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ON THE LAZER-MCKENNA CONJECTURE INVOLVING CRITICAL AND SUPERCRITICAL EXPONENTS*

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Abstract. We prove the Lazer-McKenna conjecture for an elliptic problem of Ambrosetti-Prodi type with critical and supercritical nonlinearities by constructing solutions concentrating on higher dimensional manifolds, under some partially symmetric assumption on the domain.

Key words. Elliptic equation, multiplicity, reduction method.

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1. Introduction.

In this paper, we consider the following elliptic problem:

$$\begin{cases} -\Delta u = |u|^p - s\varphi_1(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N with C^1 boundary, $p > 1$, φ_1 is a positive first eigenfunction of $-\Delta$ in Ω with Dirichlet boundary condition. Here the eigenvalues of $-\Delta$ in Ω with Dirichlet boundary condition are denoted by $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$.

Problem (1.1) is a special case of the following elliptic problem of Ambrosetti-Prodi type:

$$\begin{cases} -\Delta u = g(u) - s\varphi_1(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $g(t)$ satisfies $\lim_{t\to-\infty} \frac{g(t)}{t} = \nu < \lambda_1 < \lim_{t\to+\infty} \frac{g(t)}{t} = \mu$.

It is well known that the number of the solutions of (1.2) depends on the number of the eigenvalue λ_i that the interval (ν, μ) contains. See [3, 17, 25], and also [6, 9, 15, 18, 19, 23, 24]. A conjecture raised by Lazer and McKenna in [18] is that if $\mu = \infty$ (that is, (ν, μ) contains all the eigenvalues λ_i) and the nonlinearity $g(t)$ does not grow too fast at infinity, then the number of the solutions for (1.2) is unbounded as $s \to +\infty$. If $g(t) = t^2$ and Ω is a unit square in R^2, Bruer, McKenna and Plum [5] showed that (1.2) has at least four solutions. In [11], we proved that the Lazer-McKenna conjecture is true for (1.1) in the subcritical case $p < \frac{N+2}{N-2}$ by constructing solutions with sharp peaks (point concentration solutions) near the maximum point of $\varphi_1(y)$. A natural question is whether this conjecture is still true for (1.1) if p is critical, or even supercritical. It is almost impossible to construct point concentration solutions for (1.1) as in [21, 22, 26] for the critical case $p = \frac{N+2}{N-2}$. Therefore, we need to find different kind of solutions for (1.1) in order to prove the Lazer-McKenna conjecture for (1.1) in the critical and supercritical cases. In this paper, by constructing solutions concentrating on higher dimensional manifolds, we prove that the Lazer-McKenna

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conjecture is true for (1.1) if p is critical or supercritical under the following partially symmetric assumption on the domain Ω:

(Ω): there is an integer m, $1 < m \le N$, such that $y \in \Omega$, if and only if $(|y'|, y'') \in D$, where $y = (y', y'')$, $y' \in R^m$, $y'' \in R^{N-m}$, D is a bounded domain in R_+^{N-m+1}, and

$$R_+^{N-m+1} = \{z = (z_1, z_2 \cdots, z_{N-m+1}) : z_1 \ge 0\}.$$

The main result of this paper is the following:

Theorem 1.1. *Suppose that Ω satisfies the condition (Ω), and $p \in \left(1, \frac{N-m+3}{N-m-1}\right)$ if $1 < m \le N-2$, $p \in (1, +\infty)$ if $m = N-1, N$. For any positive integer k, there exists an $s_k > 0$, such that for $s \ge s_k$, (1.1) has at least k different solutions.*

Results on the Lazer-McKenna conjecture for (1.2) can be found in [12, 14, 21, 22, 26] for the case $g(t) = t_+^p + \lambda t$, in [10] for the case $g(t) = t_+^p + t_-^q$, $\frac{N+2}{N-2} > p > q > 1$, and in [16] for the case $g(t) = e^t$ and $N = 2$. Let us point out that [14] also contains results on the super-critical case.

Before we close this introduction, let us outline the proof of Theorem 1.1.

Let $\varepsilon^2 = s^{-(p-1)/p}$. Then it is easy to see that solving (1.1) is equivalent to solving the following elliptic problem:

$$\begin{cases} -\varepsilon^2 \Delta u = |u|^p - \varphi_1(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In view of the assumption on Ω, we will work on the following subspace of $H_0^1(\Omega)$:

$$H_s = \{u : u \in H_0^1(\Omega), u(y) = u(|y'|, y'')\}.$$

It is easy to prove that the first eigenfunction $\varphi_1(y)$ belongs to H_s. Since the first eigenfunction $\varphi_1 \in H_s$, there is a function $\bar{\varphi}_1(t, y'')$, such that $\varphi_1(y) = \bar{\varphi}_1(|y'|, y'')$. For simplicity, we still use the same notation φ_1 for this function $\bar{\varphi}_1$. Note that $s \to +\infty$ if and only if $\varepsilon \to 0$.

In Appendix A, we will show that if $\varepsilon > 0$ is small, (1.3) has a negative solution $\underline{u}_\varepsilon \in H_s$, satisfying

$$\underline{u}_\varepsilon = -\varphi_1^{1/p} + \varepsilon^2 O_\varepsilon(1),$$

where $O_\varepsilon(1)$ is uniformly bounded in any compact subset of Ω.

Let $a > 0$ be a constant. Consider the following elliptic problem:

$$\begin{cases} -\Delta U = |U - a^{1/p}|^p - a, \ U > 0, & \text{in } R^{N-m+1}, \\ U(0) = \max_{z \in R^{N-m+1}} U(z), \\ U \in H^1(R^{N-m+1}). \end{cases} \tag{1.4}$$

Since p is subcritical in R^{N-m+1}, using the standard concentration compactness argument of P.L.Lions, we can prove that (1.4) has a positive solution U_a. It is easy to see that U_a decays exponentially at infinity, and is radially symmetric. Moreover,

$$U_a(z) = a^{1/p} U\left(a^{(p-1)/2p} z\right), \tag{1.5}$$

where $U = U_1$.

In Appendix B, we will calculate the energy of $U_{\varphi_1(\bar x)}\big(\frac{|\tilde y-\bar x|}{\varepsilon}\big)$, $\tilde y = (|y'|, y'')$, $\bar x \in D$. We show that the main term in the energy expansion for $U_{\varphi_1(\bar x)}\big(\frac{|\tilde y-\bar x|}{\varepsilon}\big)$ is given by

$$A\varepsilon^{N-m+1}\bar x_1^{m-1}\varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{N-m+1}{2})}(\bar x),$$

where $A > 0$ is a constant.

Noting that

$$\bar x_1^{m-1}\varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{N-m+1}{2})}(\bar x) = 0, \quad \forall\, \bar x \in \partial D,$$

we conclude that its maximum set S is compactly contained in D.

In section 2, we will use the reduction argument to prove that for $\varepsilon > 0$ small, (1.3) has a solution

$$u \approx \underline{u}_\varepsilon + \sum_{j=1}^{k} U_{\varphi_1(x_{\varepsilon,j})}\big(\frac{|\tilde y - x_{\varepsilon,j}|}{\varepsilon}\big) \tag{1.6}$$

where $x_{\varepsilon,j} \in D$ satisfying that as $\varepsilon \to 0$,

$$x_{\varepsilon,j} \to x_j \in S, \quad \frac{|x_{\varepsilon,j} - x_{\varepsilon,i}|}{\varepsilon} \to +\infty, \ \ j \neq i.$$

Solution with the form (1.6) concentrates as $\varepsilon \to 0$ at some $m-1$ dimensional spheres. In the subcritical case $p < \frac{N+2}{N-2}$, (1.1) also has a point concentration solution, concentrating near the maximum set of φ_1. See [11]. As we pointed out earlier, in the critical case $p = \frac{N+2}{N-2}$, (1.1) may not have any point concentration solution. So, it is necessary to look for solutions concentrating at higher dimensional manifolds in order to prove the Lazer-McKenna conjecture for (1.1) in the critical case.

If the domain Ω is a ball, then, for the critical case $p = \frac{N+2}{N-2}$, (1.1) has solutions concentrating at n-dimensional spheres for $n = 1, \cdots, N-1$. Moreover, combining the result in [11] and Theorem 1.1, we conclude that if the domain Ω is a ball, then for any $p > 1$, the number of the solutions for (1.1) is unbounded as $s \to +\infty$.

Results on the solutions concentrating on higher dimensional manifolds for the singularly perturbed Dirichlet problems can be found in [8, 1, 2] in the radially symmetric case, and in [13, 4] for domains with partial symmetry, and the references therein.

In this paper, we will use the following notations. For any $\bar x \in D$, we use $B_\delta(\bar x)$ to denote the ball in R^{N-m+1}, centred at $\bar x$ with radius δ. We define

$$B_\delta^*(\bar x) = \big\{y:\ y = (y', y'') \in R^N, (|y'|, y'') \in B_\delta(\bar x)\big\}.$$

2. Solutions concentrating on manifolds. Let $\underline{u}_\varepsilon$ be the negative solution obtained in Theorem A.1. In this section, we will find solution u for (1.3), with the form $u = \underline{u}_\varepsilon + v$. Then, v satisfies

$$\begin{cases} -\varepsilon^2\Delta v + p|\underline{u}_\varepsilon|^{p-1}v = f_\varepsilon(y, v), & y \in \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where

$$f_\varepsilon(y,t) = |t + \underline{u}_\varepsilon|^p - |\underline{u}_\varepsilon|^p + p|\underline{u}_\varepsilon|^{p-1}t. \tag{2.2}$$

The functional corresponding to (2.1) is

$$I_\varepsilon(v) = \frac{1}{2}\int_\Omega \big(\varepsilon^2 |Du|^2 + p|\underline{u}_\varepsilon|^{p-1}v^2\big) - \int_\Omega F_\varepsilon(y, v), \quad v \in H_s, \tag{2.3}$$

where

$$\begin{aligned} F_\varepsilon(y,t) &= \int_0^t f_\varepsilon(y,s)\,ds \\ &= \frac{1}{p+1}|t+\underline{u}_\varepsilon|^p(t+\underline{u}_\varepsilon) + \frac{1}{p+1}|\underline{u}_\varepsilon|^{p+1} - |\underline{u}_\varepsilon|^p t + \frac{p}{2}|\underline{u}_\varepsilon|^{p-1}t^2. \end{aligned} \tag{2.4}$$

Firstly, we need to define an approximate solution for (2.1).

For any $y = (y', y'') \in R^N$, $y' \in R^m$, $y'' \in R^{N-m}$, we denote $\tilde{y} = (|y'|, y'') \in R^{N-m+1}$.

Let $\bar{W}_a(y) = U_a(\tilde{y})$, where U_a is defined in (1.4). For any $\bar{x} \in D$, let $\bar{W}_{\varepsilon,\bar{x},a}(y) = U_a\big(\frac{|\tilde{y}-\bar{x}|}{\varepsilon}\big)$. Then, $\bar{W}_{\varepsilon,\bar{x},a}$ satisfies

$$-\varepsilon^2 \Delta \bar{W}_{\varepsilon,\bar{x},a} = |\bar{W}_{\varepsilon,\bar{x},a} - a^{1/p}|^p - a + \varepsilon\frac{m-1}{|y'|}\frac{|y'|-\bar{x}_1}{|\tilde{y}-\bar{x}|}U_a'\big(\frac{|\tilde{y}-\bar{x}|}{\varepsilon}\big), \quad \text{in } \Omega. \tag{2.5}$$

Since the function in the right hand side of (2.5) may have singularity, we need to further modify $\bar{W}_{\varepsilon,\bar{x},a}$. Choose $\delta > 0$ small enough. Let $\xi(t) \geq 0$ be a smooth function, such that $\xi(t) = 0$ if $t \leq \delta$, $\xi(t) = 1$ if $t \geq 2\delta$. Define

$$W_{\varepsilon,\bar{x},a}(y) = \xi(|y'|)\bar{W}_{\varepsilon,\bar{x},a}(y).$$

Then $W_{\varepsilon,\bar{x},a}$ satisfies

$$-\varepsilon^2 \Delta W_{\varepsilon,\bar{x},a} = \xi(|y'|)\big(|\bar{W}_{\varepsilon,\bar{x},a} - a^{1/p}|^p - a\big) + \tilde{f}_{\varepsilon,\bar{x}}(y) \quad \text{in } \Omega, \tag{2.6}$$

where

$$\tilde{f}_{\varepsilon,\bar{x}}(y) = \xi\varepsilon\frac{m-1}{|y'|}\frac{|y'|-\bar{x}_1}{|\tilde{y}-\bar{x}|}U_a'\big(\frac{|\tilde{y}-\bar{x}|}{\varepsilon}\big) - 2\varepsilon D\xi DU_a\big(\frac{|\tilde{y}-\bar{x}|}{\varepsilon}\big) - \varepsilon^2 U_a\big(\frac{|\tilde{y}-\bar{x}|}{\varepsilon}\big)\Delta\xi.$$

Since $\xi = 0$ for $|y'| \leq \delta$, it is easy to see that $\tilde{f}_{\varepsilon,\bar{x}}$ is a smooth function in both y and $\bar{x}$, and satisfies

$$|\tilde{f}_{\varepsilon,\bar{x}}| \leq C\varepsilon U_a\big(\frac{|\tilde{y}-\bar{x}|}{\varepsilon}\big).$$

For any $\bar{x} \in D$, let $P_{\varepsilon,\Omega}W_{\varepsilon,\bar{x},a}$ be the solution of

$$\begin{cases} -\varepsilon^2\Delta v + pa^{(p-1)/p}v \\ = \xi(|y'|)\big(|W_{\varepsilon,x,a} - a^{1/p}|^p - a\big) + pa^{(p-1)/p}W_{\varepsilon,\bar{x},a} + \tilde{f}_{\varepsilon,\bar{x}}(y), & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

It is easy to see that $P_{\varepsilon,\Omega}W_{\varepsilon,\bar{x},a} \in H_s$. By the exponential decay of U_a, we have

$$|P_{\varepsilon,\Omega}W_{\varepsilon,\bar{x},a} - W_{\varepsilon,\bar{x},a}| \leq Ce^{-\sqrt{p}a^{(p-1)/2}d(x,\partial D)/\varepsilon}.$$

The approximate solution for (2.1) which we will use in this paper is defined as

$$V_{\varepsilon,\bar{x}} = P_{\varepsilon,\Omega} W_{\varepsilon,\bar{x},\varphi_1(\bar{x})}. \tag{2.7}$$

Denote

$$\tilde{f}(\tilde{y},t) = |t - \varphi_1^{1/p}(\tilde{y})|^p - \varphi_1(\tilde{y}) + p\varphi_1^{(p-1)/p}(\tilde{y})t. \tag{2.8}$$

Then, $V_{\varepsilon,\bar{x}}$ satisfies

$$\begin{cases} -\varepsilon^2 \Delta V_{\varepsilon,\bar{x}} + p\varphi_1^{(p-1)/p}(\bar{x}) V_{\varepsilon,\bar{x}} \\ = \tilde{f}(\bar{x}, W_{\varepsilon,\bar{x},\varphi_1(\bar{x})}) + O\Big(\big(\varepsilon + |\xi - 1|\big) W_{\varepsilon,\bar{x},\varphi_1(\bar{x})}\Big), & \text{in } \Omega, \\ V_{\varepsilon,\bar{x}} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Using Theorem A.1 and the exponentially decay of the function $W_{\varepsilon,\bar{x},\varphi_1(\bar{x})}$, we can deduce that for any $\bar{x} \in D$ with $d(\bar{x}, \partial D) \geq \bar{\delta} > 0$,

$$\big|\tilde{f}(\tilde{y}, W_{\varepsilon,\bar{x},\varphi_1(\bar{x})})(y) - f_\varepsilon(y, W_{\varepsilon,\bar{x},\varphi_1(\bar{x})}(y))\big| \leq C\varepsilon^2 W^{1-\tilde{\theta}}_{\varepsilon,\bar{x},\varphi_1(\bar{x})}(y), \quad \forall\, y \in \Omega, \tag{2.10}$$

where $\tilde{\theta} > 0$ is any small constant, $f_\varepsilon(y,t)$ is the function defined in (2.2).

Denote

$$\big\langle u, v \big\rangle_\varepsilon = \int_\Omega (\varepsilon^2 DuDv + p|\underline{u}_\varepsilon|^{p-1} uv), \quad \|u\|_\varepsilon = \big\langle u, u \big\rangle_\varepsilon^{1/2}.$$

Set

$$S = \{\bar{x} : \ \ \bar{x} \in D, \ \bar{x}_1^{m-1} \varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1} - \frac{N-m+1}{2})}(\bar{x}) = M\}, \tag{2.11}$$

where

$$M = \max_{z \in D} z_1^{m-1} \varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1} - \frac{N-m+1}{2})}(z). \tag{2.12}$$

From $z_1^{m-1} \varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1} - \frac{N-m+1}{2})}(z) = 0$ in ∂D, we know that $S \subset\subset D$.

Let

$$D_{k,\varepsilon} = \{x : \quad x = (x_1, \cdots, x_k), |x_{j,1}^{m-1} \varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1} - \frac{N-m+1}{2})}(x_j) - M| \leq \varepsilon^{1-\tau}, \\ V_{\varepsilon,x_i}(x_j) \leq \varepsilon^{1-\tau}, \ \ i \neq j, \ \ i, j = 1, \cdots, k\}, \tag{2.13}$$

where $\tau > 0$ is a small constant. The set $D_{k,\varepsilon}$ is not empty, because for $x_j \in D$, satisfying

$$|x_j - x_0| = L\varepsilon |\ln \varepsilon|, \quad |x_i - x_j| \geq \frac{2\pi L}{k} \varepsilon |\ln \varepsilon|, \quad i \neq j, \ i,j = 1, \cdots, k,$$

where $x_0 \in S$ and $L > 0$ is large, $(x_1, \cdots, x_k) \in D_{k,\varepsilon}$.

Let H be the completion of the space $C_0^\infty(\Omega)\cap H_s$ with respect to the norm $\|v\|_\varepsilon$, and let

$$E_{\varepsilon,x,k}=\{\omega\in H:\big\langle\omega,\frac{\partial V_{\varepsilon,x_j}}{\partial x_{jl}}\big\rangle_\varepsilon=0,\ l=1,\cdots,N-m+1,\ j=1,\cdots,k\}.$$

In this section, using the reduction argument, we will prove

THEOREM 2.1. *Let $k>0$ be an integer. There is an $\varepsilon_k>0$, such that for any $\varepsilon\in(0,\varepsilon_k]$,* (2.1) *has a solution of the form*

$$\tilde u_\varepsilon=\sum_{j=1}^k V_{\varepsilon,x_{\varepsilon,j}}+\omega_\varepsilon, \tag{2.14}$$

where $x_{\varepsilon,j}\in D_{k,\varepsilon}$, and $\omega_\varepsilon\in E_{\varepsilon,x,k}$ satisfies

$$\int_\Omega\big(\varepsilon^2|D\omega_\varepsilon|^2+p|\underline{u}_\varepsilon|^{p-1}|\omega_\varepsilon|^2\big)=o\big(\varepsilon^{N-m+1}\big).$$

Before we can carry out the reduction procedure, we need to do some preparation. We have the following non-degeneracy result for U, which is essential for us to construct solutions concentrating at some higher dimensional manifolds:

PROPOSITION 2.2. *Let U be a solution of* (1.4) *with $a=1$. Then U is unique and non-degenerate. That is, the kernel of the operator $-\Delta u-p|U-1|^{p-2}(U-1)u$ in $H^1(R^{N-m+1})$ is spanned by $\{\frac{\partial U}{\partial z_1},\cdots,\frac{\partial U}{\partial z_{N-m+1}}\}$.*

Proof. The readers can refer to Proposition 3.2 in [11] for the proof of this proposition. ∎

LEMMA 2.3. *Let*

$$l_{\varepsilon,x}(\omega)=\sum_{j=1}^k\int_\Omega\big(\varepsilon^2DV_{\varepsilon,x_j}D\omega+p|\underline{u}_\varepsilon|^{p-1}V_{\varepsilon,x_j}\omega\big)-\int_\Omega f_\varepsilon(y,\sum_{j=1}^k V_{\varepsilon,x_j})\omega,$$

where $f_\varepsilon(y,t)$ is defined in (2.2). *Then, $l_{\varepsilon,x}(\omega)$ is a bounded linear operator from $E_{\varepsilon,x,k}$ to R^1. Moreover, there is a constant $\sigma>0$, such that*

$$\|l_{\varepsilon,x}\|_\varepsilon=\varepsilon^{(N-m+1)/2}O\Big(\varepsilon+\sum_{i\neq j}V_{\varepsilon,x_j}^{(1+\sigma)/2}(x_i)\Big).$$

In particular, there is a $l_{\varepsilon,x}\in E_{\varepsilon,x,k}$, such that

$$\big\langle l_{\varepsilon,x},\omega\big\rangle_\varepsilon=l_{\varepsilon,x}(\omega),\quad\forall\omega\in E_{\varepsilon,x,k}.$$

Proof. Let

$$\Omega^*=\cup_{j=1}^k B_\delta^*(x_j).$$

Note that

$$\int_{\Omega_\theta}\omega^2\le Cp\int_{\Omega_\theta}|\underline{u}_\varepsilon|^{p-1}\omega^2\le C\|\omega\|_\varepsilon^2,$$

because $|\underline{u}_\varepsilon| \geq c' > 0$ if $y \in \Omega_\theta$.

Noting that V_{ε,x_j} is exponentially small outside $B^*_\delta(x_j)$, using (2.9), we have

$$\begin{aligned}
&l_{\varepsilon,x}(\omega)\\
=&p\sum_{j=1}^{k}\int_\Omega \big(|\underline{u}_\varepsilon|^{p-1} - \varphi_1^{(p-1)/p}(x_j)\big)V_{\varepsilon,x_j}\omega\\
&-\int_\Omega \Big(f_\varepsilon(y,\sum_{j=1}^{k} V_{\varepsilon,x_j}) - \sum_{j=1}^{k}\tilde{f}(x_j, W_{\varepsilon,x_j,\varphi_1(x_j)})\Big)\omega + \varepsilon^{(N-m+1)/2}O(\varepsilon)\|\omega\|_\varepsilon,
\end{aligned}$$

since for any $q \in [1, 2^*]$,

$$e^{-\sigma/\varepsilon}\big(\int_\Omega |\omega|^q\big)^{1/q} \leq Ce^{-\sigma/\varepsilon}\varepsilon^{-1}\|\omega\|_\varepsilon = \varepsilon^{(N-m+1)/2}O(\varepsilon)\|\omega\|_\varepsilon.$$

By Theorem A.1,

$$\begin{aligned}
&\int_\Omega \big(|\underline{u}_\varepsilon|^{p-1} - \varphi_1^{(p-1)/p}(x_j)\big)V_{\varepsilon,x_j}\omega\\
=&\int_{\Omega_\theta} \big(|\underline{u}_\varepsilon|^{p-1} - \varphi_1^{(p-1)/p}(\tilde{y})\big)V_{\varepsilon,x_j}\omega\\
&+\int_{\Omega_\theta} \big(\varphi_1^{(p-1)/p}(\tilde{y}) - \varphi_1^{(p-1)/p}(x_j)\big)V_{\varepsilon,x_j}\omega + O\big(e^{-\sigma/\varepsilon}\big)\int_{\Omega\setminus\Omega_\theta}|\omega|\\
=&O\Big(\big(\int_{\Omega_\theta}|\tilde{y}-x_j|^2V^2_{\varepsilon,x_j}\big)^{1/2} + \varepsilon^{(N-m+1)/2}\varepsilon^2\Big)\|\omega\|_\varepsilon = \varepsilon^{(N-m+1)/2}O(\varepsilon)\|\omega\|_\varepsilon.
\end{aligned}$$

Similarly, using (2.10), we find

$$\begin{aligned}
&\int_\Omega \big(f_\varepsilon(y,\sum_{j=1}^{k} V_{\varepsilon,x_j}) - \sum_{j=1}^{k}\tilde{f}(x_j, W_{\varepsilon,x_j,\varphi_1(x_j)})\big)\omega\\
=&\int_\Omega \big(f_\varepsilon(y,\sum_{j=1}^{k} V_{\varepsilon,x_j}) - \sum_{j=1}^{k}\tilde{f}(x_j, V_{\varepsilon,x_j})\big)\omega + O(e^{-\sigma/\varepsilon})\|\omega\|_\varepsilon\\
=&\int_\Omega \big(f_\varepsilon(y,\sum_{j=1}^{k} V_{\varepsilon,x_j}) - \sum_{j=1}^{k}f_\varepsilon(y, V_{\varepsilon,x_j})\big)\omega + \sum_{j=1}^{k}\int_\Omega\big(f_\varepsilon(y,V_{\varepsilon,x_j}) - \tilde{f}(x_j,V_{\varepsilon,x_j})\big)\omega\\
=&\int_\Omega \big(f_\varepsilon(y,\sum_{j=1}^{k} V_{\varepsilon,x_j}) - \sum_{j=1}^{k}f_\varepsilon(y, V_{\varepsilon,x_j})\big)\omega + \varepsilon^{(N-m+1)/2}O(\varepsilon^2)\|\omega\|_\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \big(f_\varepsilon(y,\sum_{j=1}^{k} V_{\varepsilon,x_j}) - \sum_{j=1}^{k}f_\varepsilon(y, V_{\varepsilon,x_j})\big)\omega\\
=&O\Big(\sum_{i\neq j}\int_\Omega V^{(1+\sigma)/2}_{\varepsilon,x_i}V^{(1+\sigma)/2}_{\varepsilon,x_j}|\omega|\Big) = \varepsilon^{(N-m+1)/2}O\Big(\sum_{i\neq j}V^{(1+\sigma)/2}_{\varepsilon,x_j}(x_i)\Big)\|\omega\|_\varepsilon.
\end{aligned}$$

□

Denote $f_{\varepsilon,t}(y,t) = \frac{\partial}{\partial t} f_\varepsilon(y,t)$.

LEMMA 2.4. *Let*

$$Q_{\varepsilon,x}(\omega,\eta) = \int_\Omega \big(\varepsilon^2 D\eta D\omega + p|\underline{u}_\varepsilon|^{p-1}\eta\omega\big) - \int_\Omega f_{\varepsilon,t}(y, \sum_{j=1}^k V_{\varepsilon,x_j})\eta\omega.$$

Then, we have

$$|Q_{\varepsilon,x}(\omega,\eta)| \le C\|\omega\|_\varepsilon \|\eta\|_\varepsilon.$$

In particular, there is a bounded linear operator $Q_{\varepsilon,x}$ from $E_{\varepsilon,x,k}$ to $E_{\varepsilon,x,k}$, such that

$$\big\langle Q_{\varepsilon,x}\omega, \eta\big\rangle_\varepsilon = Q_{\varepsilon,x}(\omega,\eta).$$

Proof. It is easy to see that

$$\Big|\int_\Omega \big(\varepsilon^2 D\eta D\omega + p|\underline{u}_\varepsilon|^p \eta\omega\big)\Big| \le \|\omega\|_\varepsilon \|\eta\|_\varepsilon.$$

On the other hand, we have

$$\begin{aligned}
&\Big|\int_\Omega f_{\varepsilon,t}(y, \sum_{j=1}^k P_{\varepsilon,\Omega}U_{\varepsilon,x_j})\eta\omega\Big| \\
\le&\Big|\int_{\Omega_\theta} f_{\varepsilon,t}(y, \sum_{j=1}^k P_{\varepsilon,\Omega}U_{\varepsilon,x_j})\eta\omega\Big| + O(e^{-\sigma/\varepsilon})\Big(\int_\Omega \omega^2\Big)^{1/2}\Big(\int_\Omega \eta^2\Big)^{1/2} \\
\le&C\Big(\int_{\Omega_\theta} \omega^2\Big)^{1/2}\Big(\int_{\Omega_\theta} \eta^2\Big)^{1/2} + O(e^{-\sigma/\varepsilon})\Big(\int_\Omega |D\omega|^2\Big)^{1/2}\Big(\int_\Omega |D\eta|^2\Big)^{1/2} \\
\le&C\Big(\int_{\Omega_\theta} p|\underline{u}_\varepsilon|^{p-1}\omega^2\Big)^{1/2}\Big(\int_{\Omega_\theta} p|\underline{u}_\varepsilon|^{p-1}\eta^2\Big)^{1/2} + O(e^{-\sigma/\varepsilon})\varepsilon^{-2}\|\omega\|_\varepsilon\|\eta\|_\varepsilon \\
\le&C\|\omega\|_\varepsilon\|\eta\|_\varepsilon.
\end{aligned}$$

Thus the result follows. □

LEMMA 2.5. *There is a constant $\rho > 0$, independent of ε and $x \in D_{k,\varepsilon}$, such that*

$$\|Q_{\varepsilon,x}\omega\|_\varepsilon \ge \rho\|\omega\|_\varepsilon, \quad \forall \omega \in E_{\varepsilon,x,k},\ x \in D_{k,\varepsilon}.$$

Proof. The proof of this lemma is standard. We just sketch the proof.

We argue by contradiction. Suppose that there are $\varepsilon_n \to 0$, $x_{j,n} \in D_{k,\varepsilon}$ with $x_{j,n} \to x_j \in S$, $\omega_n \in E_{\varepsilon_n,x_n,k}$, such that

$$\|\omega_n\|_{\varepsilon_n} = \varepsilon_n^{(N-m+1)/2},$$

and

$$\|Q_{\varepsilon,x_n}\omega_n\|_{\varepsilon_n} = o(\varepsilon^{(N-m+1)/2}). \tag{2.15}$$

We claim that for any fixed $R > 0$, $j = 1, \cdots, k$,

$$\int_{B^*_{\varepsilon_n R}(x_{j,n})} |\omega_n|^2 = o(\varepsilon^{N-m+1}). \tag{2.16}$$

In fact, for any fixed $j = 1, \cdots, k$, let $\tilde{\omega}_{j,n}(z) = \omega_n(\varepsilon_n z + x_{j,n})$, $D_n = \{z : \varepsilon_n z + x_{j,n} \in D\}$. $\tilde{U}_{i,n}(y) = V_{\varepsilon_n, x_{i,n}}(\varepsilon_n \tilde{y} + x_{j,n})$, Then we may assume that there is an $\omega_j \in H^1(R^{N-m+1})$, such that

$$D\tilde{\omega}_{j,n} \rightharpoonup D\omega_j, \quad \text{weakly in } L^2(R^{N-m+1}),$$

and

$$\tilde{\omega}_{j,n} \to \omega_j, \quad \text{in } L^2_{\text{loc}}(R^{N-m+1}),$$

as $n \to +\infty$.

From (2.15), we can prove that ω_j satisfies

$$-\Delta\omega_j - p|U_{\varphi_1(x_j)} - \varphi_1^{1/p}(x_j)|^{p-2}(U_{\varphi_1(x_j)} - \varphi_1^{1/p}(x_j))\omega_j = 0, \quad \text{in } R^{N-m+1}. \tag{2.17}$$

By Proposition 2.2, we have

$$\omega_j = \sum_{a=1}^{N-m+1} b_h \frac{\partial U_{\varphi_1(x_j)}}{\partial z_h}, \tag{2.18}$$

for some $b_h \in R^1$.

On the other hand, differentiating (2.9), we find

$$\begin{aligned}
&\Big\langle \frac{\partial V_{\varepsilon, x_{j,n}}}{\partial x_{jh}}, \omega_n \Big\rangle_\varepsilon \\
=&\int_\Omega \big(D\frac{\partial V_{\varepsilon,x_{j,n}}}{\partial x_{jh}} D\omega_n + p\varphi_1^{(p-1)/p}(x_{j,n})\frac{\partial V_{\varepsilon,x_{j,n}}}{\partial x_{jh}}\omega_n\big) \\
&+ p\int_\Omega \big(|\underline{u}_\varepsilon|^{p-1} - \varphi_1^{(p-1)/p}(x_{j,n})\big)\omega_n \frac{\partial V_{\varepsilon,x_{j,n}}}{\partial x_{jh}} \\
=&\int_\Omega \tilde{f}_t(x_{j,n}, W_{\varepsilon,x_{j,n},\varphi_1(x_{j,n})})\frac{\partial V_{\varepsilon,x_j}}{\partial x_{jh}}\omega_n + O\Big(\int_\Omega W_{\varepsilon,x_{j,n},\varphi_1(x_{j,n})}|\omega_n|\Big) \\
=&\varepsilon^{N-m} p\int_{R^{N-m+1}} \big|U_{\varphi_1(x_{j,n})} - \varphi_1^{1/p}(x_{j,n})\big|^{p-2}\big(U_{\varphi_1(x_{j,n})} - \varphi_1^{1/p}(x_{j,n})\big)\frac{\partial U_{\varphi_1(x_{j,n})}}{\partial z_h}\tilde{\omega}_{j,n} \\
&+ \varepsilon^{N-m} p\int_{R^{N-m+1}} \varphi_1^{(p-1)/p}(x_{j,n})\frac{\partial U_{\varphi_1(x_{j,n})}}{\partial z_h}\tilde{\omega}_{j,n} + O\big(\varepsilon^{N-m+1}\big),
\end{aligned}$$

from which, together with $\omega_n \in E_{\varepsilon_n, x_n, k}$, we deduce

$$\begin{aligned}
&\int_{R^{N-m+1}} \big(D\omega_j D\frac{\partial U_{\varphi_1(x_j)}}{\partial z_h} + p\varphi_1^{(p-1)/p}(x_j)\omega_j\frac{\partial U_{\varphi_1(x_j)}}{\partial z_h}\big) \\
=&p\int_{R^{N-m+1}} \Big(\big|U_{\varphi_1(x_j)} - \varphi_1^{1/p}(x_j)\big|^{p-2}\big(U_{\varphi_1(x_j)} - \varphi_1^{1/p}(x_j)\big) + \varphi_1^{(p-1)/p}(x_j)\Big) \\
&\times \frac{\partial U_{\varphi_1(x_j)}}{\partial z_h}\omega_j = 0.
\end{aligned} \tag{2.19}$$

Combining (2.18) and (2.19), we find that $\omega_j = 0$. Thus, (2.16) follows.

It follows from (2.16) that

$$
\begin{aligned}
&o(\varepsilon^{N-m+1}) = \|Q_{\varepsilon,x_n}\omega_n\|_{\varepsilon_n}\|\omega_n\|_{\varepsilon_n} \ge \left|\langle Q_{\varepsilon,x}\omega_n,\omega_n\rangle_{\varepsilon_n}\right| \\
\ge&\|\omega_n\|_{\varepsilon_n}^2 - \int_\Omega f_{\varepsilon_n,t}(y,\sum_{j=1}^k V_{\varepsilon,x_{j,n}})\omega_n^2 \\
=&\|\omega_n\|_{\varepsilon_n}^2 - \Big(\int_{\Omega\setminus\Omega_\theta} + \int_{\Omega_\theta\setminus\cup_{j=1}^k B^*_{\varepsilon_n R}(x_{j,n})} + \int_{\cup_{j=1}^k B^*_{\varepsilon_n R}(x_{j,n})}\Big)|f_{\varepsilon_n,t}(y,\sum_{j=1}^k V_{\varepsilon,x_{j,n}})|\omega_n^2 \\
=&\|\omega_n\|_{\varepsilon_n}^2 - O(e^{-\sigma/\varepsilon}) - o_R(1)\int_{\Omega_\theta\setminus\cup_{j=1}^k B^*_{\varepsilon_n R}(x_{j,n})}\omega_n^2 + o(\varepsilon^{N-m+1}) \\
=&\|\omega_n\|_{\varepsilon_n}^2 - O(e^{-\sigma/\varepsilon}) + o(\varepsilon^{N-m+1}) \ge \frac{1}{2}\varepsilon^{N-m+1}.
\end{aligned}
$$

This is a contradiction. □

The following proposition allows us to reduce the problem of finding a solution with the form (2.14) to a finite dimensional problem.

PROPOSITION 2.6. *There is an $\varepsilon_k > 0$, such that for each $\varepsilon \in (0,\varepsilon_k]$, there is a C^1-map $\omega_{\varepsilon,x}$: $D_{k,\varepsilon} \to H$, such that $\omega_{\varepsilon,x} \in E_{\varepsilon,x,k}$, and*

$$
I'(\sum_{j=1}^k V_{\varepsilon,x_j} + \omega_{\varepsilon,x}) = \sum_{j=1}^k \sum_{h=1}^{N-m+1} A_{jh}\frac{\partial V_{\varepsilon,x_j}}{\partial x_{jh}}, \tag{2.20}
$$

where A_{jh} are some constants, $j = 1,\cdots,k$, $h = 1,\cdots,N-m+1$. Moreover, we have

$$
\|\omega_{\varepsilon,x}\|_\varepsilon = \varepsilon^{(N-m+1)/2}O(\varepsilon), \quad x \in D_{k,\varepsilon}.
$$

Proof. Since p may be supercritical, $I(u)$ may not be well defined in the whole space H. To carry out the reduction argument, we first need to choose a subset of $E_{\varepsilon,x,k}$. Define

$$
\begin{aligned}
\tilde E_{\varepsilon,x,k} = \big\{\omega : \omega \in E_{\varepsilon,x,k},\ &\|\omega\|_\varepsilon \le \varepsilon^{(N-m+1)/2}\varepsilon^{1/2}, \\
&|\omega(z)| \le \varepsilon^{1/2}, \quad |\omega(z)| \le \sum_{j=1}^k e^{-\theta|z-x_j|/\varepsilon},\ \ z \in \cup_{j=1}^k B_\delta(x_j), \\
&|\omega(z)| \le k e^{-\theta\delta/\varepsilon},\ \ z \in D\setminus\cup_{j=1}^k B_\delta(x_j)\big\},
\end{aligned}
$$

where $\theta > 0$ is a fixed small constant.

Let

$$
\bar K(x,\omega) = I(\sum_{j=1}^k V_{\varepsilon,x_j} + \omega), \quad x \in D_{k,\varepsilon},\ \omega \in \tilde E_{\varepsilon,x,k}.
$$

Because $\underline{u}_\varepsilon \ge c' > 0$ in $\cup_{j=1}^k B_\delta(x_j)$, and $1 < p < \frac{N-m+3}{N-m-1}$, it is easy to check $\bar K(x,\omega)$ is well defined in $x \in D_{k,\varepsilon}$, $\omega \in \tilde E_{\varepsilon,x,k}$.

Expand $\bar{K}(x,\omega)$ near $\omega=0$ as follows:

$$\bar{K}(x,\omega)=\bar{K}(x,0)+\big\langle l_{\varepsilon,x},\omega\big\rangle_\varepsilon+\frac{1}{2}\big\langle Q_{\varepsilon,x}\omega,\omega\big\rangle_\varepsilon+R_\varepsilon(\omega),$$

where $l_{\varepsilon,x}$ and $Q_{\varepsilon,x}$ are defined in Lemma 2.3 and Lemma 2.4 respectively, and

$$R_\varepsilon(\omega)=-\int_\Omega\Big(F_\varepsilon(y,\sum_{j=1}^k V_{\varepsilon,x_j}+\omega)-F_\varepsilon(y,\sum_{j=1}^k V_{\varepsilon,x_j}) \\ -f_\varepsilon(y,\sum_{j=1}^k V_{\varepsilon,x_j})\omega-\frac{1}{2}f_{\varepsilon,t}(y,\sum_{j=1}^k V_{\varepsilon,x_j})\omega^2\Big).$$

Thus, finding a critical point for $\bar{K}(x,\omega)$ in $E_{\varepsilon,x,k}$ is equivalent to solving

$$l_{\varepsilon,x}+Q_{\varepsilon,x}\omega+R'_\varepsilon(\omega)=0. \tag{2.21}$$

Denote $\bar{p}=\min(3,p+1)$. Then

$$|R_\varepsilon(\omega)|\le C\int_\Omega|\omega|^{\bar{p}}.$$

For any $\omega\in\bar{E}_{\varepsilon,x,k}$, we have

$$\begin{aligned}&\int_{\Omega\setminus\cup_{j=1}^k B_\delta^*(x_j)}|\omega|^{\bar{p}}\le e^{-(\bar{p}-2)\theta\delta/\varepsilon}\int_{\Omega\setminus\cup_{j=1}^k B_\delta^*(x_j)}|\omega|^2\\ \le& e^{-(\bar{p}-2)\theta\delta/\varepsilon}\int_\Omega|\omega|^2\le Ce^{-(\bar{p}-2)\theta\delta/\varepsilon}\varepsilon^{-2}\|\omega\|_\varepsilon^2,\end{aligned} \tag{2.22}$$

$$\begin{aligned}&\int_{\Omega\setminus\cup_{j=1}^k B_\delta^*(x_j)}|\omega|^{\bar{p}-1}|\eta|\le e^{-(\bar{p}-2)\theta\delta/\varepsilon}\int_{\Omega\setminus\cup_{j=1}^k B_\delta^*(x_j)}|\omega||\eta|\\ \le& Ce^{-(\bar{p}-2)\theta\delta/\varepsilon}\varepsilon^{-2}\|\omega\|_\varepsilon\|\eta\|_\varepsilon,\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}&\int_{\Omega\setminus\cup_{j=1}^k B_\delta^*(x_j)}|\omega|^{\bar{p}-2}|\eta_1||\eta_2|\le Ce^{-(\bar{p}-2)\theta\delta/\varepsilon}\int_{\Omega\setminus\cup_{j=1}^k B_\delta^*(x_j)}|\eta_1||\eta_2|\\ \le& Ce^{-(\bar{p}-2)\theta\delta/\varepsilon}\varepsilon^{-2}\|\eta_1\|_\varepsilon\|\eta_2\|_\varepsilon.\end{aligned} \tag{2.24}$$

Since $\underline{u}_\varepsilon\ge c_0>0$ and $|y'|\ge c_0>0$ in $\cup_{j=1}^k B_\delta^*(x_j)$,, it is easy to check that

$$\int_{\cup_{j=1}^k B_\delta^*(x_j)}|\omega|^{\bar{p}}=\varepsilon^{N-m+1}O\big(\varepsilon^{-\bar{p}(N-m+1)/2}\|\omega\|_\varepsilon^{\bar{p}}\big),$$

$$\int_{\cup_{j=1}^k B_\delta^*(x_j)}|\omega|^{\bar{p}-1}\eta=\varepsilon^{(N-m+1)/2}O\big(\varepsilon^{-(\bar{p}-1)(N-m+1)/2}\|\omega\|_\varepsilon^{\bar{p}-1}\big)\|\eta\|_\varepsilon,$$

$$\int_{\cup_{j=1}^k B_\delta^*(x_j)}|\omega|^{\bar{p}-2}\eta_1\eta_2=O\big(\varepsilon^{-(\bar{p}-2)(N-m+1)/2}\|\omega\|_\varepsilon^{\bar{p}-2}\big)\|\eta_1\|_\varepsilon\|\eta_2\|_\varepsilon.$$

So, we obtain

$$R_\varepsilon(\omega) = \varepsilon^{N-m+1} O\big(\varepsilon^{-\bar p(N-m+1)/2}\|\omega\|_\varepsilon^{\bar p}\big), \tag{2.25}$$

$$\big\langle R'_\varepsilon(\omega), \eta\big\rangle_\varepsilon = \varepsilon^{(N-m+1)/2} O\big(\varepsilon^{-(N-m+1)(\bar p-1)/2}\|\omega\|_\varepsilon^{\bar p-1}\big)\|\eta\|_\varepsilon, \tag{2.26}$$

$$R''_\varepsilon(\omega)(\eta_1,\eta_2) = O\big(\varepsilon^{-(N-m+1)(\bar p-2)/2}\|\omega\|_\varepsilon^{\bar p-2}\big)\|\eta_1\|_\varepsilon\|\eta_2\|_\varepsilon. \tag{2.27}$$

On the other hand, using Lemma 2.5, we see that $Q_{\varepsilon,x}$ is invertible in $E_{\varepsilon,x,k}$, and there is a constant C, independent of ε and x, such that

$$\|Q_{\varepsilon,x}^{-1}\|_\varepsilon \le C. \tag{2.28}$$

Rewrite (2.21) as

$$\omega = -Q_{\varepsilon,x}^{-1} l_{\varepsilon,x} - Q_{\varepsilon,x}^{-1} R'_\varepsilon(\omega). \tag{2.29}$$

Let

$$G(\omega) = -Q_{\varepsilon,x}^{-1} l - Q_{\varepsilon,x}^{-1} R'_\varepsilon(\omega), \quad \forall\, \omega \in \tilde E_{\varepsilon,x,k}.$$

We now prove that for each l with $\|l\|_\varepsilon \le C\varepsilon^{(N-m+1)/2}\varepsilon$, G is a contraction map from $\tilde E_{\varepsilon,x,k}$ to $\bar E_{\varepsilon,x,k}$.

Step 1. For any $\omega_1 \in \tilde E_{\varepsilon,x,k}$ and $\omega_2 \in \tilde E_{\varepsilon,x,k}$, we see from (2.27) that,

$$\|G(\omega_1) - G(\omega_2)\|_\varepsilon \le C\|R'_\varepsilon(\omega_1) - R'_\varepsilon(\omega_2)\|_\varepsilon \le C\varepsilon^{\bar\sigma}\|\omega_1-\omega_2\|_\varepsilon, \tag{2.30}$$

where $\bar\sigma > 0$ is a constant. Thus, G is a contraction map.

Step 2. For each $\omega \in \tilde E_{\varepsilon,x,k}$,

$$\begin{aligned} &\|G(\omega)\|_\varepsilon \le C\|l\|_\varepsilon + C\|R'_\varepsilon(\omega)\|_\varepsilon \\ \le& C\|l\|_\varepsilon + C\varepsilon^{\sigma}\|\omega\|_\varepsilon \le C\varepsilon^{(N-m+1)/2}\varepsilon^{1/2+\sigma} \le \varepsilon^{(N-m+1)/2}\varepsilon^{1/2}. \end{aligned} \tag{2.31}$$

Step 3. For each $\omega \in \tilde E_{\varepsilon,x,k}$, we show that $\omega_1 =: G(\omega)$ satisfies

$$|\omega_1(z)| \le \varepsilon^{1/2}, \quad |\omega_1(z)| \le \sum_{j=1}^k e^{-\theta|z-x_j|/\varepsilon}, \quad z \in \cup_{j=1}^k B_\delta(x_j), \tag{2.32}$$

and

$$|\omega_1(z)| \le k e^{-\theta\delta/\varepsilon}, \quad z \in D \setminus \cup_{j=1}^k B_\delta(x_j). \tag{2.33}$$

Note that ω_1 satisfies

$$Q_{\varepsilon,x}\omega_1 = -l_{\varepsilon,x} - R'_\varepsilon(\omega),$$

which is equivalent to

$$\big\langle Q_{\varepsilon,x}\omega_1, \xi\big\rangle_\varepsilon + \big\langle l_{\varepsilon,x}, \xi\big\rangle_\varepsilon + \big\langle R'_\varepsilon(\omega), \xi\big\rangle_\varepsilon = \sum_{j=1}^k \sum_{h=1}^{N-m+1} G_{jh}\Big\langle \frac{\partial V_{\varepsilon,x_j}}{\partial x_{j,h}}, \xi\Big\rangle_\varepsilon, \tag{2.34}$$

for some $G_{jh} \in R^1$.

We claim that there is a $\sigma > 0$, such that

$$|G_{jh}| \le C\varepsilon^{\sigma+3/2}, \quad j = 1, \cdots, k,\ h = 1, \cdots, N-m+1. \tag{2.35}$$

In fact, letting $\xi = \frac{\partial V_{\varepsilon,x_i}}{\partial x_{i,\bar h}}$ in (2.34), we can solve the linear system to obtain

$$\begin{aligned} &|G_{jh}| \le C\varepsilon^{1-(N-m+1)/2}\big(\|\omega_1\|_\varepsilon + \|l_{\varepsilon,x}\|_\varepsilon + \|R'(\omega)\|_\varepsilon\big) \\ \le& C\varepsilon^{1-(N-m+1)/2}\varepsilon^{\frac{1}{2}+\sigma+(N-m+1)/2} \le C\varepsilon^{\sigma+3/2}. \end{aligned}$$

Using (2.9), we can rewrite (2.34) as

$$\begin{aligned} &-\varepsilon^2\Delta\omega_1 + p|\underline{u}_\varepsilon|^{p-1}\omega_1 - f_{\varepsilon,t}\big(\sum_{j=1}^k V_{\varepsilon,x_j}\big)\omega_1 \\ =& f_\varepsilon\Big(y, \sum_{j=1}^k V_{\varepsilon,x_j} + \omega\Big) - f_\varepsilon\Big(y, \sum_{j=1}^k V_{\varepsilon,x_j}\Big) - f_{\varepsilon,t}\Big(y, \sum_{j=1}^k V_{\varepsilon,x_j}\Big)\omega \\ &+ f_\varepsilon\Big(y, \sum_{j=1}^k V_{\varepsilon,x_j}\Big) - \sum_{j=1}^k \tilde f\Big(x_j, W_{\varepsilon,x_j,\varphi_1(x_j)}\Big) \\ &+ p\sum_{j=1}^k\Big(|\underline{u}_\varepsilon|^{p-1} - \varphi_1^{(p-1)/p}(x_j)\Big)V_{\varepsilon,x_j} + O\Big(\big(\varepsilon + |\xi(|y'|) - 1|\big)W_{\varepsilon,x_j,\varphi_1(x_j)}\Big) \\ &+ \sum_{j=1}^k\sum_{h=1}^{N-m+1} G_{jh}\Big(-\varepsilon^2\Delta\frac{\partial V_{\varepsilon,x_j}}{\partial x_{j,h}} + p|\underline{u}_\varepsilon|^{p-1}\frac{\partial V_{\varepsilon,x_j}}{\partial x_{j,h}}\Big) \\ =:& G_{\varepsilon,x}(\tilde y, \omega). \end{aligned} \tag{2.36}$$

By (2.35), we have the following estimate for $G_{\varepsilon,x}(y,\omega)$:

$$\begin{aligned} &|G_{\varepsilon,x}(y,\omega)| \le C|\omega|^{\bar p-1} + C\varepsilon^{\frac{1}{2}+\sigma}\sum_{j=1}^k V_{\varepsilon,x_j}^{1/2} + C\sum_{i\ne j} V_{\varepsilon,x_i}^{\frac{1}{2}+\sigma}V_{\varepsilon,x_j}^{\frac{1}{2}+\sigma} \\ \le& C|\omega|^{\bar p-1} + C\varepsilon^{\frac{1}{2}+\sigma}\sum_{j=1}^k V_{\varepsilon,x_j}^{1/2}, \quad \forall\, x \in D_{\varepsilon,k}. \end{aligned} \tag{2.37}$$

Let i be fixed. For any function $\omega(z)$, we denote $\tilde\omega(z) = \omega(\varepsilon z + x_i)$. Then, $\tilde\omega_1$ satisfies

$$-\Delta\tilde\omega_1 - \varepsilon\frac{m-1}{\varepsilon|y'| + x_{i,1}}\frac{z_1}{|z|}\tilde\omega_1 + p|\underline{\tilde u}_\varepsilon|^{p-1}\tilde\omega_1 - f_{\varepsilon,t}\big(\sum_{j=1}^k \tilde V_{\varepsilon,x_j}\big)\tilde\omega_1 = G_{\varepsilon,x}(\varepsilon\tilde y + x_i, \tilde\omega). \tag{2.38}$$

From $\|\omega_1\|_\varepsilon \le C\varepsilon^{(N-m+2+\sigma)/2}$, we find

$$\int_{B_2(z)} |\tilde\omega_1|^2 \le C\varepsilon^{1+\sigma}, \quad \forall\, z \in B_{2\delta/\varepsilon}(0).$$

Using the Moser iteration for (2.38), and using (2.37), we can deduce

$$\begin{aligned}&|\tilde\omega_1(z)|\le C\|\tilde\omega_1\|_{L^2(B_1(z))}+C\|G_{\varepsilon,x}(\varepsilon\tilde y+x_i,\tilde\omega)\|_{L^2(B_1(z))}\\ \le &C\varepsilon^{(\sigma+1)/2}+\varepsilon^{(\bar p-2)/2}\|\tilde\omega\|_{L^2(B_1(z))}\le C\varepsilon^{(\sigma+1)/2}\le\varepsilon^{1/2},\quad \varepsilon z+x_i\in B_\delta(x_i).\end{aligned}$$

So, we have proved

$$|\omega_1(z)|\le\varepsilon^{1/2},\quad z\in B_\delta(x_i),\ i=1,\cdots,k. \tag{2.39}$$

By (2.39), we can deduce

$$f_{\varepsilon,t}\big(\sum_{j=1}^k V_{\varepsilon,x_j}\big)\omega_1=O(\varepsilon^{1/2})\sum_{j=1}^k e^{-\sigma|z-x_j|/\varepsilon},\quad z\in D,$$

for some $\sigma>0$. As a result, (2.36) becomes

$$-\varepsilon^2\Delta\omega_1+p|\underline{u}_\varepsilon|^{p-1}\omega_1=O\Big(\varepsilon^\sigma\sum_{j=1}^k e^{-\sigma|z-x_j|/\varepsilon}+|\omega|^{\bar p-1}\Big). \tag{2.40}$$

There is a constant $b>0$, such that

$$p|\underline{u}_\varepsilon|^{p-1}\ge 2b^2>0,\quad \text{in } \cup_{j=1}^k B^*_{2\delta}(x_j).$$

Denote $G_{\varepsilon,b}(Y,y)$ be the Green's function of $-\varepsilon^2\Delta+b^2$ in Ω with Dirichlet boundary condition. Then

$$0<G_{\varepsilon,b}(Y,y)\le Ce^{-b|Y-y|/\varepsilon}.$$

Consider the following problem:

$$\begin{cases}-\varepsilon^2\Delta w+b^2w=\sum_{j=1}^k e^{-\theta(1+10\bar\theta)|\tilde y-x_j|/\varepsilon}, & y\in\Omega;\\ w=0, & y\in\partial\Omega,\end{cases} \tag{2.41}$$

where $\bar\theta>0$ is a small constant with $0<\bar\theta<<\theta$. Then the solution w_1 of (2.41) satisfies

$$0\le w_1(y)=\int_\Omega G_{\varepsilon,b}(Y,y)\sum_{j=1}^k e^{-\theta(1+10\bar\theta)|\tilde Y-x_j|/\varepsilon}\,dY\le C\sum_{j=1}^k e^{-\theta(1+9\bar\theta)|\tilde y-x_j|/\varepsilon}.$$

Denote $v=\varepsilon^{\sigma/2}w_1-\omega_1$. Then, from (2.40),

$$\begin{aligned}&-\varepsilon^2\Delta v+p|\underline{u}_\varepsilon|^{p-1}v\\ =&\varepsilon^{\sigma/2}\sum_{j=1}^k e^{-\theta(1+10\bar\theta)|\tilde y-x_j|/\varepsilon}+\varepsilon^{\sigma/2}\big(p|\underline{u}_\varepsilon|^{p-1}-b^2\big)w_1-O\Big(\varepsilon^\sigma\sum_{j=1}^k e^{-\sigma|z-x_j|/\varepsilon}+|\omega|^{\bar p-1}\Big)\\ =:&\tilde g_\varepsilon(y).\end{aligned}$$

Choose $\eta \in C_0^2(\Omega)$ with $\eta = 1$ in $\cup_{j=1}^k B^*_{(1-\bar\theta)\delta}(x_j)$, $\eta = 0$ in $D \setminus \cup_{j=1}^k B^*_\delta(x_j)$, $0 \le \eta \le 1$. Let v_1 be the solution of

$$\begin{cases} -\varepsilon^2 \Delta v + p|\underline{u}_\varepsilon|^{p-1} v = \eta \tilde g_\varepsilon(y), & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.42}$$

and let v_2 be the solution of

$$\begin{cases} -\varepsilon^2 \Delta v + p|\underline{u}_\varepsilon|^{p-1} v = (1-\eta) \tilde g_\varepsilon(y), & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.43}$$

Since for any $y \in \cup_{j=1}^k B^*_\delta(x_j)$,

$$|\omega(y)|^{\bar p-1} = |\omega(y)|^{\bar p-2-\sigma}|\omega(y)|^{1+\sigma} \le \varepsilon^{(\bar p-2-\sigma)/2} \sum_{j=1}^k e^{-\theta(1+\sigma)|\tilde y - x_j|/\varepsilon},$$

we see $\eta \tilde g_\varepsilon(y) \ge 0$. As a result, $v_1 \ge 0$.

On the other hand, by Lemma A.2, we have

$$\begin{aligned} c\varepsilon^{2(p-1)/(3p-1)} \int_\Omega v_2^2 \le \|v_2\|_\varepsilon^2 = \int_\Omega (1-\eta)\tilde g_\varepsilon(y) v_2 \\ \le C e^{-\theta(1+9\bar\theta)(1-\bar\theta)\delta/\varepsilon} \Big(\int_\Omega v_2^2\Big)^{1/2} \le C e^{-\theta(1+8\bar\theta)\delta/\varepsilon} \Big(\int_\Omega v_2^2\Big)^{1/2}. \end{aligned}$$

So,

$$\int_\Omega v_2^2 \le C e^{-2\theta(1+7\bar\theta)/\varepsilon}.$$

Thus, using the Moser iteration, similar to (2.39), we find

$$|v_2| \le C\varepsilon^{-N/2} e^{-\theta(1+7\bar\theta)\delta/\varepsilon} \le C e^{-\theta(1+6\bar\theta)\delta/\varepsilon}.$$

As a result,

$$\begin{aligned} \omega_1 = \varepsilon^{\sigma/2} w_1 - v \le \varepsilon^{\sigma/2} w_1 - v_2 \le \varepsilon^{\sigma/2} w_1 + C e^{-\theta(1+6\bar\theta)\delta/\varepsilon} \\ \le C e^{-\theta(1+6\bar\theta)\delta/\varepsilon}, \quad \text{in } \Omega \setminus \cup_{j=1}^k B^*_\delta(x_j). \end{aligned}$$

Similarly,

$$-\omega_1 \le C e^{-\theta(1+6\bar\theta)\delta/\varepsilon}, \quad \text{in } \Omega \setminus \cup_{j=1}^k B^*_\delta(x_j).$$

As a result,

$$|\omega_1| \le C e^{-\theta(1+6\bar\theta)\delta/\varepsilon} \le e^{-\theta\delta/\varepsilon}, \quad \text{in } \Omega \setminus \cup_{j=1}^k B^*_\delta(x_j). \tag{2.44}$$

Finally, we have

$$p|\underline{u}_\varepsilon(y)|^{p-1} \ge 2b_1^2 > 0, \quad d(y, \partial\Omega) \ge \bar\theta.$$

Let $\eta_1 \in C_0^2(\Omega)$ with $\eta_1 = 1$ for any $y \in \Omega$ with $d(y, \partial\Omega) \le \bar{\theta}$. Replacing η in (2.42) and (2.43) by η_1, we can prove that

$$|\omega_1(y)| \le \varepsilon^{\sigma/2} w_1(y) + |v_2|$$

with

$$|v_2(y)| \le C \sum_{j=1}^{k} e^{-\theta(1+6\bar{\theta}) d(x_j, \partial\Omega)/\varepsilon}.$$

So,

$$|v_2(y)| \le \varepsilon \sum_{j=1}^{k} e^{-\theta(1+5\bar{\theta})|\tilde{y}-x_j|/\varepsilon}, \quad y \in \cup_{j=1}^k B_\delta^*(x_j).$$

Thus,

$$|\omega_1(y)| \le C\varepsilon^{\sigma/2} \sum_{j=1}^{k} e^{-\theta|\tilde{y}-x_j|/\varepsilon} \le \sum_{j=1}^{k} e^{-\theta|\tilde{y}-x_j|/\varepsilon}, \quad y \in \cup_{j=1}^k B_\delta^*(x_j). \tag{2.45}$$

From (2.39), (2.44) and (2.45), we finish the proof of (2.32) and (2.33).

Combining Step 1–Step 3, we see that $G(\omega)$ is a contraction map from $\bar{E}_{\varepsilon,x,k}$ to $\bar{E}_{\varepsilon,x,k}$, for any $l \in E_{\varepsilon,x,k}$ with $\|l\|_\varepsilon \le C\varepsilon^{(N-m+1)/2}\varepsilon$. By the contraction mapping theorem, we know that for any $l \in E_{\varepsilon,x,k}$ with $\|l\|_\varepsilon \le C\varepsilon^{(N-m+1)/2}\varepsilon$, there is a unique $\omega \in \bar{E}_{\varepsilon,x,k}$, such that

$$\omega = G(\omega).$$

On the other hand, for any $x \in D_{k,\varepsilon}$, we have $\|l_{\varepsilon,x}\|_\varepsilon \le C\varepsilon^{(N-m+1)/2}\varepsilon$. As a result, for each $x \in D_{k,\varepsilon}$, there is $\omega_{\varepsilon,x} \in \bar{E}_{\varepsilon,x,k}$, such that (2.29) holds. Moreover, from (2.31), we have

$$\|\omega_{\varepsilon,x}\|_\varepsilon \le C\|l_{\varepsilon,x}\|_\varepsilon \le C\varepsilon^{(N-m+1)/2}\varepsilon.$$

□

Proof of Theorem 2.1. We need to choose $x \in D_{\varepsilon,k}$, such that all the constants A_{jh} in (2.20) are zero. It is easy to check that if $x \in D_{\varepsilon,k}$ is a critical point of the following function:

$$K(x) = I\Big(\sum_{j=1}^{k} V_{\varepsilon,x_j} + \omega_{\varepsilon,x}\Big),$$

where $\omega_{\varepsilon,x}$ is the function obtained in Proposition 2.6, then, $A_{jh} = 0$, $j = 1, \cdots, k$, $h = 1, \cdots, N-m+1$.

Consider

$$\max_{x \in D_{k,\varepsilon}} K(x).$$

Then it follows from Propositions 2.6 and B.2, we have for any $x \in D_{k,\varepsilon}$,

$$\begin{aligned} K(x) =& I(\sum_{j=1}^{k} V_{\varepsilon,x_j}) + O\big(\|l_{\varepsilon,x}\|_\varepsilon \|\omega_\varepsilon\|_\varepsilon + \|\omega_\varepsilon\|_\varepsilon^2 + R_\varepsilon(\omega_{\varepsilon,x_\varepsilon})\big) \\ =& I(\sum_{j=1}^{k} V_{\varepsilon,x_j}) + \varepsilon^{N-m+1} O(\varepsilon) \\ =& \varepsilon^{N-m+1} A \sum_{j=1}^{k} x_{j,1}^{m-1} \varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{N-m+1}{2})}(x_j) \\ & - \varepsilon^{N-m+1} \sum_{i\neq j} \big(c(x_i)+o(1)\big) V_{\varepsilon,x_j}(x_i) + O(\varepsilon^{N-m+2}), \end{aligned} \tag{2.46}$$

Let $x_\varepsilon \in D_{k,\varepsilon}$ is a maximum point of $K(x)$ in $D_{k,\varepsilon}$. Choose $\tilde{x}_\varepsilon = (\tilde{x}_{\varepsilon,1}, \cdots, \tilde{x}_{\varepsilon,k})$, such that

$$d(\tilde{x}_{\varepsilon,j}, S) = L\varepsilon \ln\frac{1}{\varepsilon}, \quad j = 1, \cdots, k,$$

and

$$|\tilde{x}_{\varepsilon,j} - \tilde{x}_{\varepsilon,j}| \geq \frac{L}{k}\varepsilon \ln\frac{1}{\varepsilon}, \quad i \neq j,$$

where $L > 0$ is large. Then if $L > 0$ is large, we see that $\tilde{x}_\varepsilon \in D_{k,\varepsilon}$, and

$$\tilde{x}_{\varepsilon,j,1}^{m-1} \varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{N-m+1}{2})}(\tilde{x}_{\varepsilon,j}) = M + O\big(\varepsilon^{N-m+2} \ln\frac{1}{\varepsilon}\big), \tag{2.47}$$

and

$$V_{\varepsilon,\tilde{x}_{\varepsilon,j}}(\tilde{x}_{\varepsilon,i}) = O\big(\varepsilon^{N-m+2}\big). \tag{2.48}$$

So, it follows from (2.46), (2.47) and (2.48) that

$$K(\tilde{x}_\varepsilon) = \varepsilon^{N-m+1} kAM + \varepsilon^{N-m+1} O\big(\varepsilon \ln\frac{1}{\varepsilon}\big). \tag{2.49}$$

¿From $K(\tilde{x}_\varepsilon) \leq K(x_\varepsilon)$, together with (2.49) and (2.46), we obtain

$$\sum_{j=1}^{k} \Big(x_{\varepsilon,j,1}^{m-1} \varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{N-m+1}{2})}(x_{\varepsilon,j}) - M \Big) - \sum_{i\neq j} c(x_{\varepsilon,i}) V_{\varepsilon,x_{\varepsilon,j}}(x_{\varepsilon,i}) \geq O\big(\varepsilon \ln\frac{1}{\varepsilon}\big).$$

Thus,

$$0 \leq M - x_{\varepsilon,j,1}^{m-1} \varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{N-m+1}{2})}(x_{\varepsilon,j}) \leq C\varepsilon \ln\frac{1}{\varepsilon} < \varepsilon^{1-\tau},$$

and

$$V_{\varepsilon,x_{\varepsilon,j}}(x_{\varepsilon,i}) \leq C\varepsilon \ln\frac{1}{\varepsilon} < \varepsilon^{1-\tau}.$$

That is, x_ε is an interior point of $D_{k,\varepsilon}$. Hence, x_ε is a critical point of $K(x)$. □

Appendix A. Existence of a local minimizer. In this section, we show that (1.3) has a negative solution, which is a function in H_s, and is a local minimizer of the corresponding functional in H_s. One can use the subsolution and supersolution techniques as in [7] to find a negative solution for (1.3). But it is not easy to find a good asymptotic estimate for the solution obtained via the subsolution and supersolution techniques. In this section, we will proceed as in [20, 11].

THEOREM A.1. *There is an* $\varepsilon_0 > 0$, *such that for each* $\varepsilon \in (0, \varepsilon_0]$, (1.3) *has a solution* $\underline{u}_\varepsilon$, *such that* $0 > \underline{u}_\varepsilon > -\varphi_1^{1/p}$, $\forall\, y \in \Omega$, $\underline{u}_\varepsilon \in H_s$, *and*

$$\underline{u}_\varepsilon(y) = -\varphi_1^{1/p}(y) - \varepsilon^2 \frac{\Delta \varphi_1^{1/p}(y)}{p\varphi_1^{(p-1)/p}(y)} + o(\varepsilon^2),$$

where $\varepsilon^{-2}o(\varepsilon^2) \to 0$ *uniformly on any compact subset of* Ω *as* $\varepsilon \to 0$.

Proof of Theorem A.1. Let $u = -w$. Then (1.3) becomes

$$\begin{cases} -\varepsilon^2 \Delta w = \varphi_1(y) - |w|^p, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{A.1}$$

Let

$$h(y,t) = \begin{cases} 0, & t \ge \varphi_1^{1/p}(y), \\ \varphi_1(y) - t^p, & 0 \le t < \varphi_1^{1/p}(y), \\ \varphi_1(y), & t < 0. \end{cases}$$

Consider

$$\begin{cases} -\varepsilon^2 \Delta w = h(y,w), & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{A.2}$$

It is easy to check that any solution of (A.2) is positive. Direct calculation shows that $\varphi_1^{1/p}(y) > 0$ is a supersolution of (A.2). As a result, we obtain that any solution w_ε of (A.2) satisfies

$$0 < w_\varepsilon \le \varphi_1^{1/p}.$$

Thus w_ε is also a solution of (A.1). On the other hand, since $\frac{\partial h(y,t)}{\partial t} \le 0$ for any $y \in \Omega$ and $t \in (0, \varphi_1^{1/p}(y)]$, we see that the solution of (A.2) is unique. Denote

$$J_\varepsilon(w) = \frac{\varepsilon^2}{2} \int_\Omega |Dw|^2 - \int_\Omega H(y,w),$$

where $H(y,t) = \int_0^t h(y,\tau)\, d\tau$.

Let w_ε be a minimizer of

$$\min\bigl\{ J_\varepsilon(w) : w \in H_0^1(\Omega) \bigr\}. \tag{A.3}$$

Then, w_ε is a solution of (A.2). On the other hand, $J_\varepsilon(w)$ also has a minimizer in H_s. By the uniqueness, $w_\varepsilon \in H_s$. Moreover, the asymptotic expansion follows from Theorem 2.1 in [11]. □

Let $\underline{u}_\varepsilon$ be the solution obtained in Theorem A.1. Consider the following eigenvalue problem:

$$\begin{cases} -\varepsilon^2 \Delta \eta + p|\underline{u}_\varepsilon|^{p-1}\eta = \lambda \eta, & \text{in } \Omega, \\ \eta \in H_0^1(\Omega). \end{cases} \tag{A.4}$$

We have

Lemma A.2. *Let λ_ε be the first eigenvalue of* (A.4). *Then*

$$\lambda_\varepsilon \geq c_0 \varepsilon^{2(p-1)/(3p-1)},$$

where $c_0 > 0$ is a constant, independent of ε.

Proof. For the proof of this lemma, the readers can refer to the proof of Lemma 3.6 in [11]. □

Remark A.3. Lemma A.2 shows that $\underline{u}_\varepsilon$ is a local minimizer of the corresponding functional.

Remark A.4. We need to assume that the boundary of Ω is C^1 to prove Lemma A.2. This is the only place that we need this assumption.

Appendix B. Energy expansion. Let $V_{\varepsilon,\bar{x}}$ be define in (2.7) and let $I_\varepsilon(v)$ be the functional defined in (2.3). In this section, we will expand $I_\varepsilon(V_{\varepsilon,x_j})$.

Lemma B.1. *We have*

$$I_\varepsilon(V_{\varepsilon,x_j}) = \varepsilon^{N-m+1} A x_{j,1}^{m-1} \varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1} - \frac{N-m+1}{2})}(x_j) + \varepsilon^{N-m+1} O(\varepsilon),$$

where $A > 0$ is a constant.

Proof. Firstly, let recall the definition of the function $\tilde{f}(\tilde{y}, t)$ in (2.8) and the function $f_\varepsilon(y,t)$ in (2.2). Define $\tilde{F}(\tilde{y}, t) = \int_0^t \tilde{f}(\tilde{y}, s)\, ds$.

Using the exponential decay of V_{ε,x_j}, (2.9) and (2.10), we obtain

$$\begin{aligned} I(V_{\varepsilon,x_j}) =& \frac{1}{2}\int_\Omega \Big[\tilde{f}(x_j, W_{\varepsilon,x_j,\varphi_1(x_j)}) + O\Big(\varepsilon W_{\varepsilon,x_j,\varphi_1(x_j)} + |\xi(|y'|) - 1| W_{\varepsilon,x_j,\varphi_1(x_j)}\Big)\Big] V_{\varepsilon,x_j} \\ &+ \frac{1}{2}\int_\Omega p\Big(\Big(|\underline{u}_\varepsilon|^{p-1} - \varphi_1^{(p-1)/p}(x_j)\Big) V_{\varepsilon,x_j}^2 - \int_\Omega F_\varepsilon(y, V_{\varepsilon,x_j}) \\ =& \frac{1}{2}\int_\Omega \tilde{f}(x_j, W_{\varepsilon,x_j,\varphi_1(x_j)}) V_{\varepsilon,x_j} - \int_\Omega \tilde{F}(y, V_{\varepsilon,x_j}) + O(\varepsilon^{N-m+2}) \\ =& \frac{1}{2}\int_\Omega \tilde{f}(x_j, W_{\varepsilon,x_j,\varphi_1(x_j)}) W_{\varepsilon,x_j,\varphi_1(x_j)} - \int_\Omega \tilde{F}(x_j, W_{\varepsilon,x_j,\varphi_1(x_j)}) + O(\varepsilon^{N-m+2}). \end{aligned}$$

Using (1.5), we find

$$\begin{aligned}
&\int_\Omega \tilde f(x_j,,W_{\varepsilon,x_j,\varphi_1(x_j)})W_{\varepsilon,x_j,\varphi_1(x_j)}\\
=&\varepsilon^{N-m+1}\varphi_1^{1+\frac{1}{p}-(N-m+1)\frac{p-1}{2p}}\int_{D_{\varepsilon,x_j}}(\varepsilon z_1+x_{j,1})^{m-1}\big(|U-1|^p-1+pU\big)U\\
=&\varepsilon^{N-m+1}x_{j,1}^{m-1}\varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{N-m+1}{2})}(x_j)\int_{R^{N-m+1}}\big(|U-1|^p-1+pU\big)U+O\big(\varepsilon^{N-m+2}\big)\\
=&\varepsilon^{N-m+1}x_{j,1}^{m-1}\varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{N-m+1}{2})}(x_j)\int_{R^{N-m+1}}f(U)+O\big(\varepsilon^{N-m+2}\big),
\end{aligned}$$

where $D_{\varepsilon,x_j}=\big\{z:\varepsilon z+x_j\in D\big\}$, and

$$f(t)=|t-1|^p-1+pt.$$

Similarly,

$$\begin{aligned}
&\int_\Omega \tilde F(x_j,W_{\varepsilon,x_j,\varphi_1(x_j)})\\
&=\varepsilon^{N-m+1}x_{j,1}^{m-1}\varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{N-m+1}{2})}(x_j)\int_{R^{N-m+1}}F(U)+O\big(\varepsilon^{N-m+2}\big),
\end{aligned}$$

where $F(t)=\int_0^t f(s)\,ds$.

So we have proved

$$I(V_{\varepsilon,x_j})=\varepsilon^{N-m+1}Ax_{j,1}^{m-1}\varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{N-m+1}{2})}(x_j)+\varepsilon^{N-m+1}O(\varepsilon),$$

where

$$A=\frac{1}{2}\int_{R^{N-m+1}}f(U)U-\int_{R^{N-m+1}}F(U)>0.$$

Here, we have used $f(t)\geq 0$ and the Pohozaev identity

$$\frac{N-2}{2}\int_{R^{N-m+1}}f(U)U=N\int_{R^{N-m+1}}F(u).$$

□

Proposition B.2. *For any positive integer k, we have*

$$\begin{aligned}
I(\sum_{j=1}^k V_{\varepsilon,x_j})=&\varepsilon^{N-m+1}A\sum_{j=1}^k x_{j,1}^{m-1}\varphi_1^{(1-\frac{1}{p})(\frac{p+1}{p-1}-\frac{N-m+1}{2})}(x_j)\\
&-\varepsilon^{N-m+1}\sum_{i\neq j}\big(c(x_i)+o(1)\big)V_{\varepsilon,x_j}(x_i)\\
&+\varepsilon^{N-m+1}O\Big(\varepsilon+\sum_{i\neq j}V_{\varepsilon,x_j}^{1+\sigma}(x_i)\Big),
\end{aligned}$$

where $\sigma>0$ is some constant, $o(1)\to 0$ as $\varepsilon\to 0$, and

$$c(x_i)=\frac{1}{2}x_{i,1}^{m-1}\int_{R^{N-m+1}}\tilde f\big(x_i,U_{\varphi_1(x_i)}\big)\geq c'>0.$$

Proof. We have

$$
\begin{aligned}
I(\sum_{j=1}^{k} V_{\varepsilon,x_j}) = & \sum_{j=1}^{k} I(V_{\varepsilon,x_j}) + \frac{1}{2}\sum_{i\neq j}\int_\Omega \big(\varepsilon^2 DV_{\varepsilon,x_i} DV_{\varepsilon,x_j} + p|\underline{u}_\varepsilon|^{p-1} V_{\varepsilon,x_i} V_{\varepsilon,x_j}\big) \\
& - \int_\Omega \Big(F_\varepsilon\big(y, \sum_{j=1}^{k} V_{\varepsilon,x_j}\big) - \sum_{j=1}^{k} F_\varepsilon\big(y, V_{\varepsilon,x_j}\big)\Big).
\end{aligned}
\tag{B.1}
$$

On the other hand,

$$
\begin{aligned}
& \frac{1}{2}\sum_{i\neq j}\int_\Omega \big(\varepsilon^2 DV_{\varepsilon,x_i} DV_{\varepsilon,x_j} + p|\underline{u}_\varepsilon|^{p-1} V_{\varepsilon,x_i} V_{\varepsilon,x_j}\big) \\
= & \frac{1}{2}\sum_{i\neq j}\int_\Omega \tilde f(x_i, W_{\varepsilon,x_i,\varphi_1(x_i)}) V_{\varepsilon,x_j} + \frac{1}{2}\sum_{i\neq j}\int_\Omega p\big(|\underline{u}_\varepsilon|^{p-1} - \varphi_1^{(p-1)/p}(x_i)\big) V_{\varepsilon,x_i} V_{\varepsilon,x_j} \\
& + O\Big(\int_\Omega \big(\varepsilon + |\xi(|y'|) - 1\big) W_{\varepsilon,x_i,\varphi_1(x_i)} V_{\varepsilon,x_j}\Big) \\
= & \frac{1}{2}\sum_{i\neq j}\int_\Omega \tilde f(x_i, V_{\varepsilon,x_i}) V_{\varepsilon,x_j} + \varepsilon^{N-m+1} O\Big(\varepsilon + \sum_{i\neq j} V_{\varepsilon,x_j}^{1+\sigma}(x_i)\Big),
\end{aligned}
\tag{B.2}
$$

and

$$
\begin{aligned}
& \int_\Omega \Big(F_\varepsilon\big(y, \sum_{j=1}^{k} V_{\varepsilon,x_j}\big) - \sum_{j=1}^{k} F_\varepsilon\big(y, V_{\varepsilon,x_j}\big)\Big) \\
= & \int_\Omega \Big(F_\varepsilon\big(y, \sum_{j=1}^{k} V_{\varepsilon,x_j}\big) - \sum_{j=1}^{k} F_\varepsilon\big(y, V_{\varepsilon,x_j}\big) - \sum_{i\neq j} f_\varepsilon(y, V_{\varepsilon,x_i}) V_{\varepsilon,x_j}\Big) \\
& + \sum_{i\neq j}\int_\Omega f_\varepsilon(y, V_{\varepsilon,x_i}) V_{\varepsilon,x_j} \\
= & \sum_{i\neq j}\int_\Omega \tilde f(x_i, V_{\varepsilon,x_i}) V_{\varepsilon,x_j} + \varepsilon^{N-m+1} O\Big(\varepsilon + \sum_{i\neq j} V_{\varepsilon,x_j}^{1+\sigma}(x_i)\Big).
\end{aligned}
\tag{B.3}
$$

Combining (B.1), (B.2) and (B.3), we are led to

$$
\begin{aligned}
& I(\sum_{j=1}^{k} V_{\varepsilon,x_j}) - \sum_{j=1}^{k} I(V_{\varepsilon,x_j}) \\
= & -\frac{1}{2}\sum_{i\neq j}\int_\Omega \tilde f(x_i, V_{\varepsilon,x_i}) V_{\varepsilon,x_j} + \varepsilon^{N-m+1} O\Big(\varepsilon + \sum_{i\neq j} V_{\varepsilon,x_j}^{1+\sigma}(x_i)\Big) \\
= & -\varepsilon^{N-m+1}\sum_{i\neq j}\big(c(x_i) + o(1)\big) V_{\varepsilon,x_j}(x_i) + \varepsilon^{N-m+1} O\Big(\varepsilon + \sum_{i\neq j} V_{\varepsilon,x_j}^{1+\sigma}(x_i)\Big),
\end{aligned}
\tag{B.4}
$$

where

$$
c(x_i) = \frac{1}{2} x_{i,1}^{m-1} \int_{R^{N-m+1}} \tilde f\big(x_i, U_{\varphi_1(x_i)}\big).
$$

Since

$$\tilde{f}(x_i,t) = |t - \varphi_1^{1/p}(x_i)|^p - \varphi(x_i) - \varphi_1^{(p-1)/p}(x_i)t,$$

and

$$-\Delta U_{\varphi(x_i)} = |U_{\varphi(x_i)} - \varphi_1^{1/p}(x_i)|^p - \varphi(x_i),$$

we see

$$c(x_i) = \frac{1}{2} x_{i,1}^{m-1} \int_{R^{N-m+1}} \varphi_1^{(p-1)/p}(x_i) U_{\varphi(x_i)} \geq c' > 0.$$

Thus, the result follows from Lemma B.1 and (B.4). ☐

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