

ON A METHOD OF HOLOMORPHIC FUNCTIONS TO OBTAIN SHARP REGULARIZATION RATES OF WEAK SOLUTIONS OF NAVIER-STOKES EQUATIONS*

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Abstract. We show that L^2 energy estimates combined with Cauchy integral formula for holomorphic functions can provide bounds for higher-order derivatives of smooth solutions of Navier-Stokes equations. We then extend this principle to weak solutions to improve regularization rates obtained by standard energy methods.

Key words. Compressible Navier-Stokes equations, weak solutions, time analyticity, holomorphic functions.

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1. Introduction. We show that there is a weak solution of the Navier-Stokes equations

$$(1.1) \quad v_t - u_x = 0,$$

$$(1.2) \quad u_t + p_x = \left(\varepsilon \frac{u_x}{v} \right)_x,$$

$$(1.3) \quad \left(e + \frac{1}{2}u^2 \right)_t + (up)_x = \left(\kappa \frac{e_x}{v} + \varepsilon \frac{uu_x}{v} \right)_x$$

with optimal regularization rates. Here $t \geq 0$ is time, $x \in \mathbb{R}$ is the spatial coordinate, v is the specific volume, e is the internal energy, positive constants ε, κ are the viscosity coefficient and the heat conductivity. We assume that the fluid is ideal, so that $P(v, e) = (\gamma - 1) \frac{e}{v}$, where $\gamma > 1$ is the adiabatic constant.

The questions addressed in the paper are motivated by the fact that regularization rates play a critical role in proving uniqueness and continuous dependence on initial data of weak solutions of Navier-Stokes equations. Although the question of uniqueness was resolved in Hoff & Tsyganov [2], the issue of small-time behavior of weak solutions remained. We would also like to point out that there exist other methods of proving uniqueness without knowing regularization rates (see Zlotnik & Amosov [6] and Jiang & Zlotnik [4]).

It is known that solutions of the heat equation $u_t = u_{xx}$, $u(\cdot, 0) = u_0(\cdot) \in L^2(\mathbb{R})$ become smooth for $t > 0$ and satisfy the inequalities

$$t \int_{\mathbb{R}} u_x^2(x, t) dx + t^2 \int_{\mathbb{R}} u_t^2(x, t) dx \leq \int_{\mathbb{R}} u_0^2(x) dx, \quad 0 \leq t \leq 1,$$

and the regularization rates $1/t$ for $\|u_x(\cdot, t)\|_{L^2(\mathbb{R})}^2$ and $1/t^2$ for $\|u_t(\cdot, t)\|_{L^2(\mathbb{R})}^2$ are considered to be sharp. A similar parabolic smoothing mechanism is also present in Navier-Stokes equations. As can be seen from the existence result of Chen, Hoff, Trivisa [1], u_x, e_x, u_t, e_t are in $L^2(\mathbb{R})$ for $t > 0$, but their rates of regularization are not optimal. The present paper therefore is the first result presenting an efficient method of obtaining sharp estimates for hyperbolic-parabolic differential equations with high-order nonlinear terms.

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Time analyticity of weak solutions of compressible Navier-Stokes equations, which was established in Tsyganov [5] and Hoff & Tsyganov [3], finds several important applications. In the current paper we present a method of obtaining higher-order estimates from L^2 bounds for complex values of t by means of Cauchy integral formula for holomorphic functions. This new approach allows us to overcome difficulties arising in the standard analysis.

We now give a precise formulation of our results. We say that (v, u, e) is a weak solution of (1.1)–(1.3) on $[0, T]$ if for every test function $\varphi, \psi \in \mathcal{D}(\mathbb{R} \times (-\infty, \infty))$ and all times $0 \leq t_1 < t_2 \leq T$,

$$\begin{aligned} & \int_{\mathbb{R}} v\varphi dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\mathbb{R}} [v\varphi_t - u\varphi_x] dx dt = 0, \\ & \int_{\mathbb{R}} u\psi dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\mathbb{R}} [u\psi_t + p\psi_x] dx dt = - \int_{t_1}^{t_2} \int_{\mathbb{R}} \varepsilon \frac{u_x}{v} \psi_x dx dt, \\ & \int_{\mathbb{R}} (e + \frac{1}{2}u^2)\chi dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\mathbb{R}} (e + \frac{1}{2}u^2)\chi_t dx dt - \int_{t_1}^{t_2} \int_{\mathbb{R}} up\chi_x dx dt \\ & \qquad = - \int_{t_1}^{t_2} \int_{\mathbb{R}} \left(\kappa \frac{e_x}{v} + \varepsilon \frac{uu_x}{v} \right) \chi_x dx dt. \end{aligned}$$

The following theorem contains our main result:

THEOREM 1.1. *Let $\tilde{v}, \tilde{e}, \underline{e}, M, \mu$ be positive constants such that $M^{-1} - \mu, \underline{e} - \mu > 0$. Then given initial data (v_0, u_0, e_0) for which*

$$(1.4) \quad C_0 \equiv \int_{-\infty}^{\infty} (v_0(x) - \tilde{v})^2 dx + \int_{-\infty}^{\infty} u_0^2(x) dx + \int_{-\infty}^{\infty} (e_0(x) - \tilde{e})^2 dx + \int_{-\infty}^{\infty} u_0^4(x) dx < \infty$$

and

$$(1.5) \quad v_0 \in [M^{-1}, M] \text{ a.e.}, \quad e_0 \in [\underline{e}, \infty) \text{ a.e.},$$

there is a positive T depending on $\tilde{v}, \tilde{e}, \underline{e}, \mu, M, C_0$, a positive constant C depending on $\tilde{v}, \tilde{e}, \underline{e}, \mu, M$, and a local weak solution (v, u, e) of (1.1)–(1.3) on $[0, T]$, satisfying

$$(1.6) \quad v \in [M^{-1} - \mu, M + \mu] \text{ a.e.}, \quad e \in [\underline{e} - \mu, \infty),$$

$$(1.7) \quad \sup_{t \in [0, T]} \left[\int_{\mathbb{R}} [(v(x, t) - \tilde{v})^2 + u^2(x, t) dx + (e(x, t) - \tilde{e})^2 + u^4(x, t)] dx \right] + \int_0^T \int_{\mathbb{R}} [u_x^2(x, s) + e_x^2(x, s) + u^2(x, s)u_x^2(x, s)] dx ds \leq CC_0,$$

$$(1.8) \quad \sup_{t \in [0, T]} \left[t \int_{\mathbb{R}} [u_x^2(x, t) + e_x^2(x, t) + u^2(x, t)u_x^2(x, t)] dx + t^2 \int_{\mathbb{R}} [u_t^2(x, t) + e_t^2(x, t) + u^2(x, t)u_t^2(x, t)] dx \right] \leq CC_0.$$

We now give a brief overview of the analysis presented in the paper. The weak solution (v, u, e) will be obtained as a limit of smooth solutions of (1.1)–(1.3) for mollified initial data. The main idea lies in a sequence of *a priori* bounds, which are connected as follows:

- I. We construct an open set $\mathcal{D} \subset \mathbb{C}$ with specific convexity properties, then we apply standard existence results to find a complex smooth solution of (1.1) – (1.3) for $t \in \mathcal{D}$;
- II. Assuming that v satisfies pointwise estimates (1.6) in \mathcal{D} , we show via complexification of equations (1.1) – (1.3) that $\|v(\cdot, t) - \tilde{v}\|_{L^2(\mathbb{R})}$, $\|u(\cdot, t)\|_{L^2(\mathbb{R})}$, $\|e(\cdot, t) - \tilde{e}\|_{L^2(\mathbb{R})}$, $\|u^2(\cdot, t)\|_{L^2(\mathbb{R})}$, as well as line integrals from 0 to t of $\|u_x(\cdot, s)\|_{L^2(\mathbb{R})}^2$, $\|e_x(\cdot, s)\|_{L^2(\mathbb{R})}^2$, $\|u(\cdot, s)u_x(\cdot, s)\|_{L^2(\mathbb{R})}^2$ are bounded by CC_0 for any $t \in \mathcal{D}$;
- III. We assume that the estimates in II hold and exploit convexity of \mathcal{D} to obtain bounds for $\|u_t(\cdot, t)\|_{L^2(\mathbb{R})}$, $\|e_t(\cdot, t)\|_{L^2(\mathbb{R})}$, $\|u(\cdot, t)u_t(\cdot, t)\|_{L^2(\mathbb{R})}$ and $\|u_x(\cdot, t)\|_{L^2(\mathbb{R})}$, $\|e_x(\cdot, t)\|_{L^2(\mathbb{R})}$, $\|u(\cdot, t)u_x(\cdot, t)\|_{L^2(\mathbb{R})}$ directly from Cauchy integral formula for holomorphic functions;
- IV. We derive pointwise bounds for v from the estimates in III and

$$\text{the inequality } |v(x, t) - v_0(x)| \leq C \int_0^t \|u_x(\cdot, s)\|_{L^\infty(\mathbb{R})}.$$

2. Construction of a Special Domain and L^2 Bounds. In this section we construct a special domain in \mathbb{C} and obtain *a priori* L^2 estimates for time values from this set.

We begin with construction of a specific domain in \mathbb{C} whose properties will play a key role in the analysis.

LEMMA 2.1. *For any $n > 0$ there are positive constants $C, \bar{\theta}, b$, and a convex open set \mathcal{D} in \mathbb{C} , all as small as necessary, such that*

- 1) $\mathcal{D} \subset \{z : |\arg(z)| < \bar{\theta}, \operatorname{Re}(z) > 0, |z| \leq b\}$;
- 2) \mathcal{D} is symmetric with respect to $\{z : \operatorname{Im}(z) = 0\}$;
- 3) $0 \in \partial\mathcal{D}$;
- 4) for any $z \in \mathcal{D}$ there are distinct $z_1, z_2 \in \mathcal{D}$ such that

$$|z_1| = |z_1 - z| = |z_2| = |z_2 - z|,$$

$$|z| \leq 2^{n+1} \sin^n(\angle z_1 0 z),$$

where $\angle z_1 0 z$ is the angle between two lines $0z_1$ and $0z$.

Proof. We fix an arbitrary positive $\lambda < 1$ and locate a point $A_1 = \lambda$ on the $\operatorname{Re}(z)$ axis. Let A_2 be the point with the following properties:

$$|A_2 - A_1| = |A_2|, \quad \sin^n(\angle A_2 0 A_1) = \lambda.$$

Assume now that we have constructed $k - 1$ vertices A_1, \dots, A_{k-1} . Then the $k - th$ vertex A_k satisfies

$$|A_k - A_{k-1}| = |A_k|, \quad \sin^n(\angle A_k 0 A_{k-1}) = |A_{k-1}|.$$

In addition, let A_k^* be the reflection of A_k with respect to the $Re(z)$ axis. We then define \mathcal{D} as an open set bounded by the union of line segments $A_k A_{k-1}, A_k^* A_{k-1}^*, k = 1, 2, \dots$, and $z = 0$.

First we show that $\sup_k (Arg(A_k)) \rightarrow 0$ when $\lambda \rightarrow 0$. We have for $\lambda \ll 1$ that

$$\begin{aligned} Arg(A_k) &= \sum_{m=2}^k (\angle A_m 0 A_{m-1}) = \sum_{m=2}^k \arcsin(|A_{m-1}|^{1/n}) \\ &< 2 \sum_{m=1}^k |A_m|^{1/n} < 2 \sum_{m=0}^{\infty} \left(\lambda \left(\frac{2}{3}\right)^m\right)^{1/n} = \lambda^{1/n} \frac{2}{1 - (\frac{2}{3})^{1/n}}, \end{aligned}$$

so that \mathcal{D} will satisfy condition 1) if λ is small enough. Next we take an arbitrary $z \in \mathcal{D}$. Due to symmetry of \mathcal{D} we can assume that $Im(z) > 0$. Then z belongs to a certain triangle $\triangle O A_{k-1} A_k$, where O is the origin. The absolute value $|z|$ is less than $|A_{k-1}|$ and there is $z_1 \in \triangle O A_k A_{k+1}$ such that $|z_1| = |z_1 - z|$ and $\angle z_1 0 z > \frac{1}{2} \angle A_{k+1} 0 A_k$. Therefore, $|z| < 2^{n+1} \sin^n(\angle z_1 0 z)$. \square

LEMMA 2.2. *There exists a positive $\tau < 1$ depending on the same quantities as C in the statement of Theorem 1.1 such that if \mathcal{D} is an open set with properties 1)–4) of Lemma 1.1 satisfying*

$$\mathcal{D} \subset \{z : |\arg(z)| \leq \tau, |z| \leq \tau\},$$

and if (v, u, e) is any smooth solution of (1.1)–(1.3) satisfying the following conditions:

$v(x, t), u(x, t), e(x, t)$ are holomorphic in \mathcal{D} with respect to t
for any fixed $x \in \mathbb{R}$;
for any $\theta, |\theta| = 1, |\arg(\theta)| \leq \tau$, smooth functions
 $(v(x, \theta s), u(x, \theta s), e(x, \theta s))$ of (x, s) solve

$$(2.1) \quad v_s - \bar{\theta} u_x = 0,$$

$$(2.2) \quad u_s + \bar{\theta} p_x = \bar{\theta} \left(\varepsilon \frac{u_x}{v}\right)_x,$$

$$(2.3) \quad \left(e + \frac{1}{2} u^2\right)_s + \bar{\theta} (up)_x = \bar{\theta} \left(\kappa \frac{e_x}{v} + \varepsilon \frac{uu_x}{v}\right)_x$$

at any point (x, s) for which $\theta s \in \mathcal{D}$;

$$(2.4) \quad M^{-1} - \mu \leq Re(v(x, t)) \leq M + \mu, \quad |Im(v(x, t))| \leq \mu, \quad t \in \mathcal{D},$$

then

$$(2.5) \quad \int [|v(x, \theta s) - \tilde{v}|^2 + |u(x, \theta s)|^2 + |e(x, \theta s) - \tilde{e}|^2 + |u(x, \theta s)|^4] dx + \int_0^T \int_{\mathbb{R}} [|u_x(x, \theta s)|^2 + |e_x(x, \theta s)|^2 + |u(x, \theta s) u_x(x, \theta s)|^2] dx ds \leq CC_0,$$

as long as $\theta s \in \mathcal{D}$. Here C is a positive constant as described in Theorem 1.1.

Proof. We multiply (2.1) by $\bar{v} - \tilde{v}$, (2.2) by \bar{u} , (2.3) by $\bar{e} - \tilde{e}$, add the resulting equations and integrate. Then we add the equation to its conjugate to arrive at

$$\begin{aligned}
 (2.6) \quad & \int \left[|v(x, s) - \tilde{v}|^2 + |u(x, s)|^2 + |e(x, s) - \tilde{e}|^2 \right] dx \Big|_{s=0}^{s=t} \\
 & + \int_0^t \int \left[\varepsilon \left(\frac{\bar{\theta}}{v} + \frac{\theta}{\bar{v}} \right) |u_x|^2 + \kappa \left(\frac{\bar{\theta}}{v} + \frac{\theta}{\bar{v}} \right) |e_x|^2 \right] dx ds \\
 = & - \int_0^t \int \left[u u_t (\bar{e} - \tilde{e}) + \bar{u} \bar{u}_t (e - \tilde{e}) + \varepsilon \theta \frac{u u_x \bar{e}_x}{v} + \varepsilon \bar{\theta} \frac{\bar{u} \bar{u}_x e_x}{\bar{v}} \right] dx ds \\
 & - \int_0^t \int \left[\bar{\theta} u_x (\bar{v} - \tilde{v}) + \theta \bar{u}_x (v - \tilde{v}) - \bar{\theta} (\bar{p} - \tilde{p}) u_x - \theta (p - \tilde{p}) \bar{u}_x \right] dx ds \\
 & + \int_0^t \int \left[\bar{\theta} u p \bar{e}_x + \theta \bar{u} \bar{p} e_x \right] dx ds.
 \end{aligned}$$

We then multiply (2.2) by $u \bar{u}^2$, integrate, and add to its conjugate to obtain

$$\begin{aligned}
 (2.7) \quad & \int |u(x, s)|^4 dx \Big|_{s=0}^{s=t} + \int_0^t \int 2\varepsilon \left(\frac{\bar{\theta}}{v} + \frac{\theta}{\bar{v}} \right) |u|^2 |u_x|^2 dx ds \\
 & = \int_0^t \int \left[\varepsilon \bar{\theta} \frac{u^2 \bar{u}_x^2}{v} + \varepsilon \theta \frac{\bar{u}^2 u_x^2}{\bar{v}} \right] dx ds \\
 & + \int_0^t \int \left[\bar{\theta} (p - \tilde{p}) u_x \bar{u}^2 + 2\bar{\theta} (p - \tilde{p}) u \bar{u} \bar{u}_x \right] dx ds \\
 & + \int_0^t \int \left[\theta (\bar{p} - \tilde{p}) \bar{u}_x u^2 + 2\theta (\bar{p} - \tilde{p}) \bar{u} u u_x \right] dx ds.
 \end{aligned}$$

We also multiply (2.3) by $\frac{1}{2} \bar{u}^2$ and integrate. Then we add the equation to its conjugate to arrive at

$$\begin{aligned}
 (2.8) \quad & \frac{1}{4} \int |u(x, s)|^4 dx \Big|_0^t + \frac{1}{2} \int_0^t \int \left[e_t \bar{u}^2 + \bar{e}_t u^2 \right] dx ds \\
 & = - \int_0^t \int \left[\kappa \bar{\theta} \frac{e_x \bar{u} \bar{u}_x}{v} + \kappa \theta \frac{\bar{e}_x u u_x}{\bar{v}} + \varepsilon \left(\frac{\bar{\theta}}{v} + \frac{\theta}{\bar{v}} \right) |u|^2 |u_x|^2 \right] dx ds \\
 & + \int_0^t \int \left[\bar{\theta} u p \bar{u} \bar{u}_x + \theta \bar{u} \bar{p} u u_x \right] dx ds.
 \end{aligned}$$

For a smooth solution (v, u, e) of (2.1)–(2.3) we define

$$\begin{aligned}
 (2.9) \quad A(t) = & \sup_{0 \leq s \leq t} \int \left[|v(x, \theta s) - \tilde{v}|^2 + |u(x, \theta s)|^2 + |e(x, \theta s) - \tilde{e}|^2 + |u(x, \theta s)|^4 \right] dx \\
 & + \int_0^t \int \left[|u_x(x, \theta s)|^2 + |e_x(x, \theta s)|^2 + |u(x, \theta s)|^2 |u_x(x, \theta s)|^2 \right] dx ds.
 \end{aligned}$$

We now add (2.6), (2.7), and (2.8) multiplied by a suitable constant, apply Cauchy-Schwarz inequality in an elementary way to obtain

$$(2.10) \quad A(t) \leq CC_0 + C \int_0^t \int |u|^2 |e - \tilde{e}|^2 dx ds.$$

The double integral on the right-hand side of (2.10) can be bounded by

$$\begin{aligned} & \sup_{0 \leq s \leq t} \left(\int |u(x, \theta s)|^4 dx \right)^{1/2} \left(\int_0^t \int |e(x, \theta s) - \tilde{e}|^4 dx ds \right)^{1/2} \\ & \leq CA^{1/2} \int_0^t \left(\int |e - \tilde{e}|^2 dx \right) \left(\int |e_x|^2 dx \right)^{1/2} ds \leq CC_0 t^{1/2} A(t). \end{aligned}$$

This together with (2.10) proves the lemma. \square

3. Higher-Order Estimates. In this section we obtain *a priori* higher-order estimates for smooth solutions utilizing only L^2 estimates and properties of \mathcal{D} .

LEMMA 3.1. *Let \mathcal{D} be an open set and (v, u, e) a smooth solution of (1.1)–(1.3) as described in the statement of Lemma 2.2. Then there is a positive constant C depending on the same quantities as C in the conclusion of Theorem 1.1 such that*

$$(3.1) \quad \left(\int [|u_t(x, \theta s)|^2 + |e_t(x, \theta s)|^2] dx \right)^{1/2} \leq CC_0 \left(\frac{1}{s} + \frac{1}{T-s} \right) T^{-1/2n},$$

where θ and $T = T(\theta)$ are such that $|\theta| = 1$ and $T = \sup\{s : \theta s \in \mathcal{D}\} > 0$.

Proof. First we fix θ . Property 4) of \mathcal{D} will guarantee that there exists $\beta > 0$ such that $C \text{Sin}^n(\beta) > T$ and if $s \in (0, T)$, then either the disk $B(\theta s, s\beta)$ centered at θs with radius $s\beta$ or the disk $B(\theta s, (T-s)\beta)$ centered at θs with radius $(T-s)\beta$ lies in \mathcal{D} . Then from Cauchy integral formula

$$u_t(x, \theta s) = \frac{1}{2\pi i} \int_{\partial B(\theta s, \min(s\beta, (T-s)\beta))} \frac{u(x, \xi)}{(\theta s - \xi)^2} d\xi$$

we can derive that

$$\begin{aligned} (3.2) \quad \int |u_t(x, \theta s)|^2 dx & \leq C \left(\frac{1}{(s\beta)^2} + \frac{1}{((T-s)\beta)^2} \right) \sup_{t \in \mathcal{D}} \int |u(x, t)|^2 dx \\ & \leq CC_0 \left(\frac{1}{s^2 \text{Sin}^2(\beta)} + \frac{1}{(T-s)^2 \text{Sin}^2(\beta)} \right) \\ & \leq CC_0 \left(\frac{1}{s^2} + \frac{1}{(T-s)^2} \right) T^{-1/n}. \end{aligned}$$

We can apply a similar argument to estimate $\int |e_t(x, \theta s)|^2 dx$ and thus prove (3.1). \square

LEMMA 3.2. *Let \mathcal{D} and (v, u, e) be as in the statement in Lemma 2.2. Then there is a positive constant C depending on the same quantities as C in the conclusion of Theorem 1.1 such that*

$$\begin{aligned} (3.3) \quad \int [|u_x(x, \theta s)|^2 + |e_x(x, \theta s)|^2 + |u(x, \theta s)|^2 |u_x(x, \theta s)|^2] dx \\ \leq CC_0 \left(\frac{1}{s} + \frac{1}{T-s} \right) T^{-1/n}. \end{aligned}$$

Proof. We fix θ and find β as in the proof of Lemma 3.1. Then from Cauchy integral formula we have that

$$(3.4) \quad u_x^2(x, \theta s) dx = \frac{1}{2\pi i} \int_{\partial B(\theta s, \min(s\beta, (T-s)\beta))} \frac{\int_0^\xi u_x^2(x, \zeta) d\zeta}{(\theta s - \xi)^2} d\xi,$$

where $\int_0^\xi u_x^2(x, \zeta) d\zeta$ is a line integral from 0 to ξ . It immediately follows from (3.4) and properties of β that

$$\begin{aligned} \int |u_x(x, \theta s)|^2 dx &\leq C \sup_t \left[\int_0^t \int |u_x(x, \zeta)|^2 dx d\zeta \right] \left(\frac{1}{s} + \frac{1}{T-s} \right) T^{-1/n} \\ &\leq CC_0 \left(\frac{1}{s} + \frac{1}{T-s} \right) T^{-1/n}. \end{aligned}$$

The same argument can be used to prove the estimates for $\int |e_x(x, \theta s)|^2 dx$ and $\int |u(x, \theta s)|^2 |u_x(x, \theta s)|^2 dx$. \square

LEMMA 3.3. *Let \mathcal{D} and (v, u, e) be as in the statement of Lemma 2.2 and τ as in the conclusion of Lemma 2.2. Then*

$$(3.5) \quad M^{-1} - \mu/2 \leq \operatorname{Re}(v(x, t)) \leq M + \mu/2, \quad |\operatorname{Im}(v(x, t))| \leq \mu/2, \quad t \in \mathcal{D}$$

if τ is small enough.

Proof. We divide (2.1) by v and integrate with respect to s to obtain

$$\begin{aligned} |\ln(v(x, \theta s) - \ln(v_0(x)))| &= \left| \bar{\theta} \int_0^t \frac{u_x}{v}(x, \theta s) ds \right| \\ &\leq C \int_0^t \left\| \left(\frac{u_x}{v} - p + \tilde{p} \right)(\cdot, \theta s) \right\|_{L^\infty} ds + C \int_0^t \left\| (p - \tilde{p})(\cdot, \theta s) \right\|_{L^\infty} ds \\ &\leq C \int_0^t \left(\int \left| \frac{u_x}{v} - p + \tilde{p} \right|^2 dx \right)^{1/4} \left(\int |u_t|^2 dx \right)^{1/4} ds \\ &\quad + CC_0^{1/2} T^{-1/4n} \int_0^t \left(\frac{1}{s^{1/4}} + \frac{1}{(T-s)^{1/4}} \right) ds \\ &\leq CC_0^{1/2} T^{-1/2n} \int_0^t \left(\frac{1}{s^{3/4}} + \frac{1}{(T-s)^{3/4}} \right) ds + CC_0^{1/2} T^{3/4-1/4n} \\ &\leq CC_0^{1/2} T^{1/2-1/2n}, \end{aligned}$$

so that the right-hand side of the inequality is as small as necessary if we choose τ small enough. \square

4. Existence. In this section we apply *a priori* bounds of Sections 2,3 to obtain the existence of weak solution (v, u, e) .

Let (v_0, u_0, e_0) be initial data as described in Theorem 1.1. We define $v_0^\delta = j_\delta * v_0$, $u_0^\delta = j_\delta * u_0$, $e_0^\delta = j_\delta * e_0$, where j_δ is the standard mollifier. In the following lemma we state local existence for smooth initial data.

LEMMA 4.1. *There exists $\tilde{\theta} \in \mathbb{C}$, $|\tilde{\theta}| = 1$, $\operatorname{Re}(\tilde{\theta}) > 0$ depending on $\tilde{v}, \tilde{e}, \underline{e}, \mu, M$, a positive T depending on $\tilde{v}, \tilde{e}, \underline{e}, M, \mu, \delta, C_0$, and a positive constant C depending on $\tilde{v}, \tilde{e}, \underline{e}, \mu, M$ such that there is a smooth solution $(v^\delta, u^\delta, e^\delta)$ satisfying the following*

regularity conditions:

- (4.1) $v^\delta(x, \cdot), u^\delta(x, \cdot), e^\delta(x, \cdot)$ are holomorphic in
 $\mathcal{P} = \{z : |z| \leq T, |\arg(z)| \leq \tilde{\theta}\}$ for any $x \in \mathbb{R}$;
- (4.2) $(v^\delta(x, \theta s), u^\delta(x, \theta s), e^\delta(x, \theta s))$ as functions of (x, s)
satisfy (2.1)–(2.3) for any $\theta \in \mathbb{C}, |\theta| = 1, |\arg(\theta)| \leq \arg(\tilde{\theta})$;
- (4.3) $M^{-1} - \mu \leq |v(x, t)| \leq M + \mu, |\operatorname{Im}(v(x, t))| \leq \mu$
for any $t \in \mathbb{C}, |t| \leq T, |\arg(t)| \leq \arg(\tilde{\theta})$;
- (4.4) $\|v^\delta(\cdot, t) - \tilde{v}\|_{H^3(\mathbb{R})} + \|u^\delta(\cdot, t)\|_{H^4(\mathbb{R})} + \|e^\delta(\cdot, t) - \tilde{e}\|_{H^4(\mathbb{R})}$
 $\leq C\left(\|v_0^\delta - \tilde{v}\|_{H^3(\mathbb{R})} + \|u_0^\delta\|_{H^4(\mathbb{R})} + \|e_0^\delta - \tilde{e}\|_{H^4(\mathbb{R})}\right)$
for any $t \in \mathbb{C}, |t| \leq T, |\arg(t)| \leq \arg(\tilde{\theta})$.

Proof. The proof is conceptually very close to the one that can be found in Hoff & Tsyganov [3] for local smooth solution of multidimensional Navier-Stokes equations. We omit the details, pointing out that (v, u, e) is obtained as a limit of solutions of semi-discrete difference schemes that approximate equations (2.1)–(2.3) when the mesh parameter goes to zero. \square

LEMMA 4.2. *Let positive constants $\tilde{v}, \tilde{e}, e, M, C_0$ be given. Then there exists an open set \mathcal{D} as described in Lemma 2.1 depending on these parameters and a smooth solution $(v^\delta, u^\delta, e^\delta)$ with initial data $(v_0^\delta, u_0^\delta, e_0^\delta)$ satisfying (1.4) as described in the statement of Lemma 2.2.*

Proof. First we apply Lemma 4.1 to find $\tilde{\theta}, T$, and a smooth solution $(v^\delta, u^\delta, e^\delta)$ in $\mathbb{R} \times \mathcal{P}$. Then we can find an open set $\mathcal{D} \subset \mathcal{P}$ with properties 1)–4) as described in Lemma 2.1 and prove (2.5), (3.1), (3.3), (3.5) for $t \in \mathcal{D}$. We can reapply local existence theorem for smooth initial data to show that $(v^\delta, u^\delta, e^\delta)$ can be extended into a bigger domain $\alpha\mathcal{D}, \alpha > 1$. It can also be shown by standard energy estimates that (4.4) holds as long as (2.4), (2.5), and (4.3) are satisfied, so that we can repeat the entire argument again. \square

In the following lemma we extract a subsequence $\delta_k \rightarrow 0$ for which $(v^{\delta_k}, u^{\delta_k}, e^{\delta_k}) \rightarrow (v, u, e)$ in an appropriate sense.

LEMMA 4.3. *There is a sequence $\delta = \delta_k \rightarrow 0$ and limiting (v, u, e) for which*

- (4.5) $u^\delta(\cdot, t) - u(\cdot, t), e^\delta(\cdot, t) - e(\cdot, t) \rightarrow 0$ in $L^2_{loc}(\mathbb{R}), t \in \mathcal{D}$;
- (4.6) $u^\delta - u, e^\delta - e \rightarrow 0$ uniformly on compact subsets of $\mathbb{R} \times \mathcal{D}$;
- (4.7) $u_x^\delta(\cdot, t) - u_x(\cdot, t), e_x^\delta(\cdot, t) - e_x(\cdot, t) \rightarrow 0$ in $L^2(\mathbb{R}), t \in \mathcal{D}$;
- (4.8) $v^\delta - v \rightarrow 0$ in $L^2_{loc}(\mathbb{R})$,

where \mathcal{D} is as in the statement of Lemma 4.2.

Proof. The compactness results (4.5), (4.6), (4.7) follow immediately from the bounds (2.5), (3.1), (3.3), (3.5). In order to prove (4.8) we need to introduce the effective viscous flux F by

$$F = \varepsilon \frac{u_x}{v} - p(v, e) + p(\tilde{v}, \tilde{e}).$$

From the equalities

$$F_x^\delta = \theta u_t^\delta, \quad F_t^\delta = \varepsilon \frac{u_{tx}^\delta}{v^\delta} - \varepsilon \frac{u_x^{\delta^2}}{v^{\delta^2}} - p_v(v^\delta, e^\delta) u_x^\delta - p_e(v^\delta, u^\delta) e_t^\delta$$

and the estimates (2.5), (3.1), (3.3), (3.5) we can conclude that F is Hölder continuous in x and t so that there is a sequence $\delta = \delta_k \rightarrow 0$ and a limiting F such that

$$F^\delta \rightarrow F \text{ uniformly on compact subsets of } \mathbb{R} \times \mathcal{D}.$$

If we rewrite (2.1) as

$$(\ln(v))_t = \bar{\theta} \frac{1}{\varepsilon} F - \bar{\theta} \frac{1}{\varepsilon} (p - \tilde{p})$$

then

$$\begin{aligned} \int_{-L}^L |v^{\delta_1}(x, \theta t) - v^{\delta_2}(x, \theta t)|^2 dx &\leq C \int_{-L}^L |v_0^{\delta_1}(x) - v_0^{\delta_2}(x)|^2 dx \\ &+ C \int_0^t \int_{-L}^L |v^{\delta_1}(x, \theta s) - v^{\delta_2}(x, \theta s)|^2 dx ds \\ &+ C \int_0^t \int_{-L}^L |e^{\delta_1}(x, \theta s) - e^{\delta_2}(x, \theta s)|^2 dx ds \\ &+ C \int_0^t \int_{-L}^L |F^{\delta_1}(x, \theta s) - F^{\delta_2}(x, \theta s)|^2 dx ds, \end{aligned}$$

for any $L > 0$. Therefore, $v^\delta(\cdot, t)$ is Cauchy in $L^2_{loc}(\mathbb{R})$ for any $t \in \mathcal{D}$. \square

We conclude the proof of Theorem 1.1 by observing that compactness properties (4.5)–(4.8) will guarantee that (v, u, e) is a weak solution of (1.1)–(1.3) satisfying regularity estimates (1.6)–(1.8). The proof of the lower bound for e in (1.6) is standard and can be found in Wong [6].

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