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LOCAL AND GLOBAL EXACT SHOCK RECONSTRUCTION*

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Abstract. By solving the inverse generalized Riemann problem for the quasilinear hyperbolic system of conservation laws, the exact shock reconstruction is realized in both local and global sense.

Key words. Quasilinear hyperbolic system of conservation laws, inverse generalized Riemann problem, exact shock reconstruction

AMS subject classifications. 35L45, 35L60, 35L65, 35R30

1. Introduction. In aviation industry one may be asked to make the design in such a way that the shock produced can be controlled to a given position.

In order to provide a method to solve this kind of shock control problem, as a general observation, we investigate the inverse generalized Riemann problem, the resolution framework of which can be in principle applied to other shock control problems.

Consider the quasilinear hyperbolic system of conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad t \geq 0,\ x \in \mathbb{R}, \tag{1.1}$$

where $u = (u_1, \cdots, u_n)$ is the unknown vector function of t and x, and $f(u) = (f_1(u), \cdots, f_n(u))$ is a given C^2 vector function of u.

Suppose that on the domain under consideration,

(H_1) System (1.1) is strictly hyperbolic, i.e., the matrix $\nabla f(u)$ has n distinct real eigenvalues:

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

(H_2) System (1.1) is genuinely nonlinear in the sense of P.D.Lax: for $i = 1, \cdots, n$,

$$\nabla\lambda_i(u) r_i(u) \equiv 1, \tag{1.3}$$

where $r_i(u)$ stands for a right eigenvector of $\nabla f(u)$, corresponding to $\lambda_i(u)$:

$$\nabla f(u) r_i(u) = \lambda_i(u) r_i(u). \tag{1.4}$$

For the Riemann problem of system (1.1) with the following piecewise constant initial data

$$t = 0: \qquad u = \begin{cases} u_-, & x \leq 0, \\ u_+, & x \geq 0 \end{cases} \tag{1.5}$$

with $|u_+ - u_-|$ small enough, it is well-known ([1]) that there exists a unique self-similar solution with small amplitude:

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$$u = U(\xi), \qquad \xi = \frac{x}{t}, \tag{1.6}$$

composed of $n+1$ constant states and n elementary waves passing through the origin and connecting two neighbouring constant states in succession.

Here, the i-th elementary wave is either the i-th typical shock

$$x = s_i t \tag{1.7}$$

or the i-th centered rarefaction wave

$$\xi = \lambda_i(U(\xi)), \qquad a_i \le \xi = \frac{x}{t} \le b_i. \tag{1.8}$$

In what follows, we rule out the possibility of appearing both centered rarefaction waves and degenerate typical shocks by assuming

(H_3) The self-similar solution $u = U(\frac{x}{t})$ to Riemann problem (1.1) and (1.5) is composed of $n+1$ constant states $\hat{u}^{(0)} = u_-, \hat{u}^{(1)}, \cdots, \hat{u}^{(n-1)}$ and $\hat{u}^{(n)} = u_+$ and n non-degenerate typical shocks $x = s_i t (i = 1, \cdots, n)$ with

$$s_1 < s_2 < \cdots < s_n, \tag{1.9}$$

connecting two neighbouring constant states in succession, namely,

$$u = U(\frac{x}{t}) = \begin{cases} \hat{u}^{(0)} = u_-, & x \le s_1 t, \\ \hat{u}^{(i)}, & s_i t \le x \le s_{i+1} t \quad (i = 1, \cdots, n-1), \\ \hat{u}^{(n)} = u_+, & s_n t \le x. \end{cases} \tag{1.10}$$

Here, for $i = 1, \cdots, n, \quad x = s_i t$ is the i-th non-degenerate typical shock connecting $\hat{u}^{(i-1)}$ and $\hat{u}^{(i)}$, on which we have the Rankine-Hugoniot conditions

$$f(\hat{u}^{(i)}) - f(\hat{u}^{(i-1)}) = s_i(\hat{u}^{(i)} - \hat{u}^{(i-1)}) \tag{1.11}$$

and the entropy condition

$$\begin{cases} \lambda_i(\hat{u}^{(i)}) < s_i < \lambda_i(\hat{u}^{(i-1)}), \\ \lambda_{i-1}(\hat{u}^{(i-1)}) < s_i < \lambda_{i+1}(\hat{u}^{(i)}). \end{cases} \tag{1.12}$$

We now consider the following piecewise smooth initial data as a perturbation of the original piecewise constant initial data (1.5):

$$t = 0: \qquad u = \begin{cases} u_-^0(x), & x \le 0, \\ u_+^0(x), & x \ge 0, \end{cases} \tag{1.13}$$

where $u_-^0(x)$ and $u_+^0(x)$ are C^1 functions on $x \le 0$ and $x \ge 0$ respectively and

$$u_\pm^0(0) = u_\pm. \tag{1.14}$$

Thus, we get a corresponding generalized Riemann problem.

This kind of perturbation is non-trivial from the mathematical point of view. In fact, the generalized Riemann problem is a real nonlinear PDE problem and, in the case of containing shocks, this is a complicated free boundary problem for quasilinear hyperbolic systems.

The existence and uniqueness of local piecewise smooth solution can be given by

PROPOSITION A ([2]). *When $|u_+ - u_-| \ll 1$, under assumptions (H1)–(H3), the generalized Riemann problem (1.1) and (1.13) admits a unique local piecewise C^1 solution*

$$u = u(t,x) = \begin{cases} u^{(0)}(t,x) & x \le x_1(t), \\ u^{(i)}(t,x) & x_i(t) \le x \le x_{i+1}(t) \quad (i = 1, \cdots, n-1), \\ u^{(n)}(t,x) & x_n(t) \le x. \end{cases} \tag{1.15}$$

Here $u^{(i)}(t,x) \in C^1$ satisfies system (1.1) on its angular domain in the classical sense $(i = 0, 1, \cdots, n)$ and

$$u^{(i)}(0,0) = \hat{u}^{(i)} \qquad (i = 0, 1, \cdots, n); \tag{1.16}$$

$x_i(t) \in C^2 (i = 1, \cdots, n)$ and

$$x_i(0) = 0, \qquad x_i'(0) = s_i \quad (i = 1, \cdots, n). \tag{1.17}$$

Moreover, for $i = 1, \cdots, n$, $x = x_i(t)$ is the i-th non-degenerate shock connecting $u^{(i-1)}(t,x)$ and $u^{(i)}(t,x)$, on which we have the Rankine-Hugoniot conditions

$$f(u^{(i)}) - f(u^{(i-1)}) = x_i'(t)(u^{(i)} - u^{(i-1)}) \tag{1.18}$$

and the entropy condition

$$\begin{cases} \lambda_i(u^{(i)}) < x_i'(t) < \lambda_i(u^{(i-1)}), \\ \lambda_{i-1}(u^{(i-1)}) < x_i'(t) < \lambda_{i+1}(u^{(i)}), \end{cases} \tag{1.19}$$

where $u^{(i)} = u^{(i)}(t, x_i(t))$ and $u^{(i-1)} = u^{(i-1)}(t, x_i(t))$.

Obviously, in a neighbourhood of the origin, solution (1.15) possesses a structure completely similar to the self-similar solution (1.10) to Riemann problem (1.1) and (1.5).

Then, Proposition A shows the local structural stability of the self-similar solution (1.10) to Riemann problem (1.1) and (1.5) under the perturbation (1.13) satisfying (1.14).

On the other hand, the existence and uniqueness of global piecewise smooth solution can be given by

PROPOSITION B ([3]). *When $|u_+ - u_-| \ll 1$, under assumptions (H_1)-(H_3), suppose that $u_-^0(x)$ and $u_+^0(x)$ are C^1 functions on $x \le 0$ and $x \ge 0$ respectively. Then there exists a positive constant $\varepsilon_0 > 0$ so small that for any given ε with $0 < \varepsilon \le \varepsilon_0$, if*

$$|u_-^0(x) - u_-^0(0)|, \ |u_-^{0'}(x)| \le \frac{\varepsilon}{1+|x|}, \quad \forall x \le 0 \tag{1.20}$$

$$|u_+^0(x) - u_+^0(0)|, \ |u_+^{0'}(x)| \le \frac{\varepsilon}{1+|x|}, \quad \forall x \ge 0, \tag{1.21}$$

then, the generalized Remann problem (1.1) and (1.13) admits a unique global piecewise C^1 solution on $t \geq 0$:

$$u = u(t,x) = \begin{cases} u^{(0)}(t,x), & x \leq x_1(t), \\ u^{(i)}(t,x), & x_i(t) \leq x \leq x_{i+1}(t) \quad (i = 1, \cdots, n-1), \\ u^{(n)}(t,x), & x \geq x_n(t), \end{cases} \tag{1.22}$$

containing only n non-degenerate shocks $x = x_i(t)$ $(i = 1, \cdots, n)$ passing through the origin, in which $u^{(i)}(t,x) \in C^1$ with

$$u^{(i)}(0,0) = \hat{u}^{(i)} \quad (i = 0, 1, \cdots, n) \tag{1.23}$$

satisfies system (1.1) in the classical sense on the domain D^i $(i = 0, 1, \cdots, n)$ respectively, where

$$\begin{cases} D^0 = \{(t,x) | t \geq 0, \quad x \leq x_1(t)\}, \\ D^i = \{(t,x) | t \geq 0, \quad x_i(t) \leq x \leq x_{i+1}(t)\} \quad (i = 1, \cdots, n-1), \\ D^n = \{(t,x) | t \geq 0, \quad x \geq x_n(t)\} \end{cases} \tag{1.24}$$

and $x_i(t) \in C^2$ on $t \geq 0$ with

$$x_i(0) = 0, \ x_i'(0) = s_i \quad (i = 1, \cdots, n). \tag{1.25}$$

On $x = x_i(t)$ $(i = 1, \cdots, n)$ we have the Rankine-Hugoniot conditions (1.18) and the entropy condition (1.19) Moreover, we have

$$|u^{(i)}(t,x) - \hat{u}^{(i)}| \leq \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in D^i \quad (i = 0, 1, \cdots, n), \tag{1.26}$$

$$|\frac{\partial u^{(i)}(t,x)}{\partial x}|, \ |\frac{\partial u^{(i)}(t,x)}{\partial t}| \leq \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in D^i \quad (i = 0, 1, \cdots, n) \tag{1.27}$$

and

$$|x_i'(t) - x_i'(0)| \leq \frac{K\varepsilon}{1+t}, \quad \forall t \geq 0 \quad (i = 1, \cdots, n), \tag{1.28}$$

$$|x_i''(t)| \leq \frac{K\varepsilon}{1+t}, \quad \forall t \geq 0 \quad (i = 1, \cdots, n), \tag{1.29}$$

where K is a positive constant independent of ε, t and x.

From Proposition B we have the global structural stability of the self-similar solution $u = U(\frac{x}{t})$ to Riemann problem (1.1) and (1.5) under perturbation (1.13) satisfying (1.14).

By Propositions A and B, for any given initial data (1.13) satisfying (1.14), we can solve the corresponding generalized Riemann problem for system (1.1) to determine in a unique manner n non-degenerate shocks $x = x_i(t)$ with $x_i(0) = 0$ and $x_i'(0) = s_i (i = 1, \cdots, n)$ either in a neighbourhood of the origin or on the whole upper plane $t \geq 0$, respectively.

In Sections 2 and 3 of this paper we will consider the following inverse problem: Corresponding to Riemann problem (1.1) and (1.5) with the self-similar solution

(1.10), if we know the position of n non-degenerate shocks in the piecewise C^1 solution to the corresponding generalized Riemann problem (1.1) and (1.13):

$$x = x_i(t) \in C^2 \qquad (i = 1, \cdots, n) \tag{1.30}$$

with

$$x_i(0) = 0 \quad \text{and} \quad x_i'(0) = s_i \quad (i = 1, \cdots, n), \tag{1.31}$$

in which s_i satisfies (1.9) and (1.11)–(1.12) $(i = 1, \cdots, n)$, to what degree can we determine the initial data (1.13) satisfying (1.14)?

2. Local exact shock reconstruction. In this section we consider the previous inverse problem locally and we have the following

THEOREM A ([4]). *When $|u_+ - u_-| \ll 1$, under assumptions (H_1)–(H_3), if the position of n non-degenerate shocks $x = x_i(t)(i = 1, \cdots, n)$ satisfying (1.30)–(1.31) is prescribed, for any given $u_-^0(x) \in C^1$ with $u_-^0(0) = u_-$, in a neighbourhood of the origin, we can uniquely determine $u_+^0(x) \in C^1$ with $u_+^0(0) = u_+$, such that the corresponding generalized Riemann problem (1.1) and (1.13) admits a unique piecewise C^1 solution (1.15) in which n non-degenerate shocks passing through the origin are just given by $x = x_i(t)(i = 1, \cdots, n)$.*

Proof. First of all, by the entropy condition, we can show that for $i = 1, \cdots, n$, in a neighbourhood of any given non-degenerate i-th typical shock, the Rankine-Hugoniot conditions

$$[f(u)] = s_i[u] \tag{2.1}$$

can be equivalently rewritten as

$$u_+ = G_i(u_-, s). \tag{2.2}$$

On the other hand, by the entropy condition, for the local piecewise smooth solution to the generalized Riemann problem (1.1) and (1.13), given by Proposition A, we have that, in a neighbourhood of the origin,

a. For $i = 1, \cdots, n$, any i-th non-degenerate shock $x = x_i(t)$ is non-characteristic.

b. The 1st non-degenerate shock $x = x_1(t)$ passing through the origin lies in the interior of the maximum determinate domain for the Cauchy problem with the C^1 initial data $u_-^0(x)$ $(x \le 0)$ with $u_-^0(0) = u_-$.

c. For $i = 2, \cdots, n$, the i-th non-degenerate shock $x = x_i(t)$ lies in the interior of the maximum determinate domain for the generalized Cauchy problem with the value of solution on the right side of $x = x_{i-1}(t)$ as initial data.

d. The positive x-axis lies in the interior of the maximum determinate domain for the generalized Cauchy problem with the value of solution on the right side of $x = x_n(t)$ as initial data.

Thus, we can divide our proof in several steps.

First, for any given $u_-^0(x) \in C^1$ with $u_-^0(0) = u_-$, by solving the Cauchy problem for system (1.1) with the initial data

$$t = 0: \quad u = u_-^0(x), \quad x \le 0, \tag{2.3}$$

we get a unique local C^1 solution $u = u^{(0)}(t,x)$ with $u^{(0)}(0,0) = \hat{u}^{(0)} = u_-$ on the corresponding maximum determinate domain, the right boundary of which is the 1st characteristic passing through the origin.

By the entropy condition, in a neighbourhood of the origin, $x = x_1(t)$ lies in the interior of this maximum determinate domain, then the value of solution on the left side of $x = x_1(t)$ should be $u(t, x_1(t) - 0) = u^{(0)}(t, x_1(t)) \in C^1$, which takes the value $\hat{u}^{(0)} = u_-$ at $t = 0$.

Hence, by the Rankine-Hugoniot conditions (2.2), the value of solution on the right side of $x = x_1(t)$ is then $u(t, x_1(t) + 0) = G(u^{(0)}(t, x_1(t)), x_1'(t)) \in C^1$, which takes the value $\hat{u}^{(1)}$ at $t = 0$.

Noting the entropy condition, $x = x_1(t)$ is non-characteristic.

Second, by solving the generalized Cauchy problem for system (1.1) with the value of solution on the right side of $x = x_1(t)$, we get a unique local C^1 solution $u = u^{(1)}(t,x)$ with $u^{(1)}(0,0) = \hat{u}^{(1)}$ on the corresponding maximum determinate domain, the right boundary of which is the 2nd characteristic passing through the origin.

By the entropy condition, in a neighbourhood of the origin, $x = x_2(t)$ is included in the interior of this maximum determinate domain, then the value of solution on the left side of $x_2(t)$ must be $u(t, x_2(t) - 0) = u^{(1)}(t, x_2(t)) \in C^1$, which takes the value $\hat{u}^{(1)}$ at $t = 0$.

Hence, by the Rankine-Hugoniot conditions (2.2), the value of solution on the right side of $x = x_2(t)$ is $u(t, x_2(t) + 0) = G(u^{(1)}(t, x_2(t)), x_2'(t)) \in C^1$, which takes the value $\hat{u}^2$ at $t = 0$.

Noting the entropy condition, $x = x_2(t)$ is non-characteristic.

Generally speaking, for $i = 2, \cdots, n$, by solving the generalized Cauchy problem for system (1.1) with the value of solution on the right side of $x = x_{i-1}(t)$, we get a unique local C^1 solution $u = u^{(i-1)}(t,x)$ with $u^{(i-1)}(0,0) = \hat{u}^{(i-1)}$ on the corresponding maximum determinate domain, the right boundary of which is the i-th characteristic passing through the origin.

By the entropy condition, in a neighbourhood of the origin, $x = x_i(t)$ is included in the interior of this maximum determinate domain, then the value of solution on the left side of $x = x_i(t)$ must be $u(t, x_i(t) - 0) = u^{(i-1)}(t, x_i(t)) \in C^1$, which takes the value $\hat{u}^{(i-1)}$ at $t = 0$.

Hence, by the Rankine-Hugoniot conditions (2.2), the value of solution on the right side of $x_i(t)$ is $u(t, x_i(t) + 0) = G(u^{(i-1)}(t, x_i(t)),\ x_i'(t)) \in C^1$, which takes the value $\hat{u}^{(i)}$ at $t = 0$.

$x = x_i(t)$ is still non-characteristic.

Finally, by solving the generalized Cauchy problem for system (1.1) with the value of solution on the right side of $x = x_n(t)$, we get a unique local C^1 solution $u = u^{(n)}(t,x)$ with $u^{(n)}(0,0) = \hat{u}^{(n)} = u_+$ on the corresponding maximum determinate domain, the right boundary of which is the 1st characteristic passing through the origin.

By the entropy condition, in a neighbourhood of the origin, the positive x-axis lies in the interior of this maximum determinate domain, then we get $u_+^0(x) = u^{(n)}(0,x) \in C^1$ with $u_+^0(0) = u_+$.

Theorem A is then proved. $\square$

3. Global exact shock reconstruction. We now consider the corresponding inverse problem globally. We have

THEOREM B ([5]). *When $|u_+ - u_-| \ll 1$, under assumptions (H_1)-(H_3), there exists an $\varepsilon_0 > 0$ so small that, for any given ε with $0 < \varepsilon \leq \varepsilon_0$, if one knows the position of n non-degenerate shocks $x = x_i(t) \in C^2$ $(i = 1, \cdots, n)$ satisfying*

$$x_i(0) = 0, \quad x_i'(0) = s_i \quad (i = 1, \cdots, n) \tag{3.1}$$

with

$$s_1 < s_2 < \cdots < s_n \tag{3.2}$$

and

$$|x_i'(t) - x_i'(0)|,\ |x_i''(t)| \leq \frac{\varepsilon}{1+t}, \quad \forall t \geq 0 \ \ (i = 1, \cdots, n), \tag{3.3}$$

then, for any given $u_-^0(x) \in C^1$ satisfying $u_-^0(0) = u_-$ and

$$|u_-^0(x) - u_-^0(0)|,\ |u_-^{0'}(x)| \leq \frac{\varepsilon}{1+|x|}, \quad \forall x \leq 0, \tag{3.4}$$

we can uniquely determine $u_+^0(x) \in C^1$ $(x \geq 0)$ satisfying $u_+^0(0) = u_+$ and

$$|u_+^0(x) - u_+^0(0)|,\ |u_+^{0'}(x)| \leq \frac{K\varepsilon}{1+x}, \quad \forall x \geq 0, \tag{3.5}$$

where K is a positive constant independent of ε and x, such that the corresponding generalized Riemann problem (1.1) and (1.13) admits a unique global piecewise C^1 solution (1.22) in which n non-degenerate shocks passing through the origin are just $x = x_i(t)$ $(i = 1, \cdots, n)$, on which we have the Rankine-Hugoniot conditions (1.18) and the entropy condition (1.19).

In order to prove Theorem B, it is essential to consider the Cauchy problem and the generalized Cauchy problem discussed in the previous paragraph in a global sense.

Of course, in general there is no global C^1solution on the whole maximum determinate domain for these problems, however, we do have a global C^1 solution on a little bit smaller domain and it is enough for the global construction of solution.

We first consider the Cauchy problem for system (1.1) with the initial data on the negative x-axis

$$t = 0 : u = u^0(x),\ x \leq 0. \tag{3.6}$$

LEMMA 1. *There exists a positive constant ε_0 so small that, for any given ε with $0 < \varepsilon \leq \varepsilon_0$, if $u^0(x) \in C^1$ and*

$$|u^0(x) - u^0(0)|,\ |u^{0'}(x)| \leq \frac{\varepsilon}{1+|x|}, \quad \forall x \leq 0, \tag{3.7}$$

then, on the domain

$$\hat{D} = \{(t, x) | t \geq 0, x \leq g(t)\}, \tag{3.8}$$

where $g(t) \in C^1$ satisfying $g(0) = 0$,

$$|g'(t) - g'(0)| \leq \varepsilon, \quad \forall t \geq 0 \tag{3.9}$$

and

$$g'(0) < \min_{i=1,\cdots,n}\{\lambda_i(u^0(0))\}, \tag{3.10}$$

Cauchy problem (1.1) and (3.6) admits a unique global C^1 solution $u = u(t,x)$ with

$$|u(t,x) - u(0,0)| \le \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in \hat{D}, \tag{3.11}$$

$$|\frac{\partial u}{\partial x}(t,x)|,\ |\frac{\partial u}{\partial t}(t,x)| \le \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in \hat{D}, \tag{3.12}$$

where K is a positive constant independent of ε, t and x.

Proof. cf. [3]. □

Next, we consider the generalized Cauchy problem for system (1.1) with the following generalized initial data

$$x = g_1(t) : u = \psi(t), \quad \forall t \ge 0, \tag{3.13}$$

where $x = g_1(t)$ is a non-characteristic curve.

In order to get a global C^1 solution to this problem, we assume that $x = g_1(t) \in C^2$ and $x = g_2(t) \in C^2$ $(t \ge 0)$ satisfy

$$g_1(0) = g_2(0) = 0, \tag{3.14}$$

$$\lambda_r(\psi(0)) < g_1'(0) < g_2'(0) < \lambda_s(\psi(0)) \quad (r = 1,\cdots,m; s = m+1,\cdots,n), \tag{3.15}$$

$$|g_i'(t) - g_i'(0)| \le \frac{\varepsilon}{1+t}, \quad \forall t \ge 0 \ \ (i = 1,2) \tag{3.16}$$

and

$$|g_1''(t)| \le \frac{\varepsilon}{1+t}, \quad \forall t \ge 0, \tag{3.17}$$

where $0 < \varepsilon \le \varepsilon_0$ with ε_0 suitably small.

Lemma 2. *Under assumptions (3.14)-(3.17), if $\psi(t) \in C^1$ and*

$$|\psi(t) - \psi(0)|,\ |\psi'(t)| \le \frac{\varepsilon}{1+t}, \quad \forall t \ge 0, \tag{3.18}$$

then, on the domain

$$\widetilde{D} = \{(t,x) | t \ge 0,\ g_1(t) \le x \le g_2(t)\}, \tag{3.19}$$

the generalized Cauchy problem (1.1) and (3.13) admits a unique global C^1 solution $u = u(t,x)$ with

$$|u(t,x) - u(0,0)| \le \frac{K\varepsilon}{1+x-g_1(t)}, \quad \forall (t,x) \in \widetilde{D}, \tag{3.20}$$

$$|\frac{\partial u}{\partial x}(t,x)|,\ |\frac{\partial u}{\partial t}(t,x)| \le \frac{K\varepsilon}{1+x-g_1(t)}, \quad \forall (t,x)\in \widetilde{D}, \tag{3.21}$$

where K is a positive constant independent of ε, t and x.

Proof. Taking the transformation of independent variables

$$\bar{x} = -t,\ \bar{t} = x - g_1(t), \tag{3.22}$$

the original generalized Cauchy problem on $\widetilde{D}$ is reduced to the following Cauchy problem on $\bar{D} = \{(\bar{t},\bar{x}) | \bar{t}\ge 0,\ \bar{x}\le g(\bar{t})\}$:

$$\begin{cases} \dfrac{\partial u}{\partial \bar{t}} - (A(u) - g_1'(-\bar{x})I)^{-1}\dfrac{\partial u}{\partial \bar{x}} = 0, & (3.23)\\ \bar{t} = 0 : u = \psi(-\bar{x}) \quad (\bar{x}\le 0), & (3.24)\end{cases}$$

where $\bar{x} = g(\bar{t}) (\le 0) \in C^2$ with $g(0) = 0$ is determined by

$$\bar{t} = g_2(-\bar{x}) - g_1(-\bar{x}). \tag{3.25}$$

Let

$$u_{n+1} = g_1'(-\bar{x}), \tag{3.26}$$

$$U = \begin{pmatrix} u \\ u_{n+1} \end{pmatrix} \tag{3.27}$$

and

$$\Psi(\bar{x}) = \begin{pmatrix} \psi(-\bar{x}) \\ g_1'(-\bar{x}) \end{pmatrix}. \tag{3.28}$$

On the domain $\bar{D}$, (3.23)-(3.24) can be rewritten as

$$\begin{cases} \dfrac{\partial U}{\partial \bar{t}} + \bar{A}(U)\dfrac{\partial U}{\partial \bar{x}} = 0, & (3.29)\\ \bar{t} = 0 :\ U = \Psi(\bar{x}) \quad (\bar{x}\le 0), & (3.30)\end{cases}$$

where

$$\bar{A}(U) = \begin{pmatrix} -(A(u) - u_{n+1}I)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.31}$$

(3.29) is still a strictly hyperbolic system with

$$\bar{\lambda}_r(U) > \bar{\lambda}_{n+1}(U) \equiv 0 > \bar{\lambda}_s(U) \quad (r = 1,\cdots,m; s = m+1,\cdots,n). \tag{3.32}$$

By Lemma 1, Cauchy problem (3.29)-(3.30) admits a unique global C^1 solution $U = U(\bar{t},\bar{x})$ on $\bar{D}$ and

$$|U(\bar{t},\bar{x}) - U(0,0)| \le \frac{\bar{K}\varepsilon}{1+\bar{t}}, \quad \forall (\bar{t},\bar{x})\in \bar{D}, \tag{3.33}$$

$$|\frac{\partial U}{\partial \bar{x}}(\bar{t},\bar{x})|,\ |\frac{\partial U}{\partial \bar{t}}(\bar{t},\bar{x})| \le \frac{\bar{K}\varepsilon}{1+\bar{t}}, \quad \forall (\bar{t},\bar{x})\in \bar{D}, \tag{3.34}$$

henceforth, $\bar{K}$ stands for a positive constant independent of $\varepsilon,\ t$ and x.

Then, Cauchy problem (3.23)-(3.24) admits a unique global C^1 solution $u = \bar{u}(\bar{t}, \bar{x})$ on $\bar{D}$ and

$$|\bar{u}(\bar{t}, \bar{x}) - \bar{u}(0,0)| \leq \frac{\bar{K}\varepsilon}{1+\bar{t}}, \quad \forall (\bar{t}, \bar{x}) \in \bar{D}, \tag{3.35}$$

$$|\frac{\partial \bar{u}}{\partial \bar{x}}(\bar{t}, \bar{x})|,\ |\frac{\partial \bar{u}}{\partial \bar{t}}(\bar{t}, \bar{x})| \leq \frac{\bar{K}\varepsilon}{1+\bar{t}}, \quad \forall (\bar{t}, \bar{x}) \in \bar{D}. \tag{3.36}$$

As a result, the generalized Cauchy problem (1.1) and (3.13) admits a unique global C^1 solution $u = u(t,x) = \bar{u}(x - g_1(t), -t)$ on $\widetilde{D}$. Moreover, (3.20)–(3.21) hold. ▯

Remark A. *When $x = g_2(t)$ is replaced by the positive x-axis and (3.15) is replaced by*

$$\lambda_i(\psi(0)) < g_1'(0) \quad (i = 1, \cdots, n), \tag{3.37}$$

Lemma 2 still holds.

Lemma 3. *For $i = 1, \cdots, n$, suppose that, on the left side of the i-th shock $x = x_i(t)$ satisfying $x_i(0) = 0$, $x_i'(0) = s_i$ and*

$$|x_i'(t) - x_i'(0)|,\ |x_i''(t)| \leq \frac{\varepsilon}{1+t}, \quad \forall t \geq 0, \tag{3.38}$$

the value of solution $u_-^{(i)} = u_-^{(i)}(t) \in C^1$ satisfies

$$u_-^{(i)}(0) = \hat{u}^{(i-1)} \tag{3.39}$$

and

$$|u_-^{(i)}(t) - \hat{u}^{(i-1)}|,\ |\frac{du_-^{(i)}(t)}{dt}| \leq \frac{K\varepsilon}{1+t}, \quad \forall t \geq 0. \tag{3.40}$$

Then, on the right side of $x = x_i(t)$, by the Rankine-Hugoniot conditions, we can uniquely determine the value of solution $u_+^{(i)} = u_+^{(i)}(t)$ such that

$$u_+^{(i)}(0) = \hat{u}^{(i)} \tag{3.41}$$

and

$$|u_+^{(i)}(t) - \hat{u}^{(i)}|,\ |\frac{du_+^{(i)}(t)}{dt}| \leq \frac{K\varepsilon}{1+t}, \quad \forall t \geq 0. \qquad \square \tag{3.42}$$

By means of Lemmas 1–3, the whole procedure in the proof of Theorem A works in the global sense. Then we get Theorem B.

Remark B. *Theorem B is a joint work with Wang Li-bin.*

4. Remarks.

4.1. All the shocks under consideration should be non-degenerate. In fact, taking the Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{4.1}$$

as an example, if $x = x(t)$ degenerates to a weak discontinuity: $u_+ = u_- = u_0$, $x = x(t)$ must be a straight characteristic $x = u_0 t$, on which $u = u_0$. As a result, the position of $x = x(t)$ does not give any information on the initial data $u^0_\pm(x)$ except $u^0_\pm(0) = u_0$.

4.2. The assumption that there are no centered rarefaction waves in the self-similar solution to the Riemann problem is essential. Still taking the Burger's equation (4.1) as an example, since any characteristic is a straight line, on which the solution takes a constant value, any centered wave passing through the origin must be a centered rarefaction wave with straight characteristics. Similarly, the position of the centered wave gives no information on the initial data $u^0_\pm(x)$ except $u^0_\pm(0) = u_\pm$.

4.3. The assumption that all the characteristics are genuinely nonlinear in the sense of P. D. Lax is also essential. In fact, if there is a linearly degenerate characteristic, the corresponding elementary wave of which must be a contact discontinuity . For instance, for the scalar linear equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \tag{4.2}$$

the contact discontinuity passing through the origin must be $x = t$. Obviously, the position of this contact discontinuity is useless for determining the initial data $u^0_\pm(x)$.

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