METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 9, No. 4, pp. 563–578, December 2002

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A NEW REGULARITY CRITERION FOR THE NAVIER-STOKES EQUATIONS IN TERMS OF THE GRADIENT OF ONE VELOCITY COMPONENT *

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Abstract. In this paper we consider the regularity criteria for the weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$. It is proved that if the gradient of any one component of the velocity field belongs to $L^{\alpha,\gamma}$ with $2/\alpha + 3/\gamma = 3/2$, $3 \leq \gamma < \infty$, then the weak solution actually is strong.

1. Introduction. We consider the following Cauchy problem for the incompressible Navier-Stokes equations in $\mathbb{R}^3 \times (0,T)$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \Delta u, \\ \text{div} u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity field, $p(x,t)$ is a scalar pressure, and $u_0(x)$ with $\text{div} u_0 = 0$ in the sense of distribution is the initial velocity field.

The study of the incompressible Navier-Stokes equations in three space dimensions has a long history. In the pioneering work [10] and [7], Leray and Hopf proved the existence of its weak solutions $u(x,t) \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$ for given $u_0(x) \in L^2(\mathbb{R}^3)$. But the uniqueness and regularity of the Leray-Hopf weak solutions are still big open problems. In [12], Scheffer began to study the partial regularity theory of the Navier-Stokes equations. Deeper results were obtained by Caffarelli, Kohn and Nirenberg in [2]. Further result can be found in [17] and references there in.

On the other hand, the regularity of a given weak solution u can be shown under additional conditions. In 1962, Serrin [13] proved that if u is a Leray-Hopf weak solution belonging to $L^{\alpha,\gamma} \equiv L^\alpha(0,T;L^\gamma(\mathbb{R}^3))$ with $2/\alpha + 3/\gamma \leq 1, 2 < \alpha < \infty, 3 < \gamma < \infty$, then the solution $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T))$. From then on, there are many criterion results added on u. In [18] and [5], von Wahl and Giga showed that if u is a weak solution in $C([0,T);L^3(\mathbb{R}^3))$, then $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T))$; Struwe [16] proved the same regularity of u in $L^\infty(0,T;L^3(\mathbb{R}^3)$ provided $\sup_{0<t<T} \|u(x,t)\|_{L^3}$ is sufficiently small and Kozono and Sohr [8] obtained the regularity for the weak solution $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T))$ provided $u(x,t)$ is left continuous with respect to L^3-norm for every $t \in (0,T)$. Recently Kozono and Taniuchi [9] showed that if a Leray-Hopf weak solution $u(x,t) \in L^2(0,T;BMO)$, then $u(x,t)$is actually a strong solution of (1) on $(0,T)$. $L^{\alpha,\gamma}$ is defined by

$$\|u\|_{L^{\alpha,\gamma}} = \begin{cases} \left(\displaystyle\int_0^t \|u(.,\tau)\|^\alpha_{L^\gamma} d\tau\right)^{1/\alpha} & \text{if } 1 \leq \alpha < \infty \\ \text{ess} \sup\limits_{0<\tau<t} \|u(.,\tau)\|_{L^\gamma} & \text{if } \alpha = \infty \end{cases}$$

*Received November 21, 2002; accepted for publication January 14, 2003. This work is partically supported by Hong Kong RGC Earmarked Grants CUHK-4219-99P and CUHK-4279-00P.

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where

$$\|u(.,\tau)\|_{L^\gamma} = \begin{cases} \Big(\int_{\mathbb{R}^3} |u(x,\tau)|^\gamma dx\Big)^{1/\gamma} & \text{if } 1 \le \gamma < \infty \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^3} |u(x,\tau)| & \text{if } \gamma = \infty \end{cases}$$

The point is that $\|u_\lambda\|_{L^{\alpha,\gamma}} = \|u\|_{L^{\alpha,\gamma}}$ holds for all $\lambda > 0$ if and only if $2/\alpha + 3/\gamma = 1$, where $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, $p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$ and if (u,p) solves the Navier-Stokes equations, then so does (u_λ, p_λ) for all $\lambda > 0$. Usually we say that the norm $\|u\|_{L^{\alpha,\gamma}}$ has the scaling dimension zero for $2/\alpha + 3/\gamma = 1$ [2].

Sohr [14] extended Serrin's regularity criterion by introducing Lorentz space in both time and spatial direction, $u \in L^{s,r}(0,T;L^{q,\infty})$ with $2/s + 3/q = 1$, $3 < q < \infty$, $2 < s \le r < \infty$, here $L^{p,q}$ is Lorentz space, for weak solutions which satisfy the strong energy inequality. Later on, Sohr [15] extended Serrin's regularity class for weak solutions of the Navier-Stokes equations replacing the L^q-space by Sobolev spaces of negative order, $u \in L^s(0,T;H^{-\alpha,q})$ with $2/s + 3/q = 1 - \alpha$, $3 < q < \infty$, $2 < s < \infty$, for $0 \le \alpha < 1$.

Zhou [21] proved the regularity of the Leray-Hopf weak solution by adding the Serrin's regularity criterion only on two components of the velocity field. Also in [21], the author gave a regularity criterion by adding condition on one velocity component, say, $u_3 \in L^{\alpha,\gamma}$ with $2/\alpha + 3/\gamma \le 1/2$ for $\gamma > 6$.

One can find that if $2/\alpha + 3/\gamma = 2$, both $\|\nabla u\|_{L^{\alpha,\gamma}}$ and $\|p\|_{L^{\alpha,\gamma}}$ have scaling dimension zero. Related to this point, Beirão da Veiga [1] proposed the regularity criterion on ∇u, which states that if a weak solution $u(x,t)$ satisfies $\nabla u \in L^{\alpha,\gamma}, 2/\alpha + 3/\gamma \le 2, 3/2 < \gamma < \infty$, then $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T))$. Chae and Choe [3] improved Beirão da Veiga's condition by imposing that only on the two components of the vorticity field. Very recently, Zhou [21] proved that if a Leray-Hopf weak solution satisfies $\nabla u_3 \in L^{\alpha,\gamma}$ with $2/\alpha + 3/\gamma \le 3/2$, $2 < \gamma < 3$, or $\|\nabla u_3\|_{L^{\infty,2}}$ is sufficiently small, then the weak solution is strong.

In this paper, we want to prove the analogous result for $\gamma \ge 3$. More precisely, our main theorem reads

THEOREM 1. *Suppose $u_0 \in H^1(\mathbb{R}^3)$, and* $\mathrm{div} u_0 = 0$ *in the sense of distribution. Assume that $u(x,t)$ is a Leray-Hopf weak solution of (1) in $(0,T)$. If $\nabla u_3 \in L^{\alpha,\gamma}$ with $2/\alpha + 3/\gamma = 3/2$, $3 \le \gamma < \infty$, or $\nabla u_3 \in L^{4/3,\infty}$ then $u(x,t)$ is a strong solution on $[0,T)$.*

REMARK 1. In [6], He proved the same conclusion under a stronger condition $\nabla u_3 \in L^{\alpha,\gamma}$ with $2/\alpha + 3/\gamma = 1$.

Before going to sections, we recall the definition of Leray-Hopf weak solutions.

DEFINITION. A measurable vector u is called a Leray-Hopf weak solution to the Navier- Stokes equations (1), if u satisfies the following properties
(i) u is weakly continuous from $[0,T)$ to $L^2(\mathbb{R}^3)$.
(ii) u verifies (1) in the sense of distribution, i.e.,

$$\int_0^T \int_{\mathbb{R}^3} \Big(\frac{\partial \phi}{\partial t} + (u\cdot\nabla)\phi\Big) u dx dt + \int_{\mathbb{R}^3} u_0 \phi(x,0) dx = \int_0^T \int_{\mathbb{R}^3} \nabla u : \nabla\phi dx dt$$

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,T))$ with div$\phi = 0$.

$$\int_0^T \int_{\mathbb{R}^3} u \cdot \nabla \phi dxdt = 0$$

for every $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,T))$.
(iii) The energy inequality, i.e.,

$$\|u(.,t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(.,s)\|_{L^2}^2 ds \le \|u_0\|_{L^2}^2, \quad 0 \le t \le T.$$

By a strong solution we mean a weak solution u such that

$$u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$$

It is well-known that strong solutions are regular (say, classical) and unique in the class of weak solutions.

The constants are different from section to section.

2. Proof of the Main Theorem. The proof follows from the framework established in [20].

First, we give a very simple interpolation lemma

LEMMA 1. *Assume that a measurable function* $u(x,t) \in L^{\infty,2}$ *and* $\nabla u \in L^{2,2}$ *on* $[0,T^*)$, $T^* \le T$, *then* $u \in L^{p,q}$ *with* $p \ge 2, 2 \le q \le 6$ *and* $2/p + 3/q \ge 3/2$ *for* $0 \le t \le T^*$

$$\|u\|_{L^{p,q}} \le C_1 \|u\|_{L^{\infty,2}}^{\frac{3}{q}-\frac{1}{2}} \|\nabla u\|_{L^{2,2}}^{\frac{3}{2}-\frac{3}{q}} \tag{2}$$

where $C_1 = C_1(p,q,T)$. *If* $\frac{2}{p} + \frac{3}{q} = \frac{3}{2}$, *then*

$$\|u\|_{L^{p,q}} \le C_1(q) \|u\|_{L^{\infty,2}}^{1-\frac{2}{p}} \|\nabla u\|_{L^{2,2}}^{\frac{3}{2}-\frac{3}{q}} \tag{3}$$

The proof is simple (see Lemma 1 in [21]).

The second lemma is the following Gronwall type inequality.

LEMMA 2 [4]. $a(x)$ *and* $b(x)$ *be nonnegative functions on* $[0,A)$ *and* $0 < \delta < 1$. *Suppose a nonnegative function* $y(x)$ *satisfies the differential inequality*

$$y'(x) + b(x) \le a(x) y^\delta(x) \ \text{ on } \ [0,A), \ \ y(0) = y_0. \tag{4}$$

Then for $0 \le x < A$,

$$y(x) + \int_0^x b(s)ds \le (2^{\delta/(1-\delta)} + 1)y_0 + 2^{\delta/(1-\delta)} \big(\int_0^x a(s)ds \big)^{1/(1-\delta)}. \tag{5}$$

Proof. Solving the homogeneous differential inequality $y' \le a(x)y^\delta$, one obtains

$$y(x) \le \left\{ y_0^{1-\delta} + \int_0^x a(s)ds \right\}^{1/(1-\delta)}. \tag{6}$$

substituting (6) into (4) and integrating over $[0,x]$, we obtain

$$
\begin{aligned}
y(x)+\int_0^x b(s)ds &\le \int_0^x a(s)ds\Big\{y_0^{1-\delta}+\int_0^x a(s)ds\Big\}^{\delta/(1-\delta)}+y_0\\
&\le \Big\{y_0^{1-\delta}+\int_0^x a(s)ds\Big\}^{1/(1-\delta)}+y_0\\
&\le 2^{\delta/(1-\delta)}\Big\{y_0+\big(\int_0^x a(s)ds\big)^{1/(1-\delta)}\Big\}+y_0.
\end{aligned}
$$

This complete the proof. □

Now we go to the proof of the main theorem. Since there are some differences between the proof for $\gamma = 3$ and $\gamma > 3$, we divide the proof into two parts.

Proof of Theorem 1 for $\gamma = 3$. Now our condition is that u is a Leray-Hopf weak solution on $(0,T)$ with $\nabla u_3 \in L^{4,3}$. For the vorticity field $\omega = \mathrm{curl} u = (\omega_1, \omega_2, \omega_3)$, one has the following estimate.

Lemma 3. *Suppose $u_0 \in H^1(\mathbb{R}^3)$ with $\mathrm{div} u_0 = 0$. Assume that (u,p) is a smooth solution in $\mathbb{R}^3 \times (0,T)$, which satisfies the energy inequality, with $\nabla u \in L^{\infty,2}$ and $\Delta u \in L^{2,2}$. If $\nabla u_3 \in L^{4,3}(\mathbb{R}^3 \times (0,T))$, then for $0 \le t < T$*

$$
\begin{aligned}
&\|\omega_3(.,t)\|_{L^2}^2+\int_0^t \|\nabla\omega_3(.,\tau)\|_{L^2}^2 d\tau \qquad (7)\\
&\le 3\|\omega_3^0\|_{L^2}^2 + C_2\|\nabla u_3\|_{L^{4,3}}^2\|\nabla u\|_{L^{\infty,2}}^{\frac12}\|\Delta u\|_{L^{2,2}}^{\frac12}
\end{aligned}
$$

where $C_2 = C_2(\|u_0\|_{L^2})$ and $\omega^0(x)$ is the initial datum for ω.

Proof. Vorticity $\omega = \mathrm{curl} u$ satisfies

$$
\begin{cases}
\dfrac{\partial\omega}{\partial t}+(u\cdot\nabla)\omega=(\omega\cdot\nabla)u+\Delta\omega,\\
\mathrm{div} u = 0,\\
\mathrm{curl} u=\omega,\\
\omega(x,0)=\omega^0(x).
\end{cases} \qquad (8)
$$

Multiplying the first equation of (8) by ω_3, and integrating on $\mathbb{R}^3$, after suitable integration by parts, we obtain

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\omega_3(.,t)\|_{L^2}^2+\|\nabla\omega_3(.,t)\|_{L^2}^2\\
&\le \int_{\mathbb{R}^3}|(\omega\cdot\nabla u_3)\omega_3|dx\\
&\le 2\|\nabla u\|_{L^{\frac{12}{5}}}\|\nabla u_3\|_{L^3}\|\omega_3\|_{L^4}\ \ \big(|\omega|<2|\nabla u|\big)\\
&\le C_3\|\nabla u\|_{L^{\frac{12}{5}}}\|\nabla u_3\|_{L^3}\|\omega_3\|_{L^2}^{\frac14}\|\nabla\omega_3\|_{L^2}^{\frac34}\ \ \big(\text{Gagliardo-Nirenberg inequality}\big)\\
&\le \frac12\|\nabla\omega_3\|_{L^2}^2+C_3\|\nabla u\|_{L^{\frac{12}{5}}}^{8/5}\|\nabla u_3\|_{L^3}^{8/5}\|\omega_3\|_{L^2}^{2/5}\ \ \big(\text{Young iequality}\big). \qquad (9)
\end{aligned}
$$

Then we can apply Lemma 2 on (9) corresponding to $\delta = \frac{1}{5}$ in Lemma 2,

$$\begin{aligned}
&\|\omega_3(.,t)\|_{L^2}^2 + \int_0^t \|\nabla \omega_3(.,\tau)\|_{L^2}^2 d\tau \\
&\le 3\|\omega_3^0\|_{L^2}^2 + C_4\Big\{ \int_0^t \|\nabla u\|_{L^{\frac{12}{5}}}^{8/5} \|\nabla u_3\|_{L^3}^{8/5} d\tau \Big\}^{5/4} \\
&\le 3\|\omega_3^0\|_{L^2}^2 + C_4\Big\{ \int_0^t \|\nabla u\|_{L^2}^{4/5} \|\nabla u\|_{L^3}^{4/5} \|\nabla u_3\|_{L^3}^{8/5} d\tau \Big\}^{5/4} \\
&\le 3\|\omega_3^0\|_{L^2}^2 + C_4 \|\nabla u\|_{L^{2,2}} \|\nabla u\|_{L^{4,3}} \|\nabla u_3\|_{L^{4,3}}^2 \\
&\le 3\|\omega_3^0\|_{L^2}^2 + C_5 \|\nabla u\|_{L^{\infty,2}}^{\frac{1}{2}} \|\Delta u\|_{L^{2,2}}^{\frac{1}{2}} \|\nabla u_3\|_{L^{4,3}}^2
\end{aligned}$$

where we use the energy inequality and apply Lemma 1 on $\|\nabla u\|_{L^{4,3}}$, since $\frac{2}{4} + \frac{3}{3} = \frac{3}{2}$. So we finish the proof. □

After the a priori estimate on ω_3, we establish the following a priori estimate for the velocity field.

LEMMA 4. *Under the same condition as that in Lemma 3, we have*

$$\sup_{0 \le t < T} \|\nabla u(.,t)\|_{L^2}^2 + \int_0^T \|\Delta u(.,\tau)\|_{L^2}^2 d\tau \le C_6 \tag{10}$$

where C_6 depends on T, $\|\nabla u_0\|_{L^2}$, $\|\nabla u_0\|_{L^2}$ and $\|\nabla u_3\|_{L^{4,3}}$.

Proof. As we have done in [21] we can rewrite the first equation of the Navier-Stokes equations (1) as

$$\frac{\partial u}{\partial t} + \omega \times u + \frac{1}{2}\nabla |u|^2 + \nabla p = \Delta u. \tag{11}$$

Multiply the equation (11) by Δu and integrate on $\mathbb{R}^3 \times (0,t)$, after suitable integration by parts, one obtains

$$\begin{aligned}
&\frac{1}{2}\|\nabla u(.,t)\|_{L^2}^2 + \int_0^t \|\Delta u(.,\tau)\|_{L^2}^2 d\tau \\
&= \int_0^t \int_{\mathbb{R}^3} (\omega \times u) \cdot \Delta u dx d\tau + \frac{1}{2}\|\nabla u_0\|_{L^2}^2
\end{aligned} \tag{12}$$

let

$$\begin{aligned}
I &= \int_0^t \int_{\mathbb{R}^3} (\omega \times u) \cdot \Delta u dx d\tau \\
&\le \int_0^t \int_{\mathbb{R}^3} |\omega_2 u_3 \Delta u_1| dx d\tau + \int_0^t \int_{\mathbb{R}^3} |\omega_3 u_2 \Delta u_1| dx d\tau + \int_0^t \int_{\mathbb{R}^3} |\omega_3 u_1 \Delta u_2| dx d\tau \\
&\quad + \int_0^t \int_{\mathbb{R}^3} |\omega_2 u_3 \Delta u_2| dx d\tau + \Big| \int_0^t \int_{\mathbb{R}^3} \omega_1 u_2 \Delta u_3 dx d\tau \Big| + \Big| \int_0^t \int_{\mathbb{R}^3} \omega_2 u_1 \Delta u_3 dx d\tau \Big| \\
&\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6
\end{aligned}$$

We will estimate the terms one by one.

$$
\begin{aligned}
I_1 &= \int_0^t \int_{\mathbb{R}^3} |\omega_2 u_3 \Delta u_1| dx d\tau \\
&\le \int_0^t \|\omega_2\|_{L^4} \|u_3\|_{L^4} \|\Delta u\|_{L^2} d\tau \\
&\le C_7 \int_0^t \|\nabla u\|_{L^2}^{1/4} \|\nabla u_3\|_{L^3}^{1/2} \|u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{7/4} d\tau \\
&\quad \big(\text{Gagliardo-Nirenberg inequality, for } \omega_2 \text{ and } u_3\big) \\
&\le \frac{1}{20} \|\Delta\|_{L^{2,2}}^2 + C_8 \|\nabla u_3\|_{L^{4,3}}^4 \|\nabla u\|_{L^{\infty,2}}^2, \quad \big(\text{Young inequality}\big)
\end{aligned} \tag{13}
$$

where C_8 is a constant depending on $\|u_0\|_{L^2}$ only.

$$
\begin{aligned}
I_2 &\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + 5 \int_0^t \|u_2\|_{L^a}^2 \|\omega_3\|_{L^b}^2 d\tau \\
&\quad \Big(\text{Hölder's and Young inequality } \tfrac{1}{a} + \tfrac{1}{b} = \tfrac{1}{2}\Big) \\
&\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + 5 \|u_2\|_{L^{p,a}}^2 \|\omega_3\|_{L^{q,b}}^2 \quad \Big(\text{Hölder's inequality } \tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{2}\Big)
\end{aligned}
$$

Now we want to apply Lemma 1 on $\|w_3\|_{L^{q,b}}$, so a, b, p and q satisfies

$$
\begin{cases}
\frac{1}{q} + \frac{1}{b} = \frac{1}{2}, \\
\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \\
\frac{2}{q} + \frac{3}{b} = \frac{3}{2}
\end{cases} \tag{14}
$$

(14) can be solved as

$$
\begin{cases}
p = \infty, \;\; a = 3; \\
q = 2, \;\; b = 6.
\end{cases} \tag{15}
$$

Then Lemma 3 tells us

$$
\|\omega_3\|_{L^{2,6}}^2 \le C_9 \|\nabla u_3\|_{L^{4,3}}^2 \|\nabla u\|_{L^{\infty,2}}^{1/2} \|\Delta u\|_{L^{2,2}}^{1/2} + C_{10}, \tag{16}
$$

where C_9 and C_{10} are constants depending only on $\|\omega_3^0\|_{L^2}$.

On the other hand,

$$
\begin{aligned}
\|u_2\|_{L^{\infty,3}}^2 &\le \|u\|_{L^{\infty,3}}^2 \\
&\le \|u\|_{L^{\infty,2}} \|u\|_{L^{\infty,6}} \\
&\le C_{11} \|\nabla u\|_{L^{\infty,2}}, \quad \big(\text{Energy inequality and Sobolev inequality}\big)
\end{aligned}
$$

where C_{11} depends on $\|u_0\|_{L^2}$ only.

Therefore I_2 can be estimated as

$$
I_2 \le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{12} \|\nabla u\|_{L^{\infty,2}}^{3/2} \|\Delta u\|_{L^{2,2}}^{1/2} \|\nabla u_3\|_{L^{4,3}}^2 + C_{13} \|\nabla u\|_{L^{\infty,2}}, \tag{17}
$$

where C_{12} depends on $\|u_0\|_{L^2}$, while C_{13} depends on $\|u_0\|_{L^2}$ and $\|\omega_3^0\|_{L^2}$.

I_3 is similar to I_2,

$$I_3 \le \frac{1}{20}\|\Delta u\|_{L^{2,2}}^2 + C_{12}\|\nabla u\|_{L^{\infty,2}}^{3/2}\|\Delta u\|_{L^{2,2}}^{1/2}\|\nabla u_3\|_{L^{4,3}}^2 + C_{13}\|\nabla u\|_{L^{\infty,2}}, \tag{18}$$

and I_4 is similar to I_1,

$$I_4 \le \frac{1}{20}\|\Delta u\|_{L^{2,2}}^2 + C_8\|\nabla u_3\|_{L^{4,3}}^4\|\nabla u\|_{L^{\infty,2}}^2 \tag{19}$$

$$\begin{aligned} I_5 &= \Big|\int_0^t\int_{\mathbb{R}^3}\omega_1 u_2\Delta u_3 dxd\tau\Big| \\ &\le \int_0^t\int_{\mathbb{R}^3}|(\partial_2 u_3)u_2\Delta u_3|dxd\tau + \Big|\int_0^t\int_{\mathbb{R}^3}(\partial_3 u_2)u_2\Delta u_3 dxd\tau\Big| \equiv I_5^1 + I_5^2 \end{aligned}$$

$$\begin{aligned} I_5^1 &= \int_0^t\int_{\mathbb{R}^3}|(\partial_2 u_3)u_2\Delta u_3|dxd\tau \\ &\le C_{14}\int_0^t\|\Delta u\|_{L^2}\|\nabla u_3\|_{L^3}\|\nabla u\|_{L^2}d\tau \ \big(\text{Hölder's and Sobolev inequality}\big) \\ &\le \frac{1}{40}\|\Delta u\|_{L^{2,2}}^2 + 5C_{14}T^{1/2}\|\nabla u_3\|_{L^{4,3}}^2\|\nabla u\|_{L^{\infty,2}}^2, \end{aligned} \tag{20}$$

where in the last inequality, we use Young and Hölder's inequality. For simplicity, we denote $5C_{14}T^{1/2}$ as C_{15} which depends on T.

$$\begin{aligned} I_5^2 &= \Big|\int_0^t\int_{\mathbb{R}^3}(\partial_3 u_3)u_2\Delta u_3 dxd\tau\Big| = \Big|\int_0^t\int_{\mathbb{R}^3}\frac{1}{2}u_2^2\Delta(\partial_3 u_3)dxd\tau\Big| \\ &\le \Big|\int_0^t\int_{\mathbb{R}^3}(\partial_3 u_3)u_2\Delta u_2 dxd\tau\Big| + \Big|\int_0^t\int_{\mathbb{R}^3}(\partial_3 u_3)|\nabla u_2|^2 dxd\tau\Big| \\ &\le \frac{1}{40}\|\Delta u\|_{L^{2,2}}^2 + C_{15}\|\nabla u_3\|_{L^{4,3}}^2\|\nabla u\|_{L^{\infty,2}}^2 + \int_0^t\|\nabla u_2\|_{L^3}^2\|\nabla u_3\|_{L^3}d\tau \\ &\quad \big(\text{By (20) and Hölder's inequality respectively}\big) \\ &\le \frac{1}{40}\|\Delta u\|_{L^{2,2}}^2 + C_{15}\|\nabla u_3\|_{L^{4,3}}^2\|\nabla u\|_{L^{\infty,2}}^2 + C_{16}\int_0^t\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\|\nabla u_3\|_{L^3}d\tau \\ &\le \frac{1}{20}\|\Delta u\|_{L^{2,2}}^2 + C_{17}\|\nabla u_3\|_{L^{4,3}}^2\|\nabla u\|_{L^{\infty,2}}^2 \end{aligned} \tag{21}$$

I_6 can be treated similarly

$$I_6 \le \frac{3}{40}\|\Delta u\|_{L^{2,2}}^2 + (C_{15}+C_{17})\|\nabla u_3\|_{L^{4,3}}^2\|\nabla u\|_{L^{\infty,2}}^2 \tag{22}$$

where C_{15} and C_{17} depend only on T.

Substituting the above estimates (13), (17), (18), (19), (20), (21) and (22) into (12), it follows that

$$\begin{aligned} &\frac{1}{2}\|\nabla u(.,t)\|_{L^2}^2 + \frac{13}{20}\|\Delta u\|_{L^{2,2}}^2 - \frac{1}{2}\|\nabla u_0\|_{L^2}^2 \\ &\le 2\big(C_8\|\nabla u_3\|_{L^{4,3}}^4 + (C_{15}+C_{17})\|\nabla u_3\|_{L^{4,3}}^2\big)\|\nabla u\|_{L^{\infty,2}}^2 \\ &\quad +2C_{12}\|\nabla u\|_{L^{\infty,2}}^{3/2}\|\Delta u\|_{L^{2,2}}^{1/2}\|\nabla u_3\|_{L^{4,3}}^2 + 2C_{13}\|\nabla u\|_{L^{\infty,2}} \\ &\le 2\big(C_8\|\nabla u_3\|_{L^{4,3}}^4 + (C_{15}+C_{17})\|\nabla u_3\|_{L^{4,3}}^2\big)\|\nabla u\|_{L^{\infty,2}}^2 \\ &\quad +4C_{18}\|\nabla u_3\|_{L^{4,3}}^{8/3}\|\nabla u\|_{L^{\infty,2}}^2 + \frac{3}{20}\|\Delta u\|_{L^{2,2}}^2 + 4C_{13}^2 + \frac{1}{8}\|u\|_{L^{\infty,2}}^2 \end{aligned}$$

where C_{18} is a constant depends on $\|u_0\|_{L^2}$. Hence

$$\begin{aligned}&\|\nabla u(.,t)\|_{L^2}^2+\|\Delta u\|_{L^{2,2}}-\|\nabla u_0\|_{L^2}^2\\&\le 4\big(C_8\|\nabla u_3\|_{L^{4,3}}^4+(C_{15}+C_{17})\|\nabla u_3\|_{L^{4,3}}^2\\&\quad+2C_{18}\|\nabla u_3\|_{L^{4,3}}^{8/3}\big)\|\nabla u\|_{L^{\infty,2}}^2+8C_{13}^2+\frac{1}{4}\|u\|_{L^{\infty,2}}^2,\end{aligned}\tag{23}$$

Now we choose $0<t_0\le T$, which is small enough, such that

$$\begin{aligned}&C_8\int_0^{t_0}\|\nabla u_3(.,\tau)\|_{L^3}^4d\tau+(C_{15}+C_{17})\left(\int_0^{t_0}\|\nabla u_3(.,\tau)\|_{L^3}^4d\tau\right)^{1/2}\\&+2C_{18}\left(\int_0^{t_0}\|\nabla u_3(.,\tau)\|_{L^3}^4\right)^{2/3}\le\frac{1}{8},\end{aligned}$$

and consequently from (23), we obtain that

$$\sup_{0\le t\le t_0}\|\nabla u(.,t)\|_{L^2}^2+\int_0^{t_0}\|\Delta u(.,\tau)\|_{L^2}^2d\tau\le 32C_{13}^2+4\|\nabla u_0\|_{L^2}^2.\tag{24}$$

Then we can repeat the above process from t_0, if $t_0<T$, with $u(t_0)$ as its initial data for the problem (1) and obtain a similar estimate as (23), for $t_0\le t<T$,

$$\begin{aligned}&\|\nabla u(.,t)\|_{L^2}^2+\int_{t_0}^t\|\Delta u(.,\tau)\|_{L^2}^2d\tau\\&\le 4\left(C_8\|\nabla u_3\|_{L^{4,3}}^4+(C_{15}+C_{17})\|\nabla u_3\|_{L^{4,3}}^2+2C_{18}\|\nabla u_3\|_{L^{4,3}}^{8/3}\right)\sup_{t_0\le\tau\le t}\|\nabla u\|_{L^2}^2\\&\quad+8C_{13}^2+\frac{1}{4}\sup_{t_0\le\tau\le t}\|u\|_{L^2}^2+\|\nabla u(.,t_0)\|_{L^2}^2\end{aligned}$$

There exists a number for t_1, such that

$$\begin{aligned}&C_8\int_{t_0}^{t_1}\|\nabla u_3(.,\tau)\|_{L^3}^4d\tau+(C_{15}+C_{17})\left(\int_{t_0}^{t_1}\|\nabla u_3(.,\tau)\|_{L^3}^4d\tau\right)^{1/2}\\&+2C_{18}\left(\int_{t_0}^{t_1}\|\nabla u_3(.,\tau)\|_{L^3}^4\right)^{2/3}\le\frac{1}{8},\end{aligned}$$

therefore we have

$$\begin{aligned}&\sup_{t_0\le t\le t_1}\|\nabla u(.,t)\|_{L^2}^2+\int_{t_0}^{t_1}\|\Delta u(.,\tau)\|_{L^2}^2d\tau\\&\le 32C_{13}^2+4\|\nabla u(.,t_0)\|_{L^2}^2\le 96C_{13}^2+8\|\nabla u(.,t_0)\|_{L^2}^2\end{aligned}$$

Then we can repeat the above process from t_1, if $t_1<T$. Actually, since $\nabla u_3\in L^{4,3}$ on $[0,T)$, and the coefficients involving $\|u_3\|_{L^{4,3}}$ in (23), which depend only on T, p, b, $\|u_0\|_{L^2}$, the above process only can be done for finite times. More precisely, we can get a estimate on the whole time interval.

$$\sup_{0\le t<T}\|\nabla u(.,t)\|_{L^2}^2+\int_0^T\|\Delta u(.,\tau)\|_{L^2}^2d\tau\le C_6\tag{25}$$

where C_6 depends on T, $\|\nabla u_0\|_{L^2}$, $\|u_0\|_{L^2}$ and $\|u_3\|_{L^{4,3}}$. □
After we got the a priori estimate, the proof of Theorem 1 for the case $\gamma = 3$ is simple. It is well known [19] that there is a unique strong solution $\tilde{u} \in L^\infty(0, T_0; H^1(\mathbb{R}^3)) \cap u \in L^2(0, T_0; H^2(\mathbb{R}^3))$ to (1),for some $0 < T_0$, for any given $u_0 \in L^2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ with $\mathrm{div} u_0 = 0$. Since u is a Leray-Hopf weak solution which satisfies the energy inequality, we have according to the uniqueness result, $u \equiv \tilde{u}$ on $[0, T_0)$. By the a priori estimate (25) and standard continuation argument, the local strong solution u can be extended to time T. So we have proved u actually is a strong solution on $[0, T)$.

PROOF OF THEOREM 1 FOR $\gamma > 3$. Like the proof for $\gamma = 3$, we want to give an estimate on ω_3 first. The constants are different from the above's.

LEMMA 5. *Suppose $u_0 \in H^1(\mathbb{R}^3)$ with $\mathrm{div} u_0 = 0$. Assume that (u, p) is a smooth solution in $\mathbb{R}^3 \times (0, T)$, which satisfies the energy inequality, with $\nabla u \in L^{\infty,2}$ and $\Delta u \in L^{2,2}$. If $\nabla u_3 \in L^{\alpha,\gamma}(\mathbb{R}^3 \times (0, T))$ for $\frac{2}{\alpha} + \frac{3}{\gamma} \le \frac{3}{2}$, then for $0 \le t \le T^*$*

$$\begin{aligned}&\|\omega_3(.,t)\|_{L^2}^2 + \int_0^t \|\nabla \omega_3(.,\tau)\|_{L^2}^2 d\tau \\ &\le 3\|\omega_3^0\|_{L^2}^2 + C_2 \|\nabla u_3\|_{L^{\alpha,\gamma}}^2 \|\nabla u\|_{L^{\infty,2}}^{1-\frac{3}{\gamma}} \|\Delta u\|_{L^{2,2}}^{\frac{3}{\gamma}}\end{aligned} \tag{26}$$

where $C_2 = C_2(\gamma, \|u_0\|_{L^2})$ and $\omega^0(x)$ is the initial datum for ω.

Proof. The proof is more difficult than that of Lemma 3, although the method is same. You know, when you use Hölder's inequality, how to choose the numbers which are suitable for the estimates is difficult since there are so many choices.

Multiplying the first equation of (8) by ω_3, and integrating on $\mathbb{R}^3$, after suitable integration by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\omega_3(.,t)\|_{L^2}^2 + \|\nabla \omega_3(.,t)\|_{L^2}^2 \\ &\le \int_{\mathbb{R}^3} |(\omega \cdot \nabla u_3)\omega_3| dx \\ &\le 2\|\nabla u\|_{L^{\frac{2\gamma}{\gamma-1}}} \|\nabla u_3\|_{L^\gamma} \|\omega_3\|_{L^{\frac{2\gamma}{\gamma-1}}} \quad \big(|\omega| < 2|\nabla u|\big) \\ &\le C_3 \|\nabla u\|_{L^{\frac{2\gamma}{\gamma-1}}} \|\nabla u_3\|_{L^\gamma} \|\omega_3\|_{L^2}^{1-\frac{3}{2\gamma}} \|\nabla \omega_3\|_{L^2}^{\frac{3}{2\gamma}} \quad \big(\text{Gagliardo-Nirenberg inequality}\big) \\ &\le \frac{1}{2}\|\nabla \omega_3\|_{L^2}^2 + C_3 \|\nabla u\|_{L^{\frac{2\gamma}{\gamma-1}}}^{\frac{4\gamma}{4\gamma-3}} \|\nabla u_3\|_{L^\gamma}^{\frac{4\gamma}{4\gamma-3}} \|\omega_3\|_{L^2}^{\frac{(2\gamma-3)2}{4\gamma-3}} \quad \big(\text{Young iequality}\big).\end{aligned} \tag{27}$$

Then we can apply Lemma 2 on (27) corresponding to $\delta = \frac{2\gamma-3}{4\gamma-3}$ in Lemma 2,

$$
\begin{aligned}
&\|\omega_3(.,t)\|_{L^2}^2 + \int_0^t \|\nabla\omega_3(.,\tau)\|_{L^2}^2 d\tau \\
&\le 3\|\omega_3^0\|_{L^2}^2 + C_4\Big\{\int_0^t \|\nabla u\|_{L^{\frac{2\gamma}{\gamma-1}}}^{\frac{4\gamma}{4\gamma-3}} \|\nabla u_3\|_{L^\gamma}^{\frac{4\gamma}{4\gamma-3}} d\tau\Big\}^{\frac{4\gamma-3}{2\gamma}} \\
&\le 3\|\omega_3^0\|_{L^2}^2 + C_4\Big\{\int_0^t \|\nabla u\|_{L^2}^{\frac{2\gamma}{4\gamma-3}} \|\nabla u\|_{L^{\frac{2\gamma}{\gamma-2}}}^{\frac{2\gamma}{4\gamma-3}} \|\nabla u_3\|_{L^\gamma}^{\frac{4\gamma}{4\gamma-3}} d\tau\Big\}^{\frac{4\gamma-3}{2\gamma}} \\
&\big(\text{Interpolation inequality } \tfrac{\gamma-1}{2\gamma} = \tfrac{1}{2}\tfrac{1}{2} + \tfrac{1}{2}\tfrac{\gamma-2}{2\gamma}\big) \\
&\le 3\|\omega_3^0\|_{L^2}^2 + C_4\|\nabla u\|_{L^{\infty,2}}\|\nabla u\|_{L^{\frac{2\gamma}{3},\frac{2\gamma}{\gamma-2}}}\|\nabla u_3\|_{L^{\alpha,\gamma}}^2 \\
&\big(\text{Hölder's inequality } \tfrac{\frac{2\gamma}{4\gamma-3}}{2} + \tfrac{\frac{4\gamma}{4\gamma-3}}{\alpha} + \tfrac{\frac{2\gamma}{4\gamma-3}}{2\gamma/3} = 1\big) \\
&\le 3\|\omega_3^0\|_{L^2}^2 + C_5\|\nabla u\|_{L^{\infty,2}}^{1-\frac{3}{\gamma}}\|\Delta u\|_{L^{2,2}}^{\frac{3}{\gamma}}\|\nabla u_3\|_{L^{\alpha,\gamma}}^2 \\
&\big(\text{Energy inequality and Lemma 1 since } \tfrac{2}{2\gamma/3} + \tfrac{3}{\frac{2\gamma}{\gamma-2}} = \tfrac{3}{2}\big)
\end{aligned}
$$

The proof is complete. □

LEMMA 6. *Under the same condition as that in Lemma 5, we have*

$$\sup_{0\le t<T} \|\nabla u(.,t)\|_{L^2}^2 + \int_0^T \|\Delta u(.,\tau)\|_{L^2}^2 d\tau \le C_6 \tag{28}$$

where C_6 depends on α, γ, $\|\nabla u_0\|_{L^2}$, $\|u_0\|_{L^2}$ and $\|u_3\|_{L^{\alpha,\gamma}}$.

*Proof.*Rewrite the first equation of the Navier-Stokes equations (1) as

$$\frac{\partial u}{\partial t} + \omega\times u + \frac{1}{2}\nabla|u|^2 + \nabla p = \Delta u. \tag{29}$$

Multiply the equation (29) by Δu and integrate on $\mathbb{R}^3\times(0,t)$, after suitable integration by parts, one obtains

$$
\begin{aligned}
&\frac{1}{2}\|\nabla u(.,t)\|_{L^2}^2 + \int_0^t \|\Delta u(.,\tau)\|_{L^2}^2 d\tau \qquad (30)\\
&= \int_0^t\int_{\mathbb{R}^3} (\omega\times u)\cdot\Delta u dx d\tau + \frac{1}{2}\|\nabla u_0\|_{L^2}^2
\end{aligned}
$$

let

$$
\begin{aligned}
I &= \int_0^t\int_{\mathbb{R}^3} (\omega\times u)\cdot\Delta u dx d\tau \\
&\le \int_0^t\int_{\mathbb{R}^3} |\omega_2 u_3 \Delta u_1| dx d\tau + \int_0^t\int_{\mathbb{R}^3} |\omega_3 u_2 \Delta u_1| dx d\tau + \int_0^t\int_{\mathbb{R}^3} |\omega_3 u_1 \Delta u_2| dx d\tau \\
&\quad + \int_0^t\int_{\mathbb{R}^3} |\omega_1 u_3 \Delta u_2| dx d\tau + \Big|\int_0^t\int_{\mathbb{R}^3} \omega_1 u_2 \Delta u_3 dx d\tau\Big| + \Big|\int_0^t\int_{\mathbb{R}^3} \omega_2 u_1 \Delta u_3 dx d\tau\Big| \\
&\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6
\end{aligned}
$$

We will estimate the terms one by one.

$$\begin{aligned}
I_1 &= \int_0^t \int_{\mathbb{R}^3} |\omega_2 u_3 \Delta u_1| dx d\tau \\
&\le \int_0^t \|\omega_2\|_{L^{\frac{6\alpha}{\alpha+2}}} \|u_3\|_{L^{\frac{3\alpha}{\alpha-1}}} \|\Delta u\|_{L^2} d\tau \\
&\le C_7 \int_0^t \|\nabla u\|_{L^2}^{1/\alpha} \|\nabla u_3\|_{L^\gamma}^{1/2} \|u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{2-1/\alpha} d\tau \\
&\quad \big(\text{Gagliardo-Nirenberg inequality, for } \omega_2 \text{ and } u_3\big) \\
&\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_8 \|u_3\|_{L^{\alpha,\gamma}}^\alpha \|\nabla u\|_{L^{\infty,2}}^2 \;\; \big(\text{Young inequality}\big)
\end{aligned} \tag{31}$$

$$\begin{aligned}
I_2 &\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + 5 \int_0^t \|u_2\|_{L^a}^2 \|\omega_3\|_{L^b}^2 d\tau \\
&\quad \Big(\text{Hölder's and Young inequality } \tfrac{1}{a} + \tfrac{1}{b} = \tfrac{1}{2}\Big) \\
&\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + 5 \|u_2\|_{L^{p,a}}^2 \|\omega_3\|_{L^{q,b}}^2 \quad \Big(\text{Hölder's inequality } \tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{2}\Big)
\end{aligned}$$

Just as the estimate of I_2 for $\nabla u_3 \in L^{4,3}$, we can solve p, q, a and b with

$$\begin{cases} p = \infty, \;\; a = 3; \\ q = 2, \;\; b = 6. \end{cases}$$

Then Lemma 5 tells us

$$\|\omega_3\|_{L^{2,6}} \le C_9 \|\nabla u_3\|_{L^{\alpha,\gamma}} \|\nabla u\|_{L^{\infty,2}}^{1/2 - \frac{3}{2\gamma}} \|\Delta u\|_{L^{2,2}}^{\frac{3}{2\gamma}} + C_{10}, \tag{32}$$

where C_9 depends on γ and $\|u_0\|_{L^2}$ and C_{10} depends on γ and $\|\omega_3^0\|_{L^2}$.

On the other hand,

$$\begin{aligned}
\|u_2\|_{L^{\infty,3}}^2 &\le \|u\|_{L^{\infty,3}}^2 \le \|u\|_{L^{\infty,2}} \|u\|_{L^{\infty,6}} \\
&\le C_{11} \|\nabla u\|_{L^{\infty,2}} \;\; \big(\text{Energy inequality and Sobolev inequality}\big)
\end{aligned}$$

So we have the estimate for I_2 as

$$I_2 \le C_{12} \|\nabla u\|_{L^{\infty,2}}^{2-3/\gamma} \|\Delta u\|_{L^{2,2}}^{3/\gamma} \|\nabla u_3\|_{L^{\alpha,\gamma}}^2 + \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{13} \|\nabla u\|_{L^{\infty,2}}, \tag{33}$$

where C_{12} depends on γ and $\|u_0\|_{L^2}$, while C_{13} depends on γ, $\|u_0\|_{L^2}$ and $\|\omega_3^0\|_{L^2}$.

I_3 is similar to I_2,

$$I_3 \le C_{12} \|\nabla u\|_{L^{\infty,2}}^{2-3/\gamma} \|\Delta u\|_{L^{2,2}}^{3/\gamma} \|\nabla u_3\|_{L^{\alpha,\gamma}}^2 + \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{13} \|\nabla u\|_{L^{\infty,2}}, \tag{34}$$

and I_4 is similar to I_1,

$$I_4 \le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_8 \|u_3\|_{L^{\alpha,\gamma}}^\alpha \|\nabla u\|_{L^{\infty,2}}^2. \tag{35}$$

$$\begin{aligned}
I_5 &= \Big| \int_0^t \int_{\mathbb{R}^3} \omega_1 u_2 \Delta u_3 dx d\tau \Big| \\
&\le \int_0^t \int_{\mathbb{R}^3} |(\partial_2 u_3) u_2 \Delta u_3| dx d\tau + \Big| \int_0^t \int_{\mathbb{R}^3} (\partial_3 u_2) u_2 \Delta u_3 dx d\tau \Big| \equiv I_5^1 + I_5^2
\end{aligned}$$

Since $2/\alpha + 3/\gamma = 3/2$, if $3 < \gamma < \infty$, then $4/3 < \alpha < 4$. However, the techniques are different between $4/3 < \alpha < 2$ and $2 \le \alpha < 4$. We deal with $2 < \alpha < 4$ first.

$$\begin{aligned} I_5^1 &= \int_0^t \int_{\mathbb{R}^3} |(\partial_2 u_3) u_2 \Delta u_3| dx d\tau \\ &\le \int_0^t \|\Delta u\|_{L^2} \|\nabla u_3\|_{L^\gamma} \|u_2\|_{L^t} d\tau \\ &\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{14} \|\nabla u_3\|_{L^{\alpha,\gamma}}^2 \|u\|^2_{L^{\frac{2\alpha}{\alpha-2}, \frac{2\gamma}{\gamma-2}}}, \end{aligned} \tag{36}$$

where we used Hölder's and Young inequality.

On the other hand,

$$\begin{aligned} \|u\|_{L^{\frac{2\alpha}{\alpha-2}, \frac{2\gamma}{\gamma-2}}} &\le \|u\|^{(1-\alpha/4)}_{L^{\frac{2\alpha}{\alpha-2}, \frac{6\alpha}{\alpha+4}}} \|u\|^{\alpha/4}_{L^{\frac{2\alpha}{\alpha-2}, 6}} \\ &\le C(\|u_0\|_{L^2}) \|u\|^{\alpha/4}_{L^{\frac{2\alpha}{\alpha-2}, 6}} \ \left(\text{By Lemma 1 } \ \frac{2}{\frac{2\alpha}{\alpha-2}} + \frac{3}{\frac{6\alpha}{\alpha+4}} = \frac{3}{2}\right) \\ &\le C_{15} \|\nabla u\|^{1/2}_{L^{\infty,2}} \|\nabla u\|^{\alpha/4-1/2}_{L^{2,2}} \\ &\quad \Big(\text{Sobolev inequality and interpolation inequality}\Big) \\ &\le C_{16} \|\nabla u\|^{1/2}_{L^{\infty,2}}. \end{aligned}$$

where C_{16} depends on α, γ and $\|u_0\|_{L^2}$.

For $\alpha = 2$ and $\gamma = 6$, from (36), we have

$$\begin{aligned} I_5^1 &\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{17} \|\nabla u_3\|_{L^{\alpha,\gamma}}^2 \|u\|^2_{L^{\infty,3}} \\ &\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{17} \|\nabla u_3\|_{L^{\alpha,\gamma}}^2 \|u\|_{L^{\infty,2}} \|u\|_{L^{\infty,6}} \\ &\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{18} \|\nabla u_3\|_{L^{\alpha,\gamma}}^2 \|\nabla u\|_{L^{\infty,2}}, \end{aligned}$$

where C_{18} only depends on $\|u_0\|_{L^2}$.

So for $2 \le \alpha < 4$, we have

$$I_5^1 \le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{19} \|\nabla u_3\|_{L^{\alpha,\gamma}}^2 \|\nabla u\|_{L^{\infty,2}}, \tag{37}$$

where C_{19} depends only on α, γ and $\|u_0\|_{L^2}$.

Then we turn our attention to $4/3 < \alpha < 2$. Actually it is more difficult than the previous case.

$$
\begin{aligned}
I_5^1 &= \int_0^t \int_{\mathbb{R}^3} |(\partial_2 u_3) u_2 \Delta u_3| dx d\tau \\
&\le \int_0^t \|\Delta u\|_{L^2} \|\nabla u_3\|_{L^b} \|u_2\|_{L^a} d\tau \\
&\le \int_0^t \|\Delta u\|_{L^2} \|\nabla u_3\|_{L^\gamma}^{\delta} \|\nabla u\|_{L^p}^{1-\delta} \|u\|_{L^a} d\tau \\
&\le C_{20} \int_0^t \|\Delta u\|_{L^2} \|\nabla u_3\|_{L^\gamma}^{\delta} \|\nabla u\|_{L^2}^{(1-\delta)\theta} \|\Delta u\|_{L^2}^{(1-\theta)(1-\delta)} \|u\|_{L^a} d\tau \\
&\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{21} \int_0^t \|\nabla u_3\|_{L^\gamma}^{2\delta/(\theta+\delta-\theta\delta)} \|\nabla u\|_{L^2}^{2\theta(1-\delta)/(\theta+\delta-\theta\delta)} \|u\|_{L^a}^{2/(\theta+\delta-\theta\delta)} d\tau \\
&\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{21} \|\nabla u_3\|_{L^{\alpha,\gamma}}^{2\delta/(\theta+\delta-\theta\delta)} \|\nabla u\|_{L^{\infty,2}}^{2\theta(1-\delta)/(\theta+\delta-\theta\delta)} \|u\|_{L^{q,a}}^{2/(\theta+\delta-\theta\delta)}
\end{aligned}
$$

where the constants satisfy the following system

$$
\begin{cases}
\frac{1}{a} + \frac{1}{b} = \frac{1}{2} \\
\frac{1}{b} = \frac{\delta}{\gamma} + \frac{1-\delta}{p} \\
\frac{1}{p} = (1-\theta)\left(\frac{1}{2} - \frac{1}{3}\right) + \frac{1}{2}\theta \\
\frac{2\delta}{\theta+\delta-\theta\delta} \times \frac{1}{\alpha} + \frac{2}{\theta+\delta-\theta\delta} \times \frac{1}{q} = 1
\end{cases}
\tag{38}
$$

It is obvious that the system is under determined since 4 equations and 6 unknowns. How can we find other equations?

Before solving (38), one can calculate directly from (38) that

$$
\frac{2}{q} + \frac{3}{a} = 1,
$$

therefore, as the above estimate, it is not difficult to obtain

$$
\|u\|_{L^{q,a}} \le C(q, a, \theta, \delta, \|u_0\|_{L^2}) \|\nabla u\|_{L^{\infty,2}}^{1/2},
$$

actually one can choose $q = \infty$, and a natural requirement of θ, δ is as follows

$$
\frac{1 + 2\theta(1-\delta)}{\theta + \delta - \theta\delta} = 2.
$$

Now, we can solve (38) with

$$
\begin{cases}
\delta = \frac{1}{2} \\
\theta = \frac{2}{\alpha} - 1 \\
p = \frac{6\alpha}{4-\alpha} \\
a = 3 \\
b = 6 \\
q = \infty
\end{cases}
$$

Therefore

$$
\begin{aligned}
I_5^1 &\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{21} \|\nabla u_3\|_{L^{\alpha,\gamma}}^{\alpha} \|\nabla u\|_{L^{\infty,2}}^{2-\alpha} \|u\|_{L^{\infty,3}}^{2\alpha} \\
&\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{21} \|\nabla u_3\|_{L^{\alpha,\gamma}}^{\alpha} \|\nabla u\|_{L^{\infty,2}}^{2-\alpha} \|u\|_{L^{\infty,2}}^{\alpha} \|u\|_{L^{\infty,6}}^{\alpha} \\
&\le \frac{1}{20} \|\Delta u\|_{L^{2,2}}^2 + C_{22} \|\nabla u_3\|_{L^{\alpha,\gamma}}^{\alpha} \|\nabla u\|_{L^{\infty,2}}^2
\end{aligned}
\tag{39}
$$

where C_{21} depends on α, γ and $\|u_0\|_{L^2}$.

$$\begin{aligned} I_5^2 &= \Big|\int_0^t\int_{\mathbb{R}^3}(\partial_3 u_3)u_2\Delta u_3 dxd\tau\Big| = \Big|\int_0^t\int_{\mathbb{R}^3}\frac{1}{2}u_2^2\Delta(\partial_3 u_3)dxd\tau\Big| \\ &\leq \Big|\int_0^t\int_{\mathbb{R}^3}(\partial_3 u_3)u_2\Delta u_2 dxd\tau\Big| + \Big|\int_0^t\int_{\mathbb{R}^3}(\partial_3 u_3)|\nabla u_2|^2 dxd\tau\Big| \equiv I_5^{2,1}+I_5^{2,2}\end{aligned}$$

$I_5^{2,1}$ can be treated similarly as I_5^1. For $2\leq\alpha<4$,

$$I_5^{2,1}\leq \frac{1}{20}\|\Delta u\|_{L^{2,2}}^2 + C_{19}\|\nabla u_3\|_{L^{\alpha,\gamma}}^2\|\nabla u\|_{L^{\infty,2}}, \tag{40}$$

while for $4/3<\alpha<2$,

$$I_5^{2,1}\leq \frac{1}{20}\|\Delta u\|_{L^{2,2}}^2 + C_{22}\|\nabla u_3\|_{L^{\alpha,\gamma}}^\alpha\|\nabla u\|_{L^{\infty,2}}^2. \tag{41}$$

$$\begin{aligned} I_5^{2,2} &\leq \int_0^t\int_{\mathbb{R}^3}\big|(\partial_3 u_3)|\nabla u_2|^2\big|dxd\tau \\ &\leq \|\nabla u_3\|_{L^{\alpha,\gamma}}\|\nabla u\|^2_{L^{2\alpha/(\alpha-1),2\gamma/(\gamma-1)}} \\ &\leq \|\nabla u_3\|_{L^{\alpha,\gamma}}\|\nabla u\|^{1/2}_{L^{2,2}}\|\nabla u\|^{3/2}_{L^{6\alpha/(3\alpha-4),6\gamma/(3\gamma-4)}} \\ &\leq C_{23}\|\nabla u_3\|_{L^{\alpha,\gamma}}\|\nabla u\|^{2/\alpha}_{L^{\infty,2}}\|\Delta u\|^{3/\gamma}_{L^{2,2}}. \end{aligned}\tag{42}$$

where C_{23} depends on γ and $\|u_0\|_{L^2}$.

Similarly, for $2\leq\alpha<4$,

$$\begin{aligned} I_6 &\leq \frac{1}{10}\|\Delta u\|_{L^{2,2}}^2 + 2C_{19}\|\nabla u_3\|_{L^{\alpha,\gamma}}^2\|\nabla u\|_{L^{\infty,2}} \\ &\quad C_{23}\|\nabla u_3\|_{L^{\alpha,\gamma}}\|\nabla u\|^{2/\alpha}_{L^{\infty,2}}\|\Delta u\|^{3/\gamma}_{L^{2,2}}, \end{aligned}\tag{43}$$

while for $4/3<\alpha<2$,

$$\begin{aligned} I_6 &\leq \frac{1}{10}\|\Delta u\|_{L^{2,2}}^2 + 2C_{22}\|\nabla u_3\|_{L^{\alpha,\gamma}}^\alpha\|\nabla u\|_{L^{\infty,2}}^2 \\ &\quad +C_{23}\|\nabla u_3\|_{L^{\alpha,\gamma}}\|\nabla u\|^{2/\alpha}_{L^{\infty,2}}\|\Delta u\|^{3/\gamma}_{L^{2,2}}. \end{aligned}\tag{44}$$

For $2\leq\alpha<4$, putting (31), (33), (34), (35), (37), (40), (42) and (43) into (30), one obtain

$$\begin{aligned} &\frac{1}{2}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^{2,2}}^2 \\ &\leq \frac{2}{5}\|\Delta u\|_{L^{2,2}}^2 + 2C_8\|\nabla u_3\|_{L^{\alpha,\gamma}}^\alpha\|\nabla u\|_{L^{\infty,2}}^2 \\ &\quad +2C_{12}\|\nabla u\|^{2-3/\gamma}_{L^{\infty,2}}\|\Delta u\|^{3/\gamma}_{L^{2,2}}\|\nabla u_3\|^2_{L^{\alpha,\gamma}} \\ &\quad +4C_{19}\|\nabla u_3\|^2_{L^{\alpha,\gamma}}\|\nabla u\|_{L^{\infty,2}} + \frac{1}{2}\|\nabla u_0\|^2_{L^2} \\ &\quad +2C_{23}\|\nabla u_3\|_{L^{\alpha,\gamma}}\|\nabla u\|^{2/\alpha}_{L^{\infty,2}}\|\Delta u\|^{3/\gamma}_{L^{2,2}} + 2C_{13}\|\nabla u\|_{L^{\infty,2}} \\ &\leq \frac{1}{2}\|\Delta u\|^2_{L^{2,2}} + 2\left(C_8\|\nabla u_3\|^\alpha_{L^{\alpha,\gamma}} + C_{24}\|\nabla u_3\|^{4/(2-3/\gamma)}_{L^{\alpha,\gamma}}\right)\|\nabla u\|^2_{L^{\infty,2}} \\ &\quad +4C_{19}\|\nabla u_3\|^2_{L^{\alpha,\gamma}}\|\nabla u\|_{L^{\infty,2}} \\ &\quad +2C_{25}\|\nabla u_3\|^{\frac{2\gamma}{2\gamma-3}}_{L^{\alpha,\gamma}}\|\nabla u\|^{2-\frac{\gamma}{2\gamma-3}}_{L^{\infty,2}} + 2C_{13}\|\nabla u\|_{L^{\infty,2}} + \frac{1}{2}\|\nabla u_0\|^2_{L^2}. \end{aligned}\tag{45}$$

For $2 \le \alpha < 4$, putting (31), (33), (34), (35), (39), (41), (42) and (44) into (30), one obtain

$$\begin{aligned}
&\frac{1}{2}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^{2,2}}^2 \\
&\le \frac{2}{5}\|\Delta u\|_{L^{2,2}}^2 + 2C_8\|\nabla u_3\|_{L^{\alpha,\gamma}}^{\alpha}\|\nabla u\|_{L^{\infty,2}}^2 \\
&\quad +2C_{12}\|\nabla u\|_{L^{\infty,2}}^{2-3/\gamma}\|\Delta u\|_{L^{2,2}}^{3/\gamma}\|\nabla u_3\|_{L^{\alpha,\gamma}}^2 \\
&\quad 4C_{22}\|\nabla u_3\|_{L^{\alpha,\gamma}}^{\alpha}\|\nabla u\|_{L^{\infty,2}}^2 \\
&\quad +2C_{23}\|\nabla u_3\|_{L^{\alpha,\gamma}}\|\nabla u\|_{L^{\infty,2}}^{2/\alpha}\|\Delta u\|_{L^{2,2}}^{3/\gamma} + 2C_{13}\|\nabla u\|_{L^{\infty,2}} \\
&\le \frac{1}{2}\|\Delta u\|_{L^{2,2}}^2 + 2\left((C_8+2C_{22})\|\nabla u_3\|_{L^{\alpha,\gamma}}^{\alpha} + C_{26}\|\nabla u_3\|_{L^{\alpha,\gamma}}^{4/(2-3/\gamma)}\right)\|\nabla u\|_{L^{\infty,2}}^2 \\
&\quad +4C_{19}\|\nabla u_3\|_{L^{\alpha,\gamma}}^2\|\nabla u\|_{L^{\infty,2}} \\
&\quad +2C_{27}\|\nabla u_3\|_{L^{\alpha,\gamma}}^{\frac{2\gamma}{2\gamma-3}}\|\nabla u\|_{L^{\infty,2}}^{2-\frac{\gamma}{2\gamma-3}} + 2C_{13}\|\nabla u\|_{L^{\infty,2}} + \frac{1}{2}\|\nabla u_0\|_{L^2}^2.
\end{aligned} \tag{46}$$

Just as the proof for the case $\gamma = 3$, using the integrability of $\|u_3\|_{L^\gamma}$ with respect to time variable, we can choose a sufficiently small t_0, $0 < t_0 \le T$, such that

$$C_8\int_0^{t_0}\|\nabla u_3(.,\tau)\|_{L^\gamma}^{\alpha}d\tau + C_{24}\left(\int_0^{t_0}\|\nabla u_3(.\tau)\|_{L^\gamma}^{\alpha}d\tau\right)^{\frac{4}{\alpha(2-3/\gamma)}} \le \frac{1}{4} \tag{47}$$

Due to (45) and (47),

$$\begin{aligned}
&\frac{1}{4}\|\nabla u\|_{L^{\infty,2}}^2 + \frac{1}{2}\|\Delta u\|_{L^{2,2}}^2 \\
&\le 4C_{19}\|\nabla u_3\|_{L^{\alpha,\gamma}}^2\|\nabla u\|_{L^{\infty,2}} + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 \\
&\quad +2C_{25}\|\nabla u_3\|_{L^{\alpha,\gamma}}^{\frac{2\gamma}{2\gamma-3}}\|\nabla u\|_{L^{\infty,2}}^{2-\frac{\gamma}{2\gamma-3}} + 2C_{13}\|\nabla u\|_{L^{\infty,2}}
\end{aligned} \tag{48}$$

Since the power of $\|\nabla u\|_{L^{\infty,2}}$ in the right side of (48) is strictly less than 2, so we immediately have the estimate

$$\|\nabla u\|_{L^{\infty,2}}^2 + \|\Delta u\|_{L^{2,2}}^2 \le C_{28}, \tag{49}$$

where C_{28} depends on α, γ, $\|u_0\|_{L^2}$ and $\|\nabla u_0\|_{L^2}$. Then the remaining argument is same as that in the proof of Lemma 4.

Similar argument can be done for $4/3 < \alpha < 2$. This finishes the proof of Lemma 6. □

After we have the a priori estimates Lemma 6, the proof of Theorem 1 is completely similar to the case when $\gamma = 3$.

Therefore, we finish the proof of Theorem 1.

Acknowledgment. The author would like to express sincere gratitude to his supervisor Professor Zhouping Xin for enthusiastic guidance and constant encouragement. And thanks also to Professor Shing-Tung Yau and Professor Zhouping Xin for providing an excellent study and research environment in The Institute of Mathematical Sciences.

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