulas for the zeros of the Meixner polynomials*, J. Approx. Theory, 96 (1999), pp. 281–300.

[10] I. Krasikov, *Nonnegative quadratic forms and bounds on orthogonal polynomials*, J. Approx. Theory, 111 (2001), pp. 31–49.

[11] I. Krasikov, *Discrete analogues of the Laguerre inequality*, Analysis and Applications, 1 (2003), pp. 189–198.

[12] V. I. Levenstein, *Universal bounds on codes and designes*, In: Handbook of Coding Theory, Vol.1, North-Holland, 1998, pp. 499–648.

[13] A. Mate, P. Nevai, V. Totik, *Asymptotic of the zeros of orthogonal polynomials associated with infinite intervals*, J. London. Math. Soc., 33 (1986), pp. 303–310.

[14] A. F. Nikiforov, S. K. Suslov and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, (Translated from the Russian) Springer-Verlag, Berlin, 1991.

[15] M. L. Patrick, *Some inequalities concerning Jacobi Polynomials*, SIAM J. Math. Anal., 2 (1971), pp. 213–220.

[16] M. L. Patrick, *Extension of inequalities of the Laguerre and Turan type*, Pacific J. Math., 44 (1973), pp. 675–682.

[17] B. Rui, R. Wong, *Uniform asymptotic expansion for Charlier polynomials*, Methods Appl. Anal., 1 (1994), pp. 109–134.

[18] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., v.23, Providence, RI, 1975.