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WELL-POSEDNESS OF THE IDEAL MHD SYSTEM IN CRITICAL BESOV SPACES*

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Abstract. In this paper we study the ideal incompressible magneto-hydrodynamics system, and prove the local existence and uniqueness of solutions in critical Besov spaces $B^{1+n/p}_{p,1}$ for $1 \leq p \leq \infty$.

Key words. Magneto-hydrodynamics system, critical Besov space, existence and uniqueness

AMS subject classifications. 76W05, 74H20, 74H25

1. Introduction. We are concerned with the following ideal magneto-hydrodynamics (MHD) system for the homogeneous incompressible fluid flows and magnetic fields

$$u_t + (u \cdot \nabla)u - (b \cdot \nabla)b - \nabla\pi = 0 \tag{1.1}$$

$$b_t + (u \cdot \nabla)b - (b \cdot \nabla)u = 0 \tag{1.2}$$

$$\text{div} u = 0, \quad \text{div} b = 0 \tag{1.3}$$

with initial data

$$u(x, 0) = u_0(x), \tag{1.4}$$

$$b(x, 0) = b_0(x). \tag{1.5}$$

Here $u(x,t) = (u_1(x,t), u_2(x,t), \cdots, u_n(x,t))$ is the velocity of the fluid flows , $b(x,t) = (b_1(x,t), b_2(x,t), \cdots, b_n(x,t))$ is the magnetic field, and $\pi(x,t) = p(x,t) + \frac{1}{2}|b(x,t)|^2$ is the total pressure for $x \in \mathbb{R}^n$, $t \geq 0$ and $u_0(x)$ and $b_0(x)$ are the initial velocity and initial magnetic field satisfying divu_0=0, divb_0=0, respectively. For simplicity, we have set the Reynolds number, magnetic Reynolds number and the corresponding coefficients to be constant 1 by scaling transformation.

In the case of Euler equations, the existence and uniqueness of solutions to Euler equations have been studied by many authors (see J.-Y. Chemin [3] and reference there). Recently, Vishik [11], H.C. Park and Y.J. Park [6] obtained the existence and uniqueness of solutions of the incompressible Euler equations in critical Besov spaces. Vishik considered Euler equations in space dimension 2 and proved the global well-posedness in critical Besov space $B^{1+2/p}_{p,1}$, $1 < p < \infty$ by transport equation and the invariance of vorticity. For the ideal magneto-hydrodynamics system, the method Vishik used is not valid, and it is more complicated because of the couple effect between velocity $u(x,t)$ and magneto fields $b(x,t)$. The existence of the classical solution for MHD system was shown by Kozono [4] in the bounded domain, See also [9]. In BMO^{-1} and bmo^{-1} spaces, Miao, Yuan and Zhang proved the global existence and uniqueness of solution to the incompressible MHD system for small initial data [5]. In the case of Sobolev spaces $W^{k,p}$, the existence and uniqueness results for

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the equations of ideal magneto-hydrodynamics have been established by Alexseev [1]. Moreover, in this case, Secchi [8] and Schmidt [7] proved not only existence and uniqueness results, but also the continuous dependence on the initial data.

In this paper, the local existence and uniqueness of the solution to the n-dimensional ideal incompressible MHD system (1.1)-(1.5) will be investigated. We prove that there exists a locally unique solution in the critical Besov space $B^{1+n/p}_{p,1}$ for $1 \leq p \leq \infty$ provided that the initial data $(u_0(x,t), b_0(x,t))$ is in the space. We obtain a priori estimates of solutions to the approximate equations by virtue of the couple effect between velocity $u(x,t)$ and magneto fields $b(x,t)$ subtly. Our local existence and uniqueness results are as follows:

THEOREM 1.1. *Let $u_0(x), b_0(x) \in B^{1+n/p}_{p,1}(\mathbb{R}^n)$ be divergence free vector for $1 \leq p \leq \infty$. There exists time $T > 0$ such that the Cauchy problem (1.1)-(1.5) has a unique solution $(u(x,t), b(x,t)) \in C([0,T]; B^{1+n/p}_{p,1}(\mathbb{R}^n))$.*

REMARK 1.1. *Even in the case of space dimension 2, the global well-posedness of solutions to ideal magneto-hydrodynamics in critical Besov space $B^{1+2/p}_{p,1}(\mathbb{R}^2)$ for $1 \leq p \leq \infty$ is an open problem.*

The plan of this paper is as follows: In Section 2 we recall succinctly the Littlewood-Paley dyadic decomposition, Besov spaces and Bony's para-product decomposition of two functions $f(x)$ and $g(x)$. Then we give some preliminary a priori estimates. In Section 3, we establish a priori estimates of solutions to the approximate equations, and prove that the sequence of solutions are locally bounded in the Besov space $B^{1+n/p}_{p,1}(\mathbb{R}^n)$ and that there is a Cauchy sequence of solutions to the approximate equations in $B^{n/p}_{p,1}(\mathbb{R}^n)$ for $1 \leq p \leq \infty$. The main theorem will also be proved in Section 3.

We use C to denote some positive constants which may be different from line to line and depends on parameters concerned, such as $p, q, \cdots$, but not on the involved functions. In this paper $\|\cdot\|_p$ denotes the L^p norm in $\mathbb{R}^n$ for $1 \leq p \leq \infty$.

2. Littlewood-Paley Decomposition and Preliminary estimates . We first set our notation and recall definitions of Besov spaces, and then give some preliminary estimates on Besov spaces. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}'$, the Fourier transform of $f(x)$ is defined by

$$\mathcal{F}(f) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\mathrm{d}x.$$

Let $\hat{\Phi},\ \hat{\varphi} \in C_0^\infty(\mathbb{R}^n)$ be radial functions satisfying

(I). $supp\hat{\Phi} \subset \{\xi \in \mathbb{R}^n;\ |\xi| \leq \frac{5}{6}\}$,

(II). $supp\hat{\varphi} \subset \{\xi \in \mathbb{R}^n;\ \frac{3}{5} \leq |\xi| \leq \frac{5}{3}\}$,

(III). $\hat{\Phi}(\xi) + \sum\limits_{j=0}^{\infty} \hat{\varphi}_j(\xi) = 1$, for $\xi \in \mathbb{R}^n$.

We set $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$, (i.e. $\varphi_j(x) = 2^{jn}\varphi(2^j x)$).

DEFINITION 2.1. *Let $f(x) \in \mathcal{S}'$, define*

(I). $\triangle_{-1} f = \hat{\Phi}(D)f = \Phi * f$,

(II). $\triangle_j f = \hat{\varphi}_j(D)f = \varphi_j * f$, *for* $j \geq 0$,

(III). $\triangle_j f = 0$, *for* $j \leq -2$,

(IV). $S_j f = 1 - \sum\limits_{k \geq j+1} \triangle_k f$, *for* $j \in \mathbb{Z}$.

DEFINITION 2.2. *(Triebel [10]) Let $s \in \mathbb{R}$, $1 \le p \le \infty$, and $1 \le q < \infty$. The inhomogeneous Besov norm is defined by*

$$\|f\|_{B^s_{p,q}} = \Big(\sum_{j=-1}^{\infty} 2^{jqs} \|\triangle_j f\|_p^q \Big)^{1/q} < \infty. \tag{2.1}$$

If $q = \infty$, the corresponding norm is defined by

$$\|f\|_{B^s_{p,\infty}} = \sup_{j \ge -1} 2^{js} \|\triangle_j f\|_p. \tag{2.2}$$

Below we recall the Bernstein's lemma that will be used in proofs of our results.

LEMMA 2.1. *(Bernstein's inequality)*
(a) Let $g(x) \in L^p(\mathbb{R}^n) \bigcap L^{p_1}(\mathbb{R}^n)$, and supp$\hat{g} \subset \{\xi \in \mathbb{R}^n; |\xi| \le r\}$. Then there exists a constant C such that

$$\|g\|_{p_1} \le C r^{n(\frac{1}{p} - \frac{1}{p_1})} \|g\|_p, \tag{2.3}$$

for $1 \le p \le p_1 \le \infty$.
(b) Assume that $f(x) \in L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$ and supp$\hat{f} \subset \{\xi \in \mathbb{R}^n; 2^{j-1} \le |\xi| \le 2^{j+1}\}$. There exists a constant C_k so that the following inequality holds:

$$C_k^{-1} 2^{jk} \|f\|_p \le \|D^k f\|_p \le C_k 2^{jk} \|f\|_p. \tag{2.4}$$

The proof is an immediate consequence of Young's inequality, please refer to [3] for details. The next definition describes Bony's para-product formula which gives the decomposition of the product $f \cdot g$ of two functions $f(x)$ and $g(x)$.

DEFINITION 2.3. *The para-product of two functions f and g is defined by*

$$T_g f = \sum_{i \le j-2} \triangle_i g \triangle_j f = \sum_{j \in \mathbb{Z}} S_{j-2} g \triangle_j f. \tag{2.5}$$

The remainder of the para-product is defined by

$$R(f,g) = \sum_{|i-j| \le 1} \triangle_i g \triangle_j f. \tag{2.6}$$

Then Bony's para-product formula reads

$$f \cdot g = T_g f + T_f g + R(f,g). \tag{2.7}$$

In order to obtain a priori estimates, we need the decomposition

$$\begin{aligned}(S_{j-2}u, \nabla)\triangle_j v - \triangle_j (u,\nabla) v = & -\sum_{i=1}^n \triangle_j (T_{\partial_i v} u_i) + \sum_{i=1}^n [T_{u_i}\partial_i, \triangle_j] v \\ & - \sum_{i=1}^n T_{u_i - S_{j-2}u_i} \partial_i \triangle_j v \\ & - \sum_{i=1}^n \{\triangle_j (R(u_i, \partial_i) v) - R(S_{j-2}u_i, \triangle_j \partial_i v)\} \\ = & \ I_1(u,v) + I_2(u,v) + I_3(u,v) + I_4(u,v). \end{aligned} \tag{2.8}$$

Lemma 2.2. *Let $1 \le p \le \infty$, for any divergence free vector fields $u, v \in \mathcal{S}'(\mathbb{R}^n)$, we have the estimates:*

$$\|I_1(u,v)\|_p \le C \sum_{|j-j'|\le 3} \|S_{j'-2}\nabla v\|_\infty \|\triangle_{j'} u\|_p; \tag{2.9}$$

$$\|I_2(u,v)\|_p \le C \sum_{|j-j'|\le 3} \|S_{j'-2}\nabla u\|_\infty \|\triangle_{j'} v\|_p; \tag{2.10}$$

$$\|I_3(u,v)\|_p \le C \sum_{|j-j'|\le 3} (\|\triangle_j \nabla u\|_\infty + \|\triangle_{-1} u\|_\infty) \|\triangle_{j'} v\|_p; \tag{2.11}$$

$$\begin{aligned}\|I_4(u,v)\|_p \le{}& C \sum_{|j-j'|\le 3} \sum_{|j'-j''|\le 1} (\|\triangle_{j'} \nabla u\|_\infty + \|\triangle_{-1} u\|_\infty) \|\triangle_{j''} v\|_p \\ &+ C \sum_{j'\ge j-3} 2^{j-j'} \sum_{|j'-j''|\le 1} \|\triangle_{j'}\nabla u\|_\infty \|\triangle_{j''} v\|_p. \end{aligned}\tag{2.12}$$

Proof.

$$I_1(u,v) = -\sum_{i=1}^n \sum_{j'=1}^\infty \triangle_j \{S_{j'-2}(\partial_i v) \triangle_{j'} u_i\}. \tag{2.13}$$

Taking the L^p norm, one arrives at

$$\begin{aligned}\|I_1(u,v)\|_p &\le C \sum_{i=1}^n \sum_{|j-j'|\le 3, j'\ge 1} \|S_{j'-2}(\partial_i v)\|_\infty \|\triangle_{j'} u_i\|_p \\ &\le C \sum_{|j-j'|\le 3} \|S_{j'-2}\nabla v\|_\infty \|\triangle_{j'} u\|_p. \end{aligned}\tag{2.14}$$

For $I_2(u,v)$, one has

$$\begin{aligned} I_2(u,v) &= \sum_{i=1}^n [T_{u_i}\partial_i, \triangle_j] v \\ &= \sum_{i=1}^n \sum_{|j-j'|\le 3} \{S_{j'-2}u_i \triangle_{j'}(\partial_i \triangle_j v) - \triangle_j (S_{j'-2} u_i \partial_i \triangle_{j'} v)\} \\ &= \sum_{i=1}^n \sum_{|j-j'|\le 3} \int_{\mathbb{R}^n} \varphi_j(x-y) \{S_{j'-2}u_i(x) - S_{j'-2}u_i(y)\} \partial_i \triangle_{j'} v(y) \mathrm{d}y \\ &= \sum_{i=1}^n \sum_{|j-j'|\le 3} 2^{j(n+1)} \int_{\mathbb{R}^n} \partial_i \varphi(2^j(x-y)) \{S_{j'-2}u_i(x) - S_{j'-2}u_i(y)\} \triangle_{j'} v(y) \mathrm{d}y \\ &= \sum_{i=1}^n \sum_{|j-j'|\le 3} 2^{j(n+1)} \int_{\mathbb{R}^n} \partial_i \varphi(2^j(x-y)) \int_0^1 (x-y)\cdot \nabla S_{j'-2} u_i(x+\tau(y-x)) \mathrm{d}\tau \triangle_{j'} v(y) \mathrm{d}y \\ &= \sum_{i=1}^n \sum_{|j-j'|\le 3} \int_{\mathbb{R}^n} \partial_i \varphi(z) \int_0^1 (z\cdot\nabla) S_{j'-2} u_i(x - 2^{-j}\tau z) \mathrm{d}\tau \triangle_{j'} v(x - 2^{-j} z) \mathrm{d}z. \end{aligned}\tag{2.15}$$

Taking the L^p norm, we have

$$\begin{aligned}\|I_2(u,v)\|_p &\le C\sum_{i=1}^{n}\sum_{|j-j'|\le 3}\|S_{j'-2}\nabla u_i\|_\infty\|\triangle_{j'}v\|_p\\ &\le C\sum_{|j-j'|\le 3}\|S_{j'-2}\nabla u\|_\infty\|\triangle_{j'}v\|_p. \qquad (2.16)\end{aligned}$$

Similarly, for $I_3(u,v)$, one has

$$\begin{aligned}I_3(u,v) &= -\sum_{i=1}^{n}T_{u_i-S_{j-2}u_i}\partial_i\triangle_j v\\ &= -\sum_{i=1}^{n}\sum_{|j-j'|\le 1}S_{j'-2}(u_i-S_{j-2}u_i)\triangle_{j'}(\partial_i\triangle_j v)\\ &= -\sum_{i=1}^{n}\sum_{|j-j'|\le 1}S_{j'-2}\Big(\sum_{m=j-1}^{j}\triangle_m u_i\Big)\partial_i\triangle_{j'}\triangle_j v, \qquad (2.17)\end{aligned}$$

and

$$\begin{aligned}\|I_3(u,v)\|_p &\le \sum_{i=1}^{n}\sum_{|j-j'|\le 1}\|S_{j'-2}(\triangle_{j-1}u_i+\triangle_j u_i)\|_\infty\|\partial_i(\triangle_{j'}\triangle_j v)\|_p\\ &\le \sum_{i=1}^{n}\sum_{|j-j'|\le 1}(\|\triangle_{j-1}u_i\|_\infty+\|\triangle_j u_i\|_\infty)2^j\|\triangle_{j'}v\|_p\\ &\le C\sum_{|j-j'|\le 1}2^{-j}\{\|\triangle_{j-1}\nabla u\|_\infty+\|\triangle_j\nabla u\|_\infty\}2^j\|\triangle_{j'}v\|_p\\ &\quad +C\|\triangle_{-1}u\|_\infty\sum_{|j-j'|\le 1}\|\triangle_{j'}v\|_p\\ &\le C\sum_{|j-j'|\le 1}(\|\triangle_j\nabla u\|_\infty+\|\triangle_{-1}u\|_\infty)\|\triangle_{j'}v\|_p. \qquad (2.18)\end{aligned}$$

We decompose $I_4(u,v)$ as follows:

$$\begin{aligned}I_4(u,v) &= -\sum_{i=1}^{n}\{\triangle_j R(u_i,\partial_i v)-R(S_{j-2}u_i,\triangle_j\partial_i v)\}\\ &= -\sum_{i=1}^{n}\{\triangle_j\partial_i R(u_i-S_{j-2}u_i,v)\}-\sum_{i=1}^{n}\{\triangle_j R(S_{j-2}u_i,\partial_i v)-R(S_{j-2}u_i,\triangle_j\partial_i v)\}\\ &= I_{41}+I_{42}, \qquad (2.19)\end{aligned}$$

and

$$\begin{aligned}
I_{41} &= -\sum_{i=1}^{n} \partial_i \triangle_j \sum_{|j'-j''|\le 1} \triangle_{j'}(u_i - S_{j-2}u_i)\triangle_{j''}v \\
&= -\sum_{i=1}^{n} \partial_i \triangle_j \sum_{j''=-1}^{0} \triangle_{-1}(u_i - S_{j-2}u_i)\triangle_{j''}v \\
&\quad -\sum_{i=1}^{n} \partial_i \triangle_j \sum_{j'\ge 0} \sum_{|j'-j''|\le 1} \triangle_{j'}(u_i - S_{j-2}u_i)\triangle_{j''}v \\
&= I_{411} + I_{412}.
\end{aligned} \tag{2.20}$$

Taking the L^p norm, we can estimate

$$\begin{aligned}
\|I_{411}\|_p &\le C\sum_{i=1}^{n}\sum_{j''=-1}^{0} \|\triangle_{-1}(u_i - S_{j-2}u_i)\|_\infty \|\triangle_{j''}v\|_p \\
&\le \begin{cases} C\sum\limits_{j''=-1}^{0} \|\triangle_{-1}u\|_\infty \|\triangle_{j''}v\|_p, & j=-1,\ 0,\ 1, \\ 0, & j\ge 2, \end{cases}
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\|I_{412}\|_p &\le C\sum_{i=1}^{n} 2^j \Big\| \triangle_j \sum_{|j'-j''|\le 1, j'\ge 0} \triangle_{j'}(u_i - S_{j-2}u_i)\triangle_{j''}v \Big\|_p \\
&\le C\sum_{i=1}^{n}\sum_{j'\ge j-3} 2^{j-j'} \sum_{|j'-j''|\le 1, j'\ge 0} 2^{j'}\|\triangle_{j'}u_i\|_\infty \|\triangle_{j''}v\|_p \\
&\le C\sum_{j'\ge j-3} 2^{j-j'} \sum_{|j'-j''|\le 1, j'\ge 0} \|\triangle_{j'}\nabla u\|_\infty \|\triangle_{j''}v\|_p,
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
I_{42} &= -\sum_{i=1}^{n}\sum_{j'=j-3}^{j-1}\sum_{|j'-j''|\le 1} \{\triangle_j(\triangle_{j'}S_{j-2}u_i \triangle_{j''}\partial_i v) - \triangle_{j'}S_{j-2}u_i \triangle_j\triangle_{j''}\partial_i v\} \\
&= -\sum_{i=1}^{n}\sum_{j'=j-3}^{j-1}\sum_{|j'-j''|\le 1} [\triangle_j, \triangle_{j'}S_{j-2}u_i]\triangle_{j''}\partial_i v.
\end{aligned} \tag{2.23}$$

Here

$$\begin{aligned}
&[\triangle_j, \triangle_{j'}S_{j-2}u_i]\triangle_{j''}\partial_i v \\
&= 2^{jn}\int_{\mathbb{R}^n} \varphi(2^j(x-y))\{\triangle_{j'}S_{j-2}u_i(y) - \triangle_{j'}S_{j-2}u_i(x)\}\partial_i\triangle_{j''}v(y)\mathrm{d}y \\
&= 2^{j(n+1)}\int \partial_i\varphi(2^j(x-y))\{\triangle_{j'}S_{j-2}u_i(y) - \triangle_{j'}S_{j-2}u_i(x)\}\triangle_{j''}v(y)\mathrm{d}y \\
&= 2^{j(n+1)}\int \partial_i\varphi(2^j(x-y))\Big\{\int_0^1 \triangle_{j'}((y-x)\cdot\nabla S_{j-2}u_i(x+\tau(y-x)))\mathrm{d}\tau\Big\}\triangle_{j''}v(y)\mathrm{d}y \\
&= -\int \partial_i\varphi(z)\int_0^1 \triangle_{j'}(z\cdot\nabla S_{j-2}u_i(x-2^{-i}z))\mathrm{d}\tau\triangle_{j''}v(x-2^{-j}z)\mathrm{d}z.
\end{aligned} \tag{2.24}$$

Taking the L^p norm on the both sides of Eq. (2.24) , we have

$$\|[\triangle_j, \triangle_{j'} S_{j-2} u_i] \triangle_{j''} \partial_i v\|_p \le C \|\triangle_{j'} S_{j-2} \nabla u\|_\infty \left\| \int |\partial_i \varphi(z) z \triangle_{j''} v(x - 2^{-j} z)| \mathrm{d}z \right\|_p$$
$$\le C \|\triangle_{j'} S_{j-2} \nabla u\|_\infty \|\triangle_{j''} v\|_p. \tag{2.25}$$

Substituting Eq. (2.25) into Eq. (2.23), it follows

$$\|I_{42}\|_p \le C \sum_{j'=j-3}^{j-1} \sum_{|j'-j''| \le 1} \|\triangle_{j'} \nabla u\|_\infty \|\triangle_{j''} v\|_p. \tag{2.26}$$

Collecting Eqs. (2.21), (2.22) and (2.26), the estimate of I_4 can be obtained

$$\|I_4(u,v)\|_p \le C \sum_{j'=j-3}^{j-1} \sum_{|j'-j''| \le 1} (\|\triangle_{j'} \nabla u\|_\infty + \|\triangle_{-1} u\|_\infty) \|\triangle_{j''} v\|_p$$
$$+ C \sum_{j' \ge \max(0, j-3)} 2^{j-j'} \sum_{|j'-j''| \le 1} \|\triangle_{j'} \nabla u\|_\infty \|\triangle_{j''} v\|_p, \tag{2.27}$$

and the proof of Lemma 2.2 follows. □

LEMMA 2.3. *If $u(x,t),\ v(x,t) \in B^{1+n/p}_{p,1}$ for $1 \le p \le \infty$ are divergence free vector fields, then the following estimates hold:*

$$\sum_{j=-1}^{\infty} 2^{(1+n/p)j} \|(S_{j-2} u, \nabla) \triangle_j v - \triangle_j (u, \nabla) v\|_p \le C \|u\|_{B^{1+n/p}_{p,1}} \|v\|_{B^{1+n/p}_{p,1}}, \tag{2.28}$$

$$\sum_{j=-1}^{\infty} 2^{nj/p} \|(S_{j-2} u, \nabla) \triangle_j v - \triangle_j (u, \nabla) v\|_p \le C \|u\|_{B^{1+n/p}_{p,1}} \|v\|_{B^{n/p}_{p,1}}, \tag{2.29}$$

and

$$\sum_{j=-1}^{\infty} 2^{nj/p} \|(S_{j-2} v, \nabla) \triangle_j u - \triangle_j (v, \nabla) u\|_p \le C \|v\|_{B^{1+n/p}_{p,1}} \|u\|_{B^{n/p}_{p,1}}. \tag{2.30}$$

Proof. By Lemma 2.2 we have

$$\|I_1(u,v)\|_p \le C \sum_{|j-j'| \le 3} \|S_{j'-2} \nabla v\|_\infty \|\triangle_{j'} u\|_p$$
$$\le C \sum_{|j-j'| \le 3} \sum_{k=-1}^{j'-2} \|\triangle_k \nabla v\|_\infty \|\triangle_j u\|_p$$
$$= C \sum_{|j-j'| \le 3} \sum_{k=-1}^{j'-2} 2^k 2^{kn/p} \|\triangle_k v\|_p \|\triangle_j u\|_p$$
$$\le \|v\|_{B^{1+n/p}_{p,1}} \sum_{|j-j'| \le 3} \|\triangle_{j'} u\|_p. \tag{2.31}$$

Similarly, one can deduce

$$\|I_2(u,v)\|_p \le C \sum_{|j-j'|\le 3} \|u\|_{B^{1+n/p}_{p,1}} \|\triangle_{j'} v\|_p; \tag{2.32}$$

$$\begin{aligned}\|I_3(u,v)\|_p &\le C \sum_{|j-j'|\le 3} (2^{j(1+n/p)}\|\triangle_j u\|_p + \|\triangle_{-1} u\|_p)\|\triangle_{j'} v\|_p \\ &\le C \sum_{|j-j'|\le 3} \|\triangle_{j'} v\|_p \|u\|_{B^{1+n/p}_{p,1}};\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\|I_4(u,v)\|_p \le{}& C \sum_{|j-j'|\le 3} \sum_{|j'-j''|\le 1} \Big(2^{j'(1+n/p)}\|\triangle_{j'} u\|_p + \|\triangle_{-1} u\|_p\Big)\|\triangle_{j''} v\|_p \\ &+C \sum_{j'\ge j-3} 2^{j-j'} \sum_{|j'-j''|\le 1} 2^{j'(1+n/p)} \|\triangle_{j'} u\|_p \|\triangle_{j''} v\|_p \\ \le{}& C\|u\|_{B^{1+n/p}_{p,1}} (\sum_{|j-j''|\le 5} + \sum_{j''\ge j-3} 2^{j-j'} \sum_{j''=j'-1}^{j'+1}) \|\triangle_{j''} v\|_p.\end{aligned} \tag{2.34}$$

Multiplying $2^{j(1+n/p)}$ to Eqs. (2.31)-(2.34), and summing up by j from -1 to ∞, we have

$$\begin{aligned}&\sum_{j=-1}^{\infty} 2^{j(1+n/p)} \|(S_{j-2}u,\nabla)\triangle_j v - \triangle_j (u,\nabla) v\|_p \\ &\le \sum_{j=-1}^{\infty} 2^{j(1+n/p)} (\|I_1(u,v)\|p + \|I_2(u,v)\|p + \|I_3(u,v)\|p + \|I_4(u,v)\|p) \\ &\le C(n,p)\|u\|_{B^{1+n/p}_{p,1}} \|v\|_{B^{1+n/p}_{p,1}}.\end{aligned} \tag{2.35}$$

In the computation of $\sum\limits_{j=-1}^{\infty} 2^{j(1+n/p)} \|I_4(u,v)\|_p$, we have used

$$\begin{aligned}&\sum_{j=-1}^{\infty} 2^{j(1+n/p)} \sum_{j''\ge j-3} 2^{j-j'} \sum_{j''=j'-1}^{j'+1}) \|\triangle_{j''} v\|_p \\ &= \sum_{k\ge -3} 2^{-k} (\sum_{j=-1}^{\infty} \sum_{l=-1}^{1} 2^{j(1+n/p)} \|\triangle_{j+k+l} v\|_p) \\ &\le C \sum_{k\ge -3} 2^{-(2+n/p)k} (\sum_{j=-1}^{\infty} \sum_{l=-1}^{1} 2^{(j+k+l)(1+n/p)} \|\triangle_{j+k+l} v\|_p) \\ &\le C\|v\|_{B^{1+n/p}_{p,1}}.\end{aligned} \tag{2.36}$$

If we estimate $I_1(u,v)$ in another way, we have

$$\begin{aligned}\|I_1(u,v)\|_p &\le C \sum_{|j-j'|\le 3} \|S_{j'-2}\nabla v\|_\infty \|\triangle_{j'}u\|_p \\ &\le C \sum_{|j-j'|\le 3} \|S_{j'-2}v\|_\infty 2^{j'} \|\triangle_{j'}u\|_p \\ &= C \sum_{|j-j'|\le 3} \sum_{k=-1}^{j'-2} \|\triangle_k v\|_\infty 2^{j'} \|\triangle_{j'}u\|_p \\ &\le C\|v\|_{B^{n/p}_{p,1}} \sum_{|j-j'|\le 3} 2^{j'} \|\triangle_{j'}u\|_p. \end{aligned} \tag{2.37}$$

Here we used the embedding $B^{n/p}_{p,1}(\mathbb{R}^n) \hookrightarrow B^0_{\infty,1}(\mathbb{R}^n)$. Similarly, using the estimates (2.37), (2.32), (2.33) and (2.34), the Eq. (2.29) can also be obtained.

Similarly, we estimate $I_2(u,v)$, $I_3(u,v)$ and $I_4(u,v)$ in another way

$$\begin{aligned}\|I_2(u,v)\|_p &\le C \sum_{|j-j'|\le 3} 2^{j'} \|S_{j'-2}u\|_\infty \|\triangle_{j'}v\|_p \\ &\le C\|u\|_{B^{n/p}_{p,1}} \sum_{|j-j'|\le 3} 2^{j'} \|\triangle_{j'}v\|_p; \end{aligned} \tag{2.38}$$

$$\begin{aligned}\|I_3(u,v)\|_p &\le C \sum_{|j-j'|\le 3} (2^j \|\triangle_j u\|_\infty + \|\triangle_{-1}u\|_\infty) \|\triangle_{j'}v\|_p \\ &\le C \sum_{|j-j'|\le 3} (2^{j-j'} \|\triangle_j u\|_\infty + \|\triangle_{-1}u\|_\infty) 2^{j'} \|\triangle_{j'}v\|_p \\ &\le C \sum_{|j-j'|\le 3} \|u\|_{B^0_{\infty,1}} 2^{j'} \|\triangle_{j'}v\|_p \\ &\le C\|u\|_{B^{n/p}_{p,1}} \sum_{|j-j'|\le 3} 2^{j'} \|\triangle_{j'}v\|_p; \end{aligned} \tag{2.39}$$

and

$$\begin{aligned}\|I_4(u,v)\|_p &\le C \sum_{|j-j'|\le 3} \sum_{|j'-j''|\le 1} (2^{j'} \|\triangle_{j'}u\|_\infty + \|\triangle_{-1}u\|_\infty) \|\triangle_{j''}v\|_p \\ &\quad + C \sum_{j'\ge j-3} 2^{j-j'} \sum_{|j'-j''|\le 1} 2^{j'} \|\triangle_{j'}u\|_\infty \|\triangle_{j''}v\|_p \\ &\le C\|u\|_{B^{n/p}_{p,1}} \Bigg(\sum_{|j-j''|\le 5} 2^{j''} + \sum_{j'\ge j-3} 2^j \sum_{|j'-j''|\le 1} \Bigg) \|\triangle_{j''}v\|_p. \end{aligned} \tag{2.40}$$

Using the estimates (2.31), (2.38), (2.39) and (2.40), and noting the estimate (2.36), the estimate (2.30) can be obtained, and the proof of Lemma 2.3 is thus complete. ∎

For the total pressure we have the following estimates.

Lemma 2.4. *If $u(x,t),\ b(x,t) \in B^{1+n/p}_{p,1}$ for $1 \le p \le \infty$ are divergence free vector fields, then the pressure $\pi(u,b)$ can be estimated as follows:*

$$\|\nabla \pi(u,b)\|_{B^{1+n/p}_{p,1}} \le C(\|u\|^2_{B^{1+n/p}_{p,1}} + \|b\|^2_{B^{1+n/p}_{p,1}}), \tag{2.41}$$

and

$$\|\nabla\pi(u,b)\|_{B^{n/p}_{p,1}} \le C(\|u\|_{B^{n/p}_{p,1}}\|u\|_{B^{1+n/p}_{p,1}} + \|b\|_{B^{n/p}_{p,1}}\|b\|_{B^{1+n/p}_{p,1}}). \tag{2.42}$$

Where

$$\nabla\pi(u,b) = \sum_{i,j=1}^{n}((-\triangle)^{-1}\nabla\partial_i u_j\partial_j u_i + (-\triangle)^{-1}\nabla\partial_i b_j\partial_j b_i). \tag{2.43}$$

Proof. Let

$$\begin{aligned}\pi_1(u,v) &= \sum_{i,k=1}^{n}(-\triangle)^{-1}\partial_i u_k\partial_k v_i\\ &= (-\triangle)^{-1}\nabla\mathrm{div}((u\cdot\nabla)v).\end{aligned} \tag{2.44}$$

We only need to estimate $\triangle_j\nabla\pi_1(u,v)$.

Case 1, $j \ge 0$. Take the L^p norm of $\triangle_j\nabla\pi_1(u,v)$, it follows

$$\|\triangle_j\nabla\pi_1(u,v)\|_p \le C2^{-j}\|\triangle_j\mathrm{div}((u,\nabla)v)\|_p, \tag{2.45}$$

and

$$\begin{aligned}&\|\triangle_j\mathrm{div}((u,\nabla)v)\|_p\\ &= \|\mathrm{div}\{\triangle_j(u\cdot\nabla)v - (S_{j-2}u\cdot\nabla)\triangle_j v\} + \mathrm{div}((S_{j-2}u\cdot\nabla)\triangle_j v)\|_p\\ &\le C2^j\|\triangle_j(u\cdot\nabla)u - (S_{j-2}u\cdot\nabla)\triangle_j v\|_p + C\|\mathrm{div}(S_{j-2}u\cdot\nabla)\triangle_j v\|_p.\end{aligned} \tag{2.46}$$

Using Lemma 2.3 and the following estimate

$$\begin{aligned}\sum_{j=0}^{\infty}2^{j(1+n/p)}\|(S_{j-2}u\cdot\nabla)\triangle_j v\|_p &\le C\sum_{j=0}^{\infty}2^{jn/p}\|\mathrm{div}((S_{j-2}u\cdot\nabla)\triangle_j v)\|_p\\ &\le C\sum_{j=0}^{\infty}2^{jn/p}\sum_{k=-1}^{j-2}\|\nabla\triangle_k u\|_\infty\|\nabla\triangle_j v\|_p\\ &\le C\sum_{j=0}^{\infty}2^{j(1+n/p)}\|u\|_{B^{1+n/p}_{p,1}}\|\triangle_j v\|_p\\ &\le C\|u\|_{B^{1+n/p}_{p,1}}\|v\|_{B^{1+n/p}_{p,1}},\end{aligned} \tag{2.47}$$

we arrive at

$$\sum_{j=0}^{\infty}2^{j(1+n/p)}\|\triangle_j\nabla\pi_1(u,v)\|_p \le C\|u\|_{B^{1+n/p}_{p,1}}\|v\|_{B^{1+n/p}_{p,1}}, \tag{2.48}$$

and

$$\sum_{j=0}^{\infty}2^{jn/p}\|\triangle_j\nabla\pi_1(u,v)\|_p \le C\|u\|_{B^{1+n/p}_{p,1}}\|v\|_{B^{n/p}_{p,1}}. \tag{2.49}$$

Case 2, $j=-1$.
(a). $n\geq 3$.

$$\begin{aligned}\|\triangle_{-1}\nabla\pi_1(u,v)\|_p &= \Big\|\sum_{i,k=1}^{n}(\nabla\Phi)*(-\triangle)^{-1}\partial_i u_k\partial_k v_i\Big\|_p\\ &= \Big\|\sum_{i,k=1}^{n}((-\triangle)^{-1}\nabla\Phi)*\partial_i\partial_k(u_k v_i)\Big\|_p\\ &\leq C(n)\Big\|\sum_{i,k=1}^{n}\partial_i\partial_k(\nabla\frac{1}{|x|^{n-2}}*\Phi)\Big\|_1\|u\otimes v\|_p\\ &\leq C(n)\|u\|_{B^{1+n/p}_{p,1}}\|v\|_p. \end{aligned}\tag{2.50}$$

We used the fact that

$$\begin{aligned}&\Big\|\sum_{i,k=1}^{n}\partial_i\partial_k\Big(\nabla\frac{1}{|x|^{n-2}}*\Phi\Big)\Big\|_1\\ &\leq \sum_{i,k=1}^{n}\left(\Big\|\chi\nabla\frac{1}{|x|^{n-2}}*\partial_i\partial_k\Phi\Big\|_1+\Big\|\partial_i\partial_k\Big((1-\chi)\nabla\frac{1}{|x|^{n-2}}\Big)*\Phi\Big\|_1\right)\leq C.\end{aligned}\tag{2.51}$$

Where $\chi\in C_0^\infty(\mathbb{R}^n)$ is a cut-off function satisfying $\chi(x)=1$ for $|x|\leq 1$, and 0 for $|x|\geq 2$.
(b). $n=2$.

In this case the term $C(n)\Big\|\sum_{i,k=1}^{n}\partial_i\partial_k(\nabla\frac{1}{|x|^{n-2}}*\Phi)\Big\|_1$ ought to be replaced by the term $\frac{1}{2\pi}\Big\|\sum_{i,k=1}^{n}\partial_i\partial_k(\nabla\log|x|*\Phi)\Big\|_1$. Thus one obtains

$$\begin{aligned}2^{-(1+n/p)}\|\triangle_{-1}\nabla\pi_1(u,v)\|_p &\leq C\|u\|_{B^{1+n/p}_{p,1}}2^{-(1+n/p)}\|v\|_p\\ &\leq C\|u\|_{B^{1+n/p}_{p,1}}\|v\|_{B^{1+n/p}_{p,1}},\end{aligned}\tag{2.52}$$

and

$$\begin{aligned}2^{-n/p}\|\triangle_{-1}\nabla\pi_1(u,v)\|_p &\leq C\|u\|_{B^{1+n/p}_{p,1}}2^{-n/p}\|v\|_p\\ &\leq C\|u\|_{B^{1+n/p}_{p,1}}\|v\|_{B^{n/p}_{p,1}}.\end{aligned}\tag{2.53}$$

Combining estimates (2.48)-(2.49) with (2.52)-(2.53), we arrive at

$$\sum_{j=-1}^{\infty}2^{j(1+n/p)}\|\triangle_j\nabla\pi_1(u,v)\|_p\leq C\|u\|_{B^{1+n/p}_{p,1}}\|v\|_{B^{1+n/p}_{p,1}},\tag{2.54}$$

and

$$\sum_{j=-1}^{\infty}2^{jn/p}\|\triangle_j\nabla\pi_1(u,v)\|_p\leq C\|u\|_{B^{n/p}_{p,1}}\|v\|_{B^{1+n/p}_{p,1}}.\tag{2.55}$$

By taking $v(x,t)=u(x,t)$, the proof of Lemma 2.4 is complete. ∎

Noticing that $B^{n/p}_{p,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$, so $B^{n/p}_{p,1}(\mathbb{R}^n)$ is a Banach algebra. Thus we have the following lemma.

LEMMA 2.5. *If $u(x,t),\ v(x,t) \in B^{1+n/p}_{p,1}$ for $1 \le p \le \infty$ are any divergence free vector fields, we have*

$$\|(u\cdot\nabla)v\|_{B^{n/p}_{p,1}} \le C\|u\|_{B^{n/p}_{p,1}}\|v\|_{B^{1+n/p}_{p,1}}. \tag{2.56}$$

□

3. A Priori Estimate for the Approximate Equations. In this section, we shall construct an approximate solution sequence $\{(u^m(x,t),(b^m(x,t))\}$ with $m = 1,2\cdots$ for $u_0(x),\ b_0(x) \in B^{1+n/p}_{p,1}$ and $1 \le p \le \infty$. The approximate equations are as follows:

$$\partial_t u^m + (u^{m-1}\cdot\nabla)u^m - (b^{m-1}\cdot\nabla)b^m - \nabla\pi(u^{m-1},b^{m-1}) = 0 \tag{3.1}$$

$$\partial_t b^m + (u^{m-1}\cdot\nabla)b^m - (b^{m-1}\cdot\nabla)u^m = 0 \tag{3.2}$$

$$\mathrm{div} u^m = 0, \quad \mathrm{div} b^m = 0 \tag{3.3}$$

with initial data

$$u^m(x,0) = S_m u_0(x), \tag{3.4}$$

$$b^m(x,0) = S_m b_0(x). \tag{3.5}$$

Where

$$\nabla\pi(u^{m-1},b^{m-1}) = \sum_{i,j=1}^{n}((-\triangle)^{-1}\nabla\partial_i u_j^{m-1}\partial_j u_i^{m-1} + (-\triangle)^{-1}\nabla\partial_i b_j^{m-1}\partial_j b_i^{m-1}), \tag{3.6}$$

for $m = 1,2,\cdots$. we choose $u^0(x,t) = b^0(x,t) = 0$.

Considering the sequence of particle trajectory mapping $X^m_j(\alpha,t)$ defined by

$$\frac{\partial}{\partial t}X^m_j(\alpha,t) = (S_{j-2}(u^m - b^m))(X^m_j(\alpha,t),t), \tag{3.7}$$

$$X^m_j(\alpha,0) = \alpha. \tag{3.8}$$

for $m = 1,\ 2,\ \cdots$.

We now define the space

$$Y^a_T \triangleq C([0,T]; B^a_{p,1}(\mathbb{R}^n)). \tag{3.9}$$

In what follows, we take $a = \frac{n}{p}$ and $a = 1 + \frac{n}{p}$, respectively.

LEMMA 3.1. *Let $u_0(x),\ b_0(x) \in B^{1+n/p}_{p,1}$, $1 \le p \le \infty$. If $(u^m(x,t),b^m(x,t))$ is a solution to the Cauchy problem of the approximate equations (3.1)-(3.5). Then there exists a $T > 0$ so that the solution $(u^m(x,t),b^m(x,t))$ is bounded in $Y^{1+n/p}_T$ for $m = 0\ 1,\ 2,\ \cdots$. Precisely, there exists a constant C such that*

$$\|u^m\|_{Y^{1+n/p}_T} + \|b^m\|_{Y^{1+n/p}_T} \le C(\|u_0\|_{B^{1+n/p}_{p,1}} + \|u_0\|_{B^{1+n/p}_{p,1}}). \tag{3.10}$$

Proof. In order to prove the lemma, we take $\triangle_j$ on both sides of Eqs. (3.1) and (3.2), then add $(S_{j-2}u^{m-1}\cdot\nabla)\triangle_j u^m - (S_{j-2}b^{m-1}\cdot\nabla)\triangle_j u^m$ and $(S_{j-2}u^{m-1}\cdot$

$\nabla)\triangle_j b^m - (S_{j-2}b^{m-1}\cdot\nabla)\triangle_j b^m$ on the both sides of the result equations, respectively, we have

$$\begin{aligned}&\partial_t\triangle_j u^m + (S_{j-2}u^{m-1}\cdot\nabla)\triangle_j u^m - (S_{j-2}b^{m-1}\cdot\nabla)\triangle_j u^m\\ &= (S_{j-2}u^{m-1}\cdot\nabla)\triangle_j u^m - \triangle_j(u^{m-1}\cdot\nabla)u^m + \triangle_j(b^{m-1}\cdot\nabla)b^m \\ &-(S_{j-2}b^{m-1}\cdot\nabla)\triangle_j u^m - \triangle_j\nabla\pi(u^{m-1},b^{m-1}),\end{aligned}\tag{3.11}$$

$$\begin{aligned}&\partial_t\triangle_j b^m + (S_{j-2}u^{m-1}\cdot\nabla)\triangle_j b^m - (S_{j-2}b^{m-1}\cdot\nabla)\triangle_j b^m\\ &= (S_{j-2}u^{m-1}\cdot\nabla)\triangle_j b^m - \triangle_j(u^{m-1}\cdot\nabla)b^m + \triangle_j(b^{m-1}\cdot\nabla)u^m \\ &-(S_{j-2}b^{m-1}\cdot\nabla)\triangle_j b^m.\end{aligned}\tag{3.12}$$

Summing up Eqs. (3.11) and (3.12), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\triangle_j(u^m+b^m)(X_j^{m-1}(\alpha,t),t)\\ &= ((S_{j-2}u^{m-1}\cdot\nabla)\triangle_j u^m - \triangle_j(u^{m-1}\cdot\nabla)u^m)\\ &+((S_{j-2}u^{m-1}\cdot\nabla)\triangle_j b^m - \triangle_j(u^{m-1}\cdot\nabla)b^m)\\ &-((S_{j-2}b^{m-1}\cdot\nabla)\triangle_j b^m - \triangle_j(b^{m-1}\cdot\nabla)b^m)\\ &-((S_{j-2}b^{m-1}\cdot\nabla)\triangle_j u^m - \triangle_j(b^{m-1}\cdot\nabla)u^m) - \triangle_j\pi(u^{m-1},b^{m-1}).\end{aligned}\tag{3.13}$$

Using Lemma 2.3 and 2.4, taking the $B^{1+n/p}_{p,1}$ norm together with integrating the Eq. (3.13) with respect to t from the both sides, we arrive at

$$\begin{aligned}&\|u^m(t)+b^m(t)\|_{B^{1+n/p}_{p,1}} \le \|u^m(0)\|_{B^{1+n/p}_{p,1}} + \|b^m(0)\|_{B^{1+n/p}_{p,1}}\\ &+C\int_0^t(\|u^{m-1}(s)\|_{B^{1+n/p}_{p,1}} + \|b^{m-1}(s)\|_{B^{1+n/p}_{p,1}})(\|u^m(s)\|_{B^{1+n/p}_{p,1}} + \|b^m(s)\|_{B^{1+n/p}_{p,1}})\mathrm{d}s\\ &+C\int_0^t(\|u^{m-1}(s)\|^2_{B^{1+n/p}_{p,1}} + \|b^{m-1}(s)\|^2_{B^{1+n/p}_{p,1}})\mathrm{d}s,\end{aligned}\tag{3.14}$$

by the property of volume preserving of the mapping $X_j^{m-1}(\alpha,t)$. Taking the space-time norm on both sides of (3.14) , we have

$$\begin{aligned}&\|u^m(t)+b^m(t)\|_{Y_T^{1+n/p}} \le \|u(0)\|_{B^{1+n/p}_{p,1}} + \|b(0)\|_{B^{1+n/p}_{p,1}}\\ &+C(\|u^{m-1}\|_{Y_T^{1+n/p}} + \|b^{m-1}\|_{Y_T^{1+n/p}})\int_0^t(\|u^m(s)\|_{B^{1+n/p}_{p,1}} + \|b^m(s)\|_{B^{1+n/p}_{p,1}})\mathrm{d}s\\ &+CT(\|u^{m-1}\|^2_{Y_T^{1+n/p}} + \|b^{m-1}\|^2_{Y_T^{1+n/p}}).\end{aligned}\tag{3.15}$$

On the other hand, if we take another sequence of particle trajectory mapping $Y_j^m(\alpha,t)$ defined by

$$\frac{\partial}{\partial t}Y_j^m(\alpha,t) = (S_{j-2}(u^m+b^m))(Y_j^m(\alpha,t),t),\tag{3.16}$$

$$Y_j^m(\alpha,0)=\alpha.\tag{3.17}$$

for $m=1,\ 2,\ \cdots$. In the same way as that leading to estimate (3.15), we also arrive at

$$\begin{aligned}&\|u^m(t)-b^m(t)\|_{Y_T^{1+n/p}} \le \|u(0)\|_{B^{1+n/p}_{p,1}} + \|b(0)\|_{B^{1+n/p}_{p,1}}\\ &+C(\|u^{m-1}\|_{Y_T^{1+n/p}} + \|b^{m-1}\|_{Y_T^{1+n/p}})\int_0^t(\|u^m(s)\|_{B^{1+n/p}_{p,1}} + \|b^m(s)\|_{B^{1+n/p}_{p,1}})\mathrm{d}s\\ &+CT(\|u^{m-1}\|^2_{Y_T^{1+n/p}} + \|b^{m-1}\|^2_{Y_T^{1+n/p}}).\end{aligned}\tag{3.18}$$

Combining estimates (3.15) and (3.18), by the law of the parallelogram we have

$$\begin{aligned}&\|u^m(t)\|_{Y_T^{1+n/p}}+\|b^m(t)\|_{Y_T^{1+n/p}}\leq 2\|u(0)\|_{B_{p,1}^{1+n/p}}+2\|b(0)\|_{B_{p,1}^{1+n/p}}\\&+C(\|u^{m-1}\|_{Y_T^{1+n/p}}+\|b^{m-1}\|_{Y_T^{1+n/p}})\int_0^t(\|u^m(s)\|_{B_{p,1}^{1+n/p}}+\|b^m(s)\|_{B_{p,1}^{1+n/p}})\mathrm{d}s\\&+CT(\|u^{m-1}\|^2_{Y_T^{1+n/p}}+\|b^{m-1}\|^2_{Y_T^{1+n/p}}).\end{aligned}\tag{3.19}$$

Using Gronwall-type inequality, it follows that

$$\begin{aligned}&\|u^m(t)\|_{Y_T^{1+n/p}}+\|b^m(t)\|_{Y_T^{1+n/p}}\\&\leq 2\{\|u(0)\|_{B_{p,1}^{1+n/p}}+\|b(0)\|_{B_{p,1}^{1+n/p}}+CT(\|u^{m-1}\|^2_{Y_T^{1+n/p}}+\|b^{m-1}\|^2_{Y_T^{1+n/p}})\}\\&\quad\exp\{C(\|u^{m-1}\|_{Y_T^{1+n/p}}+\|b^{m-1}\|_{Y_T^{1+n/p}})\}\end{aligned}\tag{3.20}$$

Thus, by the standard induction arguments from estimate (3.20), one can arrive at

$$\|u^m(t)\|_{Y_T^{1+n/p}}+\|b^m(t)\|_{Y_T^{1+n/p}}\leq 4C(\|u(0)\|_{B_{p,1}^{1+n/p}}+\|b(0)\|_{B_{p,1}^{1+n/p}})e^{1/2},\tag{3.21}$$

for all $m\geq 0$, if we take

$$T\leq T_0=\frac{1}{16eC^2\left(\|u(0)\|_{B_{p,1}^{1+n/p}}+\|b(0)\|_{B_{p,1}^{1+n/p}}\right)}.\tag{3.22}$$

Thus we complete the proof of Lemma 3.1. □

Next, we prove that the solution sequence $(u^m(x,t),b^m(x,t))$ is a Cauchy sequence in the space $C([0,T];B_{p,1}^{n/p}(\mathbb{R}^n))$ for $m=0,\ 1,\ 2,\ \cdots$.

Lemma 3.2. *Let $u_0(x),\ b_0(x)\in B_{p,1}^{1+n/p}$, $1\leq p\leq\infty$. If $(u^m(x,t),b^m(x,t))$ is a solution to the Cauchy problem of the approximate equations (3.1)-(3.5). Then there exists a $T>0$ so that the solution $(u^m(x,t),b^m(x,t))$ is a Cauchy sequence in the space $Y_T^{n/p}$ for $m=0,\ 1,\ 2,\ \cdots$.*

Proof. Subtracting the m-th equations (3.1) and (3.2) from the $(m+1)$-th ones, we can obtain

$$\begin{aligned}&\frac{\partial}{\partial t}(u^{m+1}-u^m)+(u^m\cdot\nabla)((u^{m+1}-u^m))+((u^m-u^{m-1})\cdot\nabla)u^m\\&=(b^m\cdot\nabla)((b^{m+1}-b^m))+((b^m-b^{m-1})\cdot\nabla)b^m+\nabla\pi_1(u^m-u^{m-1},u^m)\\&\quad+\nabla\pi_1(u^{m-1},u^m-u^{m-1})+\nabla\pi_2(b^m-b^{m-1},b^m)+\nabla\pi_2(b^{m-1},b^m-b^{m-1}),\end{aligned}\tag{3.23}$$

$$\begin{aligned}&\frac{\partial}{\partial t}(b^{m+1}-b^m)+(u^m\cdot\nabla)((b^{m+1}-b^m))+((u^m-u^{m-1})\cdot\nabla)b^m\\&=(b^m\cdot\nabla)((u^{m+1}-u^m))+((b^m-b^{m-1})\cdot\nabla)u^m.\end{aligned}\tag{3.24}$$

Taking $\triangle_j$ on the both sides of (3.23) and (3.24), adding the term $(S_{j-2}(u^m-b^m)\cdot\nabla)\triangle_j(u^{m+1}-u^m)(X_j^m(\alpha,t),t)$ on both sides of (3.23) and adding the term $(S_{j-2}(u^m-b^m)\cdot\nabla)\triangle_j(b^{m+1}-b^m)(X_j^m(\alpha,t),t)$ on both sides of (3.24), then summing up the result equations we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\triangle_j(u^{m+1}-u^m+b^{m+1}-b^m)\\
=&\{(S_{j-2}u^m\cdot\nabla)\triangle_j(u^{m+1}-u^m)-\triangle_j((u^m\cdot\nabla)(u^{m+1}-u^m))\}\\
&-\{(S_{j-2}b^m\cdot\nabla)\triangle_j(b^{m+1}-b^m)-\triangle_j((b^m\cdot\nabla)(b^{m+1}-b^m))\}\\
&+\{(S_{j-2}u^m\cdot\nabla)\triangle_j(b^{m+1}-b^m)-\triangle_j((u^m\cdot\nabla)(b^{m+1}-b^m))\}\\
&-\{(S_{j-2}b^m\cdot\nabla)\triangle_j(u^{m+1}-u^m)-\triangle_j((b^m\cdot\nabla)(u^{m+1}-u^m))\}\\
&-\triangle_j((u^m-u^{m-1})\cdot\nabla)u^m+\triangle_j((b^m-b^{m-1})\cdot\nabla)b^m\\
&-\triangle_j((u^m-u^{m-1})\cdot\nabla)b^m+\triangle_j((b^m-b^{m-1})\cdot\nabla)u^m\\
&+\triangle_j\nabla\pi_1(u^m-u^{m-1},u^m)+\triangle_j\nabla\pi_1(u^{m-1},u^m-u^{m-1})\\
&+\triangle_j\nabla\pi_2(b^m-b^{m-1},b^m)+\triangle_j\nabla\pi_2(b^{m-1},b^m-b^{m-1}),
\end{aligned}\tag{3.25}$$

by the particle trajectory mapping $X_j^m(\alpha,t)$ defined in (3.7). Repeating the similar procedure from (3.13)-(3.15), one has

$$\begin{aligned}
&\|(u^{m+1}(t)-u^m(t))+(b^{m+1}(t)-b^m(t))\|_{B^{n/p}_{p,1}}\\
\le& C2^{-m}(\|u_0\|_{B^{1+n/p}_{p,1}}+\|b_0\|_{B^{1+n/p}_{p,1}})\\
&C\int_0^T(\|u^m(s)-u^{m-1}(s)\|_{B^{n/p}_{p,1}}+\|b^m(s)-b^{m-1}(s)\|_{B^{n/p}_{p,1}})\mathrm{d}s\\
&C\int_0^T(\|u^{m+1}(s)-u^m(s)\|_{B^{n/p}_{p,1}}+\|b^{m+1}(s)-b^m(s)\|_{B^{n/p}_{p,1}})\mathrm{d}s,
\end{aligned}\tag{3.26}$$

for $0<t\le T$.

If we take another sequence of particle trajectory mapping $Y_j^m(\alpha,t)$ defined by (3.16). Taking $\triangle_j$ on the both sides of equations (3.23) and (3.24), together with adding the term $(S_{j-2}(u^m-b^m)\cdot\nabla)\triangle_j(u^{m+1}-u^m)(X_j^m(\alpha,t),t)$ on the both sides of (3.23) and adding the term $(S_{j-2}(u^m-b^m)\cdot\nabla)\triangle_j(b^{m+1}-b^m)(X_j^m(\alpha,t),t)$ on the both sides of (3.24), then subtracting the result equations we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\triangle_j((u^{m+1}-u^m)-(b^{m+1}-b^m))\\
=&\{(S_{j-2}u^m\cdot\nabla)\triangle_j(u^{m+1}-u^m)-\triangle_j((u^m\cdot\nabla)(u^{m+1}-u^m))\}\\
&-\{(S_{j-2}b^m\cdot\nabla)\triangle_j(b^{m+1}-b^m)-\triangle_j((b^m\cdot\nabla)(b^{m+1}-b^m))\}\\
&-\{(S_{j-2}u^m\cdot\nabla)\triangle_j(b^{m+1}-b^m)-\triangle_j((u^m\cdot\nabla)(b^{m+1}-b^m))\}\\
&+\{(S_{j-2}b^m\cdot\nabla)\triangle_j(u^{m+1}-u^m)-\triangle_j((b^m\cdot\nabla)(u^{m+1}-u^m))\}\\
&-\triangle_j((u^m-u^{m-1})\cdot\nabla)u^m+\triangle_j((b^m-b^{m-1})\cdot\nabla)b^m\\
&+\triangle_j((u^m-u^{m-1})\cdot\nabla)b^m-\triangle_j((b^m-b^{m-1})\cdot\nabla)u^m\\
&+\triangle_j\nabla\pi_1(u^m-u^{m-1},u^m)+\triangle_j\nabla\pi_1(u^{m-1},u^m-u^{m-1})\\
&+\triangle_j\nabla\pi_2(b^m-b^{m-1},b^m)+\triangle_j\nabla\pi_2(b^{m-1},b^m-b^{m-1}).
\end{aligned}\tag{3.27}$$

Using the same procedure as above, one has

$$\begin{aligned}
&\|(u^{m+1}(t)-u^m(t))-(b^{m+1}(t)-b^m(t))\|_{B^{n/p}_{p,1}}\\
\le\ & C2^{-m}(\|u_0\|_{B^{1+n/p}_{p,1}}+\|b_0\|_{B^{1+n/p}_{p,1}})\\
& C\int_0^T(\|u^m(s)-u^{m-1}(s)\|_{B^{n/p}_{p,1}}+\|b^m(s)-b^{m-1}(s)\|_{B^{n/p}_{p,1}})\mathrm{d}s\\
& C\int_0^T(\|u^{m+1}(s)-u^m(s)\|_{B^{n/p}_{p,1}}+\|b^{m+1}(s)-b^m(s)\|_{B^{n/p}_{p,1}})\mathrm{d}s, \qquad (3.28)
\end{aligned}$$

for $0<t\le T$.

Combining estimates (3.26) and (3.28), by the law of the parallelogram one arrives at

$$\begin{aligned}
&\|u^{m+1}(t)-u^m(t)\|_{B^{n/p}_{p,1}}+\|b^{m+1}(t)-b^m(t)\|_{B^{n/p}_{p,1}}\\
\le\ & C_1 2^{-m}(\|u_0\|_{B^{1+n/p}_{p,1}}+\|b_0\|_{B^{1+n/p}_{p,1}})\\
& C_2\int_0^T(\|u^m(s)-u^{m-1}(s)\|_{B^{n/p}_{p,1}}+\|b^m(s)-b^{m-1}(s)\|_{B^{n/p}_{p,1}})\mathrm{d}s\\
& C_3\int_0^T(\|u^{m+1}(s)-u^m(s)\|_{B^{n/p}_{p,1}}+\|b^{m+1}(s)-b^m(s)\|_{B^{n/p}_{p,1}})\mathrm{d}s, \qquad (3.29)
\end{aligned}$$

Taking the $Y^{n/p}_T$ norm, and letting $T\le min(T_0,\ \frac{1}{2c_3})$, we have

$$\begin{aligned}
&\|u^{m+1}-u^m\|_{Y^{n/p}_T}+\|b^{m+1}-b^m\|_{Y^{n/p}_T}\\
\le\ & C_1 2^{-m}+C_2T(\|u^m-u^{m-1}\|_{Y^{n/p}_T}+\|b^m-b^{m-1}\|_{Y^{n/p}_T}). \qquad (3.30)
\end{aligned}$$

Thus we can deduce that

$$\|u^{m+1}-u^m\|_{Y^{n/p}_T}+\|b^{m+1}-b^m\|_{Y^{n/p}_T}\le C(m+C_1)2^{-m}<\varepsilon, \qquad (3.31)$$

if m is large enough. Therefore, $(u^m(x,t),b^m(x,t))$ is a Cauchy sequence. The proof of Lemma 3.2 follows. □

4. The Proof of the Main Theorem. In this section we prove the main theorem. In Section 3 we have constructed a sequence of solutions of the approximate equations (3.1)-(3.5), $(u^m(x,t),\ b^m(x,t))$ for $m=0,\ 1,\ \cdots$ which is bounded in the space $C([0,T];B^{1+n/p}_{p,1}(\mathbb{R}^n))$ and strongly convergent in the space $C([0,T];B^{n/p}_{p,1}(\mathbb{R}^n))$. In this section we prove that the strong limit $(u(x,t),\ b(x,t))$ of the sequence $(u^m(x,t),\ b^m(x,t))$ for $m=0,\ 1,\ \cdots$ is in $C([0,T];B^{1+n/p}_{p,1}(\mathbb{R}^n))$.

Since the sequence $(u^m(x,t),\ b^m(x,t))$ for $m=0,\ 1,\ \cdots$ is bounded in $C([0,T];B^{1+n/p}_{p,1}(\mathbb{R}^n))$, by means of the Banach-Alaogu theorem it weak∗-converges (up to a subsequence) to some vector function $(\tilde{u}(x,t),\ \tilde{b}(x,t))$ in $L^\infty([0,T];B^{1+n/p}_{p,1}(\mathbb{R}^n))$. But the sequence $(u^m(x,t),\ b^m(x,t))$ for $m=0,\ 1,\ \cdots$ converges in $C([0,T];B^{n/p}_{p,1}(\mathbb{R}^n))$, and, in particular, it is weak∗-convergent to $(u(x,t),\ b(x,t))$ in $C([0,T];B^{n/p}_{p,1}(\mathbb{R}^n))$. Thus we have $(u(x,t),\ b(x,t))=(\tilde{u}(x,t),\ \tilde{b}(x,t))$ by the uniqueness of the weak∗-limit, and $(u(x,t),\ b(x,t))\in$

$C([0,T]; B^{1+n/p}_{p,1}(\mathbb{R}^n))$. By the standard arguments it follows that $(u(x,t),\ b(x,t))$ satisfies the MHD system (1.1)-(1.5).

Finally we prove uniqueness of solutions to MHD system. Assume that $(u(x,t),\ b(x,t))$ and $(\tilde{u}(x,t),\ \tilde{b}(x,t))$ are two solutions of the MHD system (1.1)-(1.5) in $C([0,T]; B^{1+n/p}_{p,1}(\mathbb{R}^n))$ with the same initial data $(u_0(x), b_0(x)) \in B^{1+n/p}_{p,1}(\mathbb{R}^n))$. Taking the difference between the equations satisfied by $(u(x,t),\ b(x,t))$ and $(\tilde{u}(x,t),\ \tilde{b}(x,t))$, we find

$$\begin{aligned}
&\frac{\partial}{\partial t}(u-\tilde{u}) + (u\cdot\nabla)(u-\tilde{u}) + ((u-\tilde{u})\cdot\nabla)\tilde{u} \\
&= (b\cdot\nabla)(b-\tilde{b}) + ((b-\tilde{b})\cdot\nabla)\tilde{b} - \nabla\pi_1(u, u-\tilde{u}) \\
&\quad -\nabla\pi_1(u-\tilde{u},\tilde{u}) - \nabla\pi_2(b, b-\tilde{b}) - \nabla\pi_2(b-\tilde{b},\tilde{b}), \qquad (4.1)\\
&\frac{\partial}{\partial t}(b-\tilde{b}) + (u\cdot\nabla)(b-\tilde{b}) + ((u-\tilde{u})\cdot\nabla)\tilde{b} \\
&= (b\cdot\nabla)(u-\tilde{u}) + ((b-\tilde{b})\cdot\nabla)\tilde{u}, \qquad (4.2)
\end{aligned}$$

and

$$(u-\tilde{u})(x,0) = u_0(x) - \tilde{u}_0(x), \qquad (4.3)$$
$$(b-\tilde{b})(x,0) = b_0(x) - \tilde{b}_0(x). \qquad (4.4)$$

Where $(u_0(x), b_0(x))$ and $(\tilde{u}_0(x), \tilde{b}_0(x))$ are the initial data corresponding to the solutions $(u(x,t),\ b(x,t))$ and $(\tilde{u}(x,t),\ \tilde{b}(x,t))$, respectively. Using the same procedure as above, it can be deduced that

$$\begin{aligned}
&\|u-\tilde{u}\|_{B^{1+n/p}_{p,1}} + \|b-\tilde{b}\|_{B^{1+n/p}_{p,1}} \le 2(\|u_0-\tilde{u}_0\|_{B^{1+n/p}_{p,1}} + \|b_0-\tilde{b}_0\|_{B^{1+n/p}_{p,1}}) \\
&+C\int_0^T (\|(u,b)\|_{B^{1+n/p}_{p,1}} + \|(\tilde{u},\tilde{b})\|_{B^{1+n/p}_{p,1}})(\|u-\tilde{u}\|_{B^{1+n/p}_{p,1}} + \|b-\tilde{b}\|_{B^{1+n/p}_{p,1}})\mathrm{d}t, \quad (4.5)
\end{aligned}$$

for some $C>0$. Using the Gronwall inequality, it yields

$$\begin{aligned}
&\|u-\tilde{u}\|_{B^{1+n/p}_{p,1}} + \|b-\tilde{b}\|_{B^{1+n/p}_{p,1}} \\
&\le 2(\|u_0-\tilde{u}_0\|_{B^{1+n/p}_{p,1}} + \|b_0-\tilde{b}_0\|_{B^{1+n/p}_{p,1}}) \\
&\quad \exp\left\{C\int_0^T (\|(u,b)\|_{B^{1+n/p}_{p,1}} + \|(\tilde{u},\tilde{b})\|_{B^{1+n/p}_{p,1}})\mathrm{d}t\right\}. \qquad (4.6)
\end{aligned}$$

Thus we arrive at $u(x,t) = \tilde{u}(x,t)$ and $b(x,t) = \tilde{b}(x,t)$ for $0 \le t \le T$, and we have proved the uniqueness. The proof of the main theorem follows.

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