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MAGNETOTELLURICS IN LOCALLY-LAYERED RANDOM MEDIA*

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Abstract. Magnetotellurics (MT) is a method of determining the electrical resistivity of the earth's subsurface as a function of position by analyzing the electromagnetic (EM) field on the earth's surface. It is a passive method, in that ambient EM radiation is used as a source. In this paper we consider model subsurfaces for MT that contain small scale random stratification; that is, we introduce random microlayers and allow the earth's electrical properties to vary rapidly and randomly in space as the layer boundaries are crossed. The layers are not assumed to be plane, but are allowed to vary laterally in space in a direction that changes smoothly on the scale of an EM wavelength. By asymptotic analysis of the resulting stochastic differential equations with a small parameter we generalize previous results of White, Kohler and Srnka for plane layered media; we show that the resulting EM field may be approximated using a non-random effective medium theory, but with random corrections. These corrections are a gaussian random process which represents multiple scattering from the random microlayers. We show how the effective medium theory differs from the plane layered case, and derive a spatially varying correction for the EM field on the surface of the earth, which accounts for stratifications that are not planar.

Key words. electromagnetic exploration; geophysics; stochastic differential equations; stochastic limit theorems; effective medium theory; random scattering

AMS subject classifications. 86A25, 60H30, 78A48, 78A40

1. Introduction. In magnetotelluric (MT) surveys [14],[9], the electrical resistivity in the earth's subsurface is reconstructed from measurements on the surface of the earth, over a large range of frequencies, of the electrical impedance, *i.e.* the tangential electric field divided by the tangential magnetic field. MT is a passive exploration method, in that no active electromagnetic (EM) source is used. Instead, the EM fields that are measured are produced by ambient radiation from natural sources such as electrical activity in the ionosphere and distant thunderstorms.

Because the electrical resistivity of hydrocarbon-saturated rock is one to three orders of magnitude greater than that of the background, resistivity is an excellent indicator of the presence of hydrocarbons. Hence, a resistivity map of the the subsurface which has good spatial resolution would be invaluable for hydrocarbon exploration. However, since the earth is a conducting medium, EM waves in the earth are diffusive and only the longer EM wavelengths penetrate to depths that are important for exploration. It is therefore a challenging problem for any EM prospecting method to achieve good resolution in the reconstructed resistivity maps since they are obtained from primarily long wavelength information.

The mathematics of the MT inverse problem, *i.e.* reconstructing the subsurface resistivity from surface impedance data at a range of frequencies, is especially well developed for plane layered media [6], [11], [14], [4]. Because of sedimentary processes, plane layered media are often used as relatively simple models with some degree of geological validity. In this case, the measured impedance at the earth's surface does not depend on the exciting EM field but only on properties of the subsurface. Thus, it is not necessary to measure the ambient EM field in order to reconstruct

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the resistivity as a function of depth. Furthermore, the solution exists and is unique. However, the problem is ill-posed, in that many resistivity profiles can be shown to give approximately the same impedance data on the surface.

Because of the ill-posedness of the inverse problem, MT reconstruction is usually effected by some form of Tikhonov regularization [7]. As is usual in this method, the true solution is assumed to be smooth, or at least piecewise smooth, and it is the comparative lack of oscillation that is used to select the best candidate solution from among all solutions that match the impedance data well. Mathematically, one constructs an objective function that is a sum of two terms, one term representing the impedance data misfit and the other term representing the solution smoothness. The resistivity map is then varied iteratively to minimize the objective function, thus giving a trade-off between fitting the data and the assumed nonoscillatory nature of the solution. (Some recent developments in regularization theory have dealt with sharp geoelectric boundaries; see [13], [3], [15].)

Paradoxically, it is well known that resistivity does not vary smoothly with depth, but has order one oscillations on very small spatial scales. This is easily seen from plots of well logs, that is, resistivity measurements obtained from boreholes. These measurements typically have spatial resolution of less than a meter. In this paper and in our previous work [12], we explore the consequences of scattering from these small scale inhomogeneities when analyzing MT data.

In [12] it is shown that resistivities in plane layered media are well modeled by a random process which is rapidly-varying in depth, with a correlation length on the order of about 3 meters. The statistics have a slow spatial modulation with depth, so that the random process is nonstationary. Using this model, the solution of the forward problem is characterized as follows, using limit theorems for stochastic differential equations with a small parameter.

The surface impedances are given,to leading order, by effective medium theory, and therefore depend only on the smoothly modulated moving average of the conductivity versus depth profile. The existence of an effective medium explains how regularization methods can obtain a reasonable answer to the inverse problem. The regularized solution is close to the moving average of the conductivity, which does not oscillate rapidly with depth.

In [12] it was also shown that the next order term in the surface impedance, after effective medium theory, is a gaussian random process, with statistics that can be characterized using a central limit theorem of Khasminskii [2] for stochastic differential equations with a small parameter. This random process represents scattering noise, and is inherent to the MT method. It was shown that the magnitude of this effect is comparable to the magnitude of all other MT survey errors combined. The consequences of this source of error for the detection of hydrocarbons was then explored in some detail.

In the present paper we extend the results of [12] to media that are rapidly varying but are not plane layered. It is assumed that there are rapid oscillations of electrical resistivity in a direction that is not strictly vertical, but can vary in space. That is, our model is a smooth deformation of a plane layered medium. This is a more realistic geological model, since it allows for the sediment thicknesses and layer boundaries to vary laterally in space. We also allow for slow lateral variations of electrical rsistivity within the layers themselves.

If the lateral variations are not too severe, we demonstrate below the robustness of the plane layered theory. We show that the impedance is well approximated by effective medium theory, with an additive gaussian error. Furthermore, the statistics of the gaussian error is identical to that computed in [12]. The signature of the layer deformations is obtained as a new additive term in the effective medium theory, and is completely characterized below. In particular, we compute explicit formulas for this new term, in the case of a randomly homogeneous half space. These formulas complement the explicit formulas obtained in [12] for the random statistics of the impedance fluctuations in scattering from a randomly homogeneous half space.

The organization of this paper is as follows: In Section 2 we introduce a convenient coordinate system for locally-layered media, and write Maxwell's equations in this coordinate system. Uniqueness of the coordinate system, under appropriate hypotheses, is demonstrated in the Appendix. The locally-layered MT problem is then formulated in Section 3, where a small parameter and appropriate length scales are introduced. A perturbation hierarchy is derived in Section 4, and the general solution for the plane-layered case is identified as part of this hierarchy, thus making contact with the previous plane-layered theory. However, a new term is also identified, which embodies the effects of the 3-dimensional geometry. This new term is completely characterized in Section 5 by a two-point linear boundary value problem for a system of ordinary differential equations. The resulting equations show how waves are scattered into all angles by undulations in the layering. In Section 6 we consider the example of a locally-layered homogeneous random conducting half-space. In previous work on plane-layered media, the statistics of the homogeneous half-space were given by explicit algebraic formulas. Here we derive complementary formulas, showing the new effects caused by the layering not being plane. In Section 7 we show the corresponding effect on the surface impedance tensor, which is the primary measurement used in MT interpretation. An illustrative case is evaluated numerically in Section 8 and conclusions presented in Section 9.

This paper is dedicated to Professor George Papanicolaou, on the occasion of his sixtieth birthday. Among his numerous accomplishments, Professor Papanicolaou has for decades been a leader in the mathematics of waves in random media; in particular, he has pioneered the methods of analysis which we use in this manuscript. It is a great pleasure for us to participate in this celebration of his work.

2. Maxwell's Equations in Locally-Layered Coordinates. We assume a layered earth occupying the half-space $x_3 < 0$, with the boundaries of the layers given by the level curves of a smooth function $\overline{X}_3(x_1, x_2, x_3)$. For simplicity, we assume a flat topography at the surface so that

$$\overline{X}_3(x_1, x_2, 0) = 0. \tag{2.1}$$

We will introduce a coordinate transformation

$$\mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) \longrightarrow \mathbf{x}' = (\mathrm{x}'_1, \mathrm{x}'_2, \mathrm{x}'_3)$$

by choosing $\overline{X}_1(x_1, x_2, x_3), \overline{X}_2(x_1, x_2, x_3)$ so that $\big(\overline{X}_1(\cdot), \overline{X}_2(\cdot), \overline{X}_3(\cdot)\big) : R^3_- \longrightarrow R^3_-$. Since only $\overline{X}_3$ is given, there appears to be some arbitrariness in the choice of $\overline{X}_1, \overline{X}_2$. However, the choice becomes unique if it is assumed that gradients in the direction of the layers are proportional to $\frac{\partial}{\partial x'_3}$ in the new coordinate system, and that

$$\overline{X}_j(x_1, x_2, 0) = x_j, \qquad j = 1, 2. \tag{2.2}$$

Then, application of the chain rule gives

$$\nabla \overline{X}_j \cdot \nabla \overline{X}_3 = 0, \qquad j = 1, 2, \tag{2.3}$$

whence, in the new coordinate system

$$\frac{\nabla\overline{X}_3}{\left|\nabla\overline{X}_3\right|}\cdot\nabla = \left|\nabla\overline{X}_3\right|\frac{\partial}{\partial x_3'}. \tag{2.4}$$

Note that in general $\nabla\overline{X}_1\cdot\nabla\overline{X}_2 \neq 0$ so that in general (x_1', x_2', x_3') are not orthogonal curvilinear coordinates. In the Appendix it is shown constructively that equations (2.2) and (2.3) determine $\overline{X}_1, \overline{X}_2$ uniquely.

We next write Maxwell's equations in this coordinate system. Let

$$\begin{aligned}\alpha_{jl} &= \nabla\overline{X}_j\cdot\nabla\overline{X}_l, \qquad j,l = 1,2,3 \\ \Delta &= \alpha_{11}\alpha_{22} - (\alpha_{12})^2 \\ \overline{\alpha} &= \frac{\nabla\overline{X}_3}{|\nabla\overline{X}_3|^2}\cdot(\nabla\overline{X}_1\wedge\nabla\overline{X}_2).\end{aligned} \tag{2.5}$$

Note that $\alpha_{13} = \alpha_{23} = 0$. Also, it is assumed that $|\nabla\overline{X}_3| \neq 0, \overline{\alpha} \neq 0$, so that the jacobian of the coordinate transformation does not vanish. The electric and magnetic field vectors are written as

$$\begin{aligned}\mathbf{E} &= \sum_{j=1}^{3} E_j\nabla\overline{X}_j \\ \mathbf{H} &= \sum_{j=1}^{3} H_j\nabla\overline{X}_j.\end{aligned} \tag{2.6}$$

Note that E_1, E_2, H_1, H_2 reduce to the usual Cartesian components on the surface $x_3 = 0$. Furthermore, these components are continuous throughout $x_3 < 0$ if it is assumed that all discontinuities in the material parameters are in the layering direction $\nabla\overline{X}_3$, so that $\nabla\overline{X}_1, \nabla\overline{X}_2$ are tangential to any surface of discontinuity.

Note from the chain rule that the curl of $\mathbf{E}$ is

$$\nabla\wedge\mathbf{E} = \sum_{j,l=1}^{3}\frac{\partial E_j}{\partial x_l'}\left(\nabla\overline{X}_l\wedge\nabla\overline{X}_j\right) \tag{2.7}$$

with a similar expression for $\nabla\wedge\mathbf{H}$. From equation (2.7)

$$\nabla\overline{X}_3\cdot(\nabla\wedge\mathbf{E}) = \overline{\alpha}\alpha_{33}\left[\frac{\partial E_2}{\partial x_1'} - \frac{\partial E_1}{\partial x_2'}\right]. \tag{2.8}$$

Also,

$$\nabla\overline{X}_j\wedge(\nabla\wedge\mathbf{E}) = \sum_{l,m=1}^{3}\left[\frac{\partial E_l}{\partial x_m'} - \frac{\partial E_m}{\partial x_l'}\right]\alpha_{jl}\nabla\overline{X}_m. \tag{2.9}$$

Putting $\alpha_{13} = \alpha_{23} = 0$ into equation (2.9) yields that, for $j = 1,2$

$$\nabla\overline{X}_3\cdot\left[\nabla\overline{X}_j\wedge(\nabla\wedge\mathbf{E})\right] = \alpha_{33}\left[\left(\frac{\partial E_1}{\partial x_3'} - \frac{\partial E_3}{\partial x_1'}\right)\alpha_{j1} + \left(\frac{\partial E_2}{\partial x_3'} - \frac{\partial E_3}{\partial x_2'}\right)\alpha_{j2}\right], \quad j = 1,2. \tag{2.10}$$

Again, we have similar expressions as (2.8) - (2.10) with $\mathbf{E}$ replaced by $\mathbf{H}$.

Maxwell's equations are

$$\begin{aligned} \nabla \wedge \mathbf{E} &= i\omega\mu\mathbf{H} \\ \nabla \wedge \mathbf{H} &= (\sigma - i\omega\epsilon)\mathbf{E}. \end{aligned} \tag{2.11}$$

Dotting these equations with $\nabla\overline{X}_3$ and using equation (2.8) gives

$$H_3 = \frac{\overline{\alpha}}{i\omega\mu}\left[\frac{\partial E_2}{\partial x_1'} - \frac{\partial E_1}{\partial x_2'}\right] \tag{2.12}$$

$$E_3 = \frac{\overline{\alpha}}{(\sigma - i\omega\epsilon)}\left[\frac{\partial H_2}{\partial x_1'} - \frac{\partial H_1}{\partial x_2'}\right]. \tag{2.13}$$

We next apply the operator $\nabla\overline{X}_3 \cdot \left(\nabla\overline{X}_j \wedge \cdot\right), \;\; j = 1, 2$ to equation (2.11) and use (2.10) to get that

$$\begin{aligned} &\alpha_{11}\left(\tfrac{\partial E_1}{\partial x_3'} - \tfrac{\partial E_3}{\partial x_1'}\right) + \alpha_{12}\left(\tfrac{\partial E_2}{\partial x_3'} - \tfrac{\partial E_3}{\partial x_2'}\right) = i\omega\mu\overline{\alpha}H_2 \\ &\alpha_{11}\left(\tfrac{\partial H_1}{\partial x_3'} - \tfrac{\partial H_3}{\partial x_1'}\right) + \alpha_{12}\left(\tfrac{\partial H_2}{\partial x_3'} - \tfrac{\partial H_3}{\partial x_2'}\right) = (\sigma - i\omega\epsilon)\overline{\alpha}E_2 \\ &\alpha_{22}\left(\tfrac{\partial E_2}{\partial x_3'} - \tfrac{\partial E_3}{\partial x_2'}\right) + \alpha_{12}\left(\tfrac{\partial E_1}{\partial x_3'} - \tfrac{\partial E_3}{\partial x_1'}\right) = -i\omega\mu\overline{\alpha}H_1 \\ &\alpha_{22}\left(\tfrac{\partial H_2}{\partial x_3'} - \tfrac{\partial H_3}{\partial x_2'}\right) + \alpha_{12}\left(\tfrac{\partial H_1}{\partial x_3'} - \tfrac{\partial H_3}{\partial x_1'}\right) = -(\sigma - i\omega\epsilon)\overline{\alpha}E_1. \end{aligned} \tag{2.14}$$

Now solving equations (2.14) for $\frac{\partial E_j}{\partial x_3'}, \frac{\partial H_j}{\partial x_3'}, \;\; j = 1, 2$ gives, after substitution for E_3, H_3 from equations (2.12), (2.13)

$$\begin{aligned} \tfrac{\partial E_1}{\partial x_3'} &= \tfrac{i\omega\mu\overline{\alpha}}{\Delta}\left[\alpha_{12}H_1 + \alpha_{22}H_2\right] + \tfrac{\partial}{\partial x_1'}\left\{\tfrac{\overline{\alpha}}{(\sigma - i\omega\epsilon)}\left[\tfrac{\partial H_2}{\partial x_1'} - \tfrac{\partial H_1}{\partial x_2'}\right]\right\} \\ \tfrac{\partial H_2}{\partial x_3'} &= \tfrac{(\sigma - i\omega\epsilon)\overline{\alpha}}{\Delta}\left[-\alpha_{11}E_1 - \alpha_{12}E_2\right] + \tfrac{\partial}{\partial x_2'}\left\{\tfrac{\overline{\alpha}}{i\omega\mu}\left[\tfrac{\partial E_2}{\partial x_1'} - \tfrac{\partial E_1}{\partial x_2'}\right]\right\} \\ \tfrac{\partial E_2}{\partial x_3'} &= \tfrac{i\omega\mu\overline{\alpha}}{\Delta}\left[-\alpha_{11}H_1 - \alpha_{12}H_2\right] + \tfrac{\partial}{\partial x_2'}\left\{\tfrac{\overline{\alpha}}{(\sigma - i\omega\epsilon)}\left[\tfrac{\partial H_2}{\partial x_1'} - \tfrac{\partial H_1}{\partial x_2'}\right]\right\} \\ \tfrac{\partial H_1}{\partial x_3'} &= \tfrac{(\sigma - i\omega\epsilon)\overline{\alpha}}{\Delta}\left[\alpha_{12}E_1 + \alpha_{22}E_2\right] + \tfrac{\partial}{\partial x_1'}\left\{\tfrac{\overline{\alpha}}{i\omega\mu}\left[\tfrac{\partial E_2}{\partial x_1'} - \tfrac{\partial E_1}{\partial x_2'}\right]\right\}. \end{aligned} \tag{2.15}$$

Equations (2.15) are Maxwell's equations for the electric and magnetic fields tangential to the layering surfaces. As remarked, these components are continuous across interfaces, even if the material parameters, σ, ϵ, μ are discontinuous in the layering direction. The normal components E_3, H_3 are given in terms of the tangential components by equations (2.12) and (2.13).

3. Problem Formulation. The problem we consider is that of an electromagnetic plane wave normally incident upon the planar interface $x_3' = 0$ from the free space region $x_3' > 0$. In this upper half space the primed and unprimed coordinates are identical. A conducting (locally layered) macrolayer occupies the region $-L' < x_3' < 0$ and the half space $-\infty < x_3' < -L'$ is assumed to be a rock basement. The region $-L' < x_3' < 0$ is assumed to consist of thin undulating microlayers, with boundaries defined by $x_3' = \textit{constant}$ surfaces. The conductivity undergoes $O(1)$ variations from one microlayer to the next. Thus, σ is a rapidly varying function of x_3' within the macrolayer; we model this rapid variation as a stochastic process, using a small parameter δ. In the rock basement, conductivity is assumed to be constant. Therefore, the conductivity has the following functional form.

$$\sigma = \begin{cases} 0, & x_3' > 0 \\ \sigma\left(x_1', x_2', x_3', \frac{x_3'}{\delta}\right), & -L' \leq x_3' \leq 0 \\ \sigma_b & -\infty < x_3' < -L' \end{cases} . \tag{3.1}$$

Note that in the macrolayer the first three arguments of σ allow for a possible deterministic modulation of the conductivity; the fourth argument embodies the rapid random fluctuations.

Maxwell's equations, governing the propagation, are given in (2.15). Note in particular the $\sigma - i\omega\epsilon$ term, where ϵ represents the (free space) permittivity. In $x_3' > 0$, $\sigma = 0$ and this term reduces to $-i\omega\epsilon$. In $x_3' \leq 0$, for frequencies of interest in MT applications, $\sigma >> |i\omega\epsilon|$ and we will make the customary approximation $\sigma - i\omega\epsilon \approx \sigma$ in this region. Above the earth, in $x_3' > 0$, the incident field is a normally incident plane wave. Across the interfaces $x_3' = 0$ and $x_3' = -L'$, the field components tangential to the interfaces must be continuous. Recall, noting equations (2.6), that the dependent variables appearing in equations (2.15) differ from the actual field components since the vectors $\nabla \overline{X}_j$ are generally not unit vectors below the earth's surface. However, since the $\nabla \overline{X}_j$ are continuous vector functions, we can infer that the four transverse field components appearing in (2.15) must each be continuous across the interfaces $x_3' = 0$ and $x_3' = -L'$. The focus of our attention will be the slab region $-L' < x_3' < 0$.

In the free space region above the slab, an upward-propagating radiation condition must be imposed upon the reflected fields. A downward-propagating radiation condition must likewise be imposed upon the fields in the basement region, $-\infty < x_3' < -L'$; the fields must vanish as $x_3' \downarrow -\infty$.

Note that both σ and σ^{-1} appear in equations (2.15). Let $\langle\cdot\rangle$ denote mean or expected value and let

$$\langle\sigma\rangle = \sigma_A(x_1', x_2', x_3'), \qquad \sigma = \sigma_A(x_1', x_2', x_3') + \nu\left(x_1', x_2', x_3', \frac{x_3'}{\delta}\right), \tag{3.2}$$
$$\langle\sigma^{-1}\rangle = \sigma_H^{-1}(x_1', x_2', x_3'), \qquad \sigma^{-1} = \sigma_H^{-1}(x_1', x_2', x_3') + \eta\left(x_1', x_2', x_3', \frac{x_3'}{\delta}\right).$$

Thus ν and η represent zero mean random fluctuations of the conductivity and its reciprocal, respectively. Since $\delta^{-\frac{1}{2}}\nu$ and $\delta^{-\frac{1}{2}}\eta$ possess $O(1)$ probabilistic limits as $\delta \to 0$, these fluctuation terms are relatively small. Leading order behavior is governed by effective medium theory ([12]), wherein we replace σ and σ^{-1} in equations (2.15) by their respective mean values, σ_A and σ_H^{-1}. Note that the resulting effective medium is a deterministic anisotropic medium.

The full problem is a generalization of the type of problem studied by Khasminskii and others, ([2]). This prior work suggests that its solution will consist of the deterministic solution of the effective medium problem to which is added an $O\left(\delta^{\frac{1}{2}}\right)$ random term arising from the rapid random fluctuations. Our objective is not to study this problem in its present generality; rather, we want to assess and demonstrate the robustness of the plane layering model. Therefore, we shall consider the case where the layering undulations are small and conductivity varies slowly in the transverse primed coordinates, *i.e.* where the conductivity varies slowly as one moves along any given undulating microlayer.

Two additional length scales are needed to quantify the ideas of "small undulations" and "slow lateral variations". How these scales relate to the small length parameter δ already introduced will determine the importance of these effects relative to the $O\left(\delta^{\frac{1}{2}}\right)$ effect that we anticipate arising from the random layering.

Our goal is to identify a scaling regime in which the random fluctuations, layering deformations and lateral variations all introduce effects that are comparable in size. Therefore, if slow lateral variations are to contribute comparably, they must occur on a $\delta^{-\frac{1}{2}}$ length scale; that is, σ will be a function of $\delta^{\frac{1}{2}}x_1', \delta^{\frac{1}{2}}x_2'$. Therefore, we modify equations (3.2) as follows.

$$\begin{aligned} \langle\sigma\rangle &= \sigma_A, \qquad \sigma = \sigma_A\left(\delta^{\frac{1}{2}}x_1', \delta^{\frac{1}{2}}x_2', x_3'\right) + \nu\left(\delta^{\frac{1}{2}}x_1', \delta^{\frac{1}{2}}x_2', x_3', \frac{x_3'}{\delta}\right), \\ \langle\sigma^{-1}\rangle &= \sigma_H^{-1}, \qquad \sigma^{-1} = \sigma_H^{-1}\left(\delta^{\frac{1}{2}}x_1', \delta^{\frac{1}{2}}x_2', x_3'\right) + \eta\left(\delta^{\frac{1}{2}}x_1', \delta^{\frac{1}{2}}x_2', x_3', \frac{x_3'}{\delta}\right). \end{aligned} \tag{3.3}$$

We scale the layering undulations to produce a comparable effect. Small $O\left(\delta^{\frac{1}{2}}\right)$ undulations are introduced by assuming the locally layered coordinates have the form

$$x_j' = \overline{X}_j\left(x_1, x_2, x_3\right) = x_j + \delta^{\frac{1}{2}}\Psi_j\left(x_1, x_2, x_3\right), \qquad j = 1, 2, 3. \tag{3.4}$$

In particular, the function $\Psi_3\left(x_1, x_2, x_3\right)$ is prescribed subject to the constraint $\Psi_3\left(x_1, x_2, 0\right) = 0$. Let

$$\phi\left(x_1, x_2, x_3\right) = -\int_0^{x_3}\Psi_3\left(x_1, x_2, s\right)ds. \tag{3.5}$$

Then, orthogonality conditions (2.3) and equations (3.4), (3.5) imply that

$$\begin{aligned} x_j' &= x_j + \delta^{\frac{1}{2}}\phi_j + O\left(\delta\right), \qquad j = 1, 2 \\ x_3' &= x_3 - \delta^{\frac{1}{2}}\phi_3 \end{aligned} \tag{3.6}$$

where the subscripts applied to ϕ refer to partial derivatives. Note that

$$\phi\left(x_1, x_2, 0\right) = \phi_3\left(x_1, x_2, 0\right) = 0 \tag{3.7}$$

and that, to leading order, we can view ϕ and its partial derivatives as functions of either the primed or unprimed coordinates. Noting equations (2.5), it follows, in turn, that

$$\begin{aligned} \alpha_{ij} &= \delta_{ij} + 2\delta^{\frac{1}{2}}\phi_{ij} + O\left(\delta\right), \qquad i, j = 1, 2 \\ \alpha_{33} &= 1 - 2\delta^{\frac{1}{2}}\phi_{33} \\ \Delta &= 1 + 2\delta^{\frac{1}{2}}\left(\phi_{11} + \phi_{22}\right) + O\left(\delta\right) \\ \overline{\alpha} &= 1 + \delta^{\frac{1}{2}}\left(\phi_{11} + \phi_{22} + \phi_{33}\right) + O\left(\delta\right). \end{aligned} \tag{3.8}$$

Before assessing the impact of this scaling upon Maxwell's equations (2.15), we comment briefly upon the relevance of this scaling to MT applications. Nominal macrolayer thicknesses exist on the order of a kilometer, say 1 kilometer for definiteness; for representative values of conductivity within the earth and a frequency of 1 Hertz, this distance is comparable to a skin depth of penetration of an impinging electromagnetic wave. Well-log data likewise suggest a correlation length of conductivity fluctuations to be on the order of 3 meters. Therefore, the parameter $\delta \approx 0.003$ in such applications.

With this value of δ, layering undulations have a nominal amplitude of 60 meters; over a lateral distance of a kilometer a thin conducting microlayer will typically meander this vertical distance. Likewise, if one were to follow one such undulating microlayer over a lateral extent of a kilometer, the value of its conductivity would change by roughly 6%. It is important to note that this small change occurs as one moves along the undulating microlayer. If one moves horizontally, that is, parallel to the flat surface, $O(1)$ changes occur more rapidly, as soon as the horizontal path crosses a microlayer boundary.

4. Perturbation Expansion. We introduce a formal perturbation expansion based upon the scalings introduced in the last section. As previously noted, we neglect the $i\omega\epsilon$ term in equations (2.15) within the conducting macrolayer. The equations for the transverse electromagnetic fields within this slab then assume the form

$$\frac{\partial}{\partial x_3'}\begin{bmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{bmatrix} = L_0 \begin{bmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{bmatrix} + \delta^{\frac{1}{2}}\left(\delta^{-\frac{1}{2}} L_1 \begin{bmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{bmatrix}\right) \tag{4.1}$$

where

$$\mathbf{E}_t = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \qquad \mathbf{H}_t = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}. \tag{4.2}$$

$$L_0 = \begin{bmatrix} 0 & \frac{i\omega\mu\overline{\alpha}}{\Delta}A \\ \frac{\sigma_A\overline{\alpha}}{\Delta}A & 0 \end{bmatrix} + \begin{bmatrix} 0 & D\left(\frac{\overline{\alpha}}{\sigma_H}\right) \\ D\left(\frac{\overline{\alpha}}{i\omega\mu}\right) & 0 \end{bmatrix} \tag{4.3}$$

with

$$A = \begin{bmatrix} \alpha_{12} & \alpha_{22} \\ -\alpha_{11} & -\alpha_{12} \end{bmatrix}, \qquad D(\cdot) = \begin{bmatrix} -\frac{\partial}{\partial x_1'}(\cdot)\frac{\partial}{\partial x_2'} & \frac{\partial}{\partial x_1'}(\cdot)\frac{\partial}{\partial x_1'} \\ -\frac{\partial}{\partial x_2'}(\cdot)\frac{\partial}{\partial x_2'} & \frac{\partial}{\partial x_2'}(\cdot)\frac{\partial}{\partial x_1'} \end{bmatrix}$$

and

$$\delta^{-\frac{1}{2}} L_1 = \delta^{-\frac{1}{2}}\begin{bmatrix} 0 & 0 \\ \frac{\nu\overline{\alpha}}{\Delta}A & 0 \end{bmatrix} + \begin{bmatrix} 0 & D(\overline{\alpha}\eta) \\ 0 & 0 \end{bmatrix}. \tag{4.4}$$

The operator L_0 is deterministic and characterizes an anisotropic effective medium; the operator L_1 is a zero mean random operator that has rapid random variation and slower deterministic modulation in the depth (x_3') direction. Note that the transverse coordinate partial derivatives impact the fluctuation term η only through its slow lateral variation. the rapid random fluctuations (that is, the $\frac{x_3'}{\delta}$ dependence) is unaffected by these derivatives.

We now consider the simplifications introduced into these operators by the small undulations assumption. The operator L_0 admits a formal perturbation expansion

$$L_0 = L_{00} + \delta^{\frac{1}{2}} L_{01} + \delta^{\frac{1}{2}} L_{02} + O(\delta) \tag{4.5}$$

where

$$L_{00}=\begin{bmatrix} 0 & 0 & 0 & i\omega\mu \\ 0 & 0 & -i\omega\mu & 0 \\ 0 & \sigma_A & 0 & 0 \\ -\sigma_A & 0 & 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & -\sigma_H^{-1}\partial^2_{x_1'x_2'} & \sigma_H^{-1}\partial^2_{x_1'x_1'} \\ 0 & 0 & -\sigma_H^{-1}\partial^2_{x_2'x_2'} & \sigma_H^{-1}\partial^2_{x_2'x_1'} \\ -(i\omega\mu)^{-1}\partial^2_{x_1'x_2'} & (i\omega\mu)^{-1}\partial^2_{x_1'x_1'} & 0 & 0 \\ -(i\omega\mu)^{-1}\partial^2_{x_2'x_2'} & (i\omega\mu)^{-1}\partial^2_{x_2'x_1'} & 0 & 0 \end{bmatrix}$$

$$L_{01}=\begin{bmatrix} 0 & 0 & i2\omega\mu\phi_{12} & i\omega\mu\Phi_1 \\ 0 & 0 & -i\omega\mu\Phi_2 & -i2\omega\mu\phi_{12} \\ 2\sigma_A\phi_{12} & \sigma_A\Phi_1 & 0 & 0 \\ -\sigma_A\Phi_2 & -2\sigma_A\phi_{12} & 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} 0 & -\sigma_H^{-1}D(\phi_{11}+\phi_{22}+\phi_{33}) \\ (i\omega\mu)^{-1}D(\phi_{11}+\phi_{22}+\phi_{33}) & 0 \end{bmatrix}$$

$$L_{02}=\begin{bmatrix} 0 & 0 & -\left(\sigma_H^{-1}\right)_1\partial_{x_2'} & \left(\sigma_H^{-1}\right)_1\partial_{x_1'} \\ 0 & 0 & -\left(\sigma_H^{-1}\right)_2\partial_{x_2'} & \left(\sigma_H^{-1}\right)_2\partial_{x_1'} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.6}$$

where, for brevity, $\Phi_1=-\phi_{11}+\phi_{22}+\phi_{33}$ and $\Phi_2=\phi_{11}-\phi_{22}+\phi_{33}$; in the expression for L_{02}, the notation $\left(\sigma_H^{-1}\right)_j$ denotes the partial derivative of $(\sigma_H^{-1})_j$ with respect to x_j'. The deterministic operator L_0 therefore expands into the sum of a leading order term and two perturbation operators. The operators L_{01} and L_{02} are consequences of the undulations and slow lateral variations, respectively.

The zero mean random operator $\delta^{-\frac{1}{2}}L_1$ admits the leading order simplification

$$\begin{aligned} \delta^{-\frac{1}{2}}L_1 &= \begin{bmatrix} 0 & 0 & -\delta^{-\frac{1}{2}}\eta\partial^2_{x_1'x_2'} & \delta^{-\frac{1}{2}}\eta\partial^2_{x_1'x_1'} \\ 0 & 0 & -\delta^{-\frac{1}{2}}\eta\partial^2_{x_2'x_2'} & \delta^{-\frac{1}{2}}\eta\partial^2_{x_2'x_1'} \\ 0 & \delta^{-\frac{1}{2}}\nu & 0 & 0 \\ -\delta^{-\frac{1}{2}}\nu & 0 & 0 & \end{bmatrix} + O\left(\delta^{\frac{1}{2}}\right) \\ &\equiv \delta^{-\frac{1}{2}}L_{10}+O\left(\delta^{\frac{1}{2}}\right). \end{aligned} \tag{4.7}$$

We use L_{10} to denote the leading order random operator.

Consider the macrolayer region $-L'<x_3'<0$. If we now expand the fields in a similar perturbation expansion,

$$\begin{bmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{t0} \\ \mathbf{H}_{t0} \end{bmatrix} + \delta^{\frac{1}{2}}\begin{bmatrix} \mathbf{E}_{t1} \\ \mathbf{H}_{t1} \end{bmatrix} + \cdots \tag{4.8}$$

we obtain

$$\frac{\partial}{\partial_{x_3'}}\begin{bmatrix} \mathbf{E}_{t0} \\ \mathbf{H}_{t0} \end{bmatrix} = L_{00}\begin{bmatrix} \mathbf{E}_{t0} \\ \mathbf{H}_{t0} \end{bmatrix} \tag{4.9}$$

$$\frac{\partial}{\partial_{x_3'}}\begin{bmatrix} \mathbf{E}_{t1} \\ \mathbf{H}_{t1} \end{bmatrix} = L_{00}\begin{bmatrix} \mathbf{E}_{t1} \\ \mathbf{H}_{t1} \end{bmatrix} + L_{01}\begin{bmatrix} \mathbf{E}_{t0} \\ \mathbf{H}_{t0} \end{bmatrix} + L_{02}\begin{bmatrix} \mathbf{E}_{t0} \\ \mathbf{H}_{t0} \end{bmatrix} + \left(\delta^{-\frac{1}{2}}L_{10}\right)\begin{bmatrix} \mathbf{E}_{t0} \\ \mathbf{H}_{t0} \end{bmatrix}.$$

The continuity of the transverse fields across the macrolayer boundaries requires that $\mathbf{E}_{tj}, \mathbf{H}_{tj}, j = 0, 1$ be continuous across $x_3' = 0$ and $x_3' = -L'$. These fields must also satisfy appropriate radiation conditions in the semi-infinite regions above and below the macrolayer.

An important goal in our study of this perturbation hierarchy will be to make contact with the problem considered in [12]. In particular, we want to determine the extent to which the random plane-layered model studied in [12] remains robust under deformation. To address this question, we now simplify the model somewhat and assume that the effective medium conductivities σ_A and σ_H^{-1} are functions of x_3' only and that the corresponding zero mean fluctuations are functions of x_3' and $\frac{x_3'}{\delta}$ only. Thus, the random conducting microlayers undergo undulations but the conductivity within each of these deformed thin layers remains constant.

This simplification of the model has very significant consequences. Recall that the incident excitation is a normally-incident plane wave, a function of x_3' only. Consider now the deterministic operator L_{00} defined in (4.6). Its coefficients become functions of x_3' only; moreover, its action upon vector functions of x_3' reduces to matrix multiplication. The leading order fields, $\mathbf{E}_{t0}, \mathbf{H}_{t0}$, become functions of x_3' only. Similar observations can be made about the zero mean random operator $\delta^{-\frac{1}{2}} L_{10}$. Its coefficients become functions of x_3' and $\frac{x_3'}{\delta}$ only; its action upon vector functions of x_3' likewise reduces to matrix multiplication. Lastly, note that the action of the operator L_{02} upon a vector function of x_3' vanishes. Thus,

$$L_{02} \begin{bmatrix} \mathbf{E}_{t0} \\ \mathbf{H}_{t0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

These observations enable us to identify the plane-layered problem studied in [12] as well as the modifications introduced by the deformations. We use superposition and a regrouping of perturbation expansion (4.1). Let

$$\begin{aligned} \mathbf{E}_t' &= \mathbf{E}_{t0} + (\mathbf{E}_t - \langle \mathbf{E}_t \rangle) \\ \mathbf{H}_t' &= \mathbf{H}_{t0} + (\mathbf{H}_t - \langle \mathbf{H}_t \rangle) \end{aligned} \tag{4.10}$$

The fields defined by (4.10), the sum of the deterministic leading order fields and the zero mean perturbation fields, are solutions of the following problem. In the macrolayer $-L' < x_3' < 0$:

$$\begin{aligned} \frac{d}{dx_3'} \begin{bmatrix} \mathbf{E}_t' \\ \mathbf{H}_t' \end{bmatrix} = &\begin{bmatrix} 0 & 0 & 0 & i\omega\mu \\ 0 & 0 & -i\omega\mu & 0 \\ 0 & \sigma_A(x_3') + \nu\left(x_3', \frac{x_3'}{\delta}\right) & 0 & 0 \\ -\sigma_A(x_3') - \nu\left(x_3', \frac{x_3'}{\delta}\right) & 0 & 0 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} \mathbf{E}_t' \\ \mathbf{H}_t' \end{bmatrix} \end{aligned} \tag{4.11}$$

while in the rock basement $\infty < x_3' < -L'$:

$$\frac{d}{dx_3'} \begin{bmatrix} \mathbf{E}_t' \\ \mathbf{H}_t' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & i\omega\mu \\ 0 & 0 & -i\omega\mu & 0 \\ 0 & \sigma_b & 0 & 0 \\ -\sigma_b & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_t' \\ \mathbf{H}_t' \end{bmatrix}. \tag{4.12}$$

The transverse incident fields are specified on the boundary $x_3' = 0$; an upward-propagating radiation condition is imposed upon the reflected fields at that boundary. The transverse fields are required to be continuous across the macrolayer-basement boundary $x_3' = -L'$. A downward-propagating radiation condition in the basement completes the specification of this problem.

This is the problem studied in [12]. note that the transverse polarizations decouple. The input impedance in each case is a solution of a stochastic Riccati equation. If, for example, $\xi = \frac{E_{t1}'}{H_{t2}'}$, then

$$\begin{aligned} \frac{d}{dx_3'}\xi &= i\omega\mu + (\sigma_A + \nu)\,\xi^2, \qquad -L' < x_3' < 0, \\ \xi\,(-L') &= \xi_b \end{aligned} \tag{4.13}$$

where ξ_b is the basement input impedance. This equation was analyzed in [12]; the impedance was shown asymptotically to consist of the sum of a $O(1)$ deterministic impedance and an $O\left(\delta^{\frac{1}{2}}\right)$ zero mean gaussian fluctuation term.

The layering deformations manifest themselves in the problem satisfied by the mean perturbation fields. From (4.9):

$$\begin{aligned} \frac{\partial}{\partial_{x_3'}}\begin{bmatrix} \langle \mathbf{E}_{t1} \rangle \\ \langle \mathbf{H}_{t1} \rangle \end{bmatrix} &= L_{00}\begin{bmatrix} \langle \mathbf{E}_{t1} \rangle \\ \langle \mathbf{H}_{t1} \rangle \end{bmatrix} + L_{01}\begin{bmatrix} \mathbf{E}_{t0} \\ \mathbf{H}_{t0} \end{bmatrix}, \qquad -L' < x_3' < 0 \\ \frac{\partial}{\partial_{x_3'}}\begin{bmatrix} \langle \mathbf{E}_{t1} \rangle \\ \langle \mathbf{H}_{t1} \rangle \end{bmatrix} &= L_{00}^{b}\begin{bmatrix} \langle \mathbf{E}_{t1} \rangle \\ \langle \mathbf{H}_{t1} \rangle \end{bmatrix} + L_{01}\begin{bmatrix} \mathbf{E}_{t0} \\ \mathbf{H}_{t0} \end{bmatrix}, \qquad -\infty < x_3' < -L' \end{aligned} \tag{4.14}$$

where L_{00}^b denotes the operator L_{00} defined in (4.6) with the effective medium conductivities σ_A and σ_H both replaced by basement conductivity σ_b. For this problem, no excitation is incident from the upper half-space; the leading order fields act as effective sources within the macrolayer and basement. The mean perturbation fields satisfy an upward radiation condition at $x_3' = 0$, a continuity condition across the macrolayer-basement boundary, and a radiation condition within the basement. These fields must vanish as $x_3' \to -\infty$. We study this problem in the next section.

5. Perturbation Mean Field Analysis. We consider the problem represented by equations (4.14). To simplify notation, we will drop the $\langle\cdot\rangle$ symbols and simply refer to the perturbation mean fields as $\mathbf{E}_{t1}$ and $\mathbf{H}_{t1}$. As a first step, we introduce the following lateral Fourier transforms.

$$\hat{\mathbf{E}}_{t1}\,(k_1, k_2, x_3') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{E}_{t1}\,(x_1', x_2', x_3')\, e^{-i(k_1 x_1' + k_2 x_2')} dx_1' dx_2' \tag{5.1}$$

with $\hat{\mathbf{H}}_{t1}$similarly defined. Recall that the leading order fields, $\mathbf{E}_{t0}, \mathbf{H}_{t0}$ are functions of x_3' only. Noting equations (4.6), the transformed version of equations (4.14) becomes:

$$\begin{aligned} \frac{d}{dx_3'}\begin{bmatrix} \hat{\mathbf{E}}_{t1} \\ \hat{\mathbf{H}}_{t1} \end{bmatrix} &= \hat{L}_{00}\begin{bmatrix} \hat{\mathbf{E}}_{t1} \\ \hat{\mathbf{H}}_{t1} \end{bmatrix} + \hat{L}_{01}\begin{bmatrix} \mathbf{E}_{t0}(x_3') \\ \mathbf{H}_{t0}(x_3') \end{bmatrix}, \qquad -L' < x_3' < 0 \\ \frac{d}{dx_3'}\begin{bmatrix} \hat{\mathbf{E}}_{t1} \\ \hat{\mathbf{H}}_{t1} \end{bmatrix} &= \hat{L}_{00}^{b}\begin{bmatrix} \hat{\mathbf{E}}_{t1} \\ \hat{\mathbf{H}}_{t1} \end{bmatrix} + \hat{L}_{01}^{b}\begin{bmatrix} \mathbf{E}_{t0}(x_3') \\ \mathbf{H}_{t0}(x_3') \end{bmatrix}, \qquad -\infty < x_3' < -L' \end{aligned} \tag{5.2}$$

where $\hat{L}_{00}$ and $\hat{L}_{01}$ are the following matrices.

$$\hat{L}_{00} = \begin{bmatrix} 0 & 0 & \sigma_H^{-1}(x_3')k_1k_2 & i\omega\mu - \sigma_H^{-1}(x_3')k_1^2 \\ 0 & 0 & -i\omega\mu + \sigma_H^{-1}(x_3')k_2^2 & -\sigma_H^{-1}(x_3')k_1k_2 \\ (i\omega\mu)^{-1}k_1k_2 & \sigma_A(x_3') - (i\omega\mu)^{-1}k_1^2 & 0 & 0 \\ -\sigma_A(x_3') + (i\omega\mu)^{-1}k_2^2 & -(i\omega\mu)^{-1}k_1k_2 & 0 & 0 \end{bmatrix}$$
$$\hat{L}_{01} = \begin{bmatrix} 0 & 0 & i2\omega\mu\hat{\phi}_{12} & i\omega\mu\hat{\Phi}_1 \\ 0 & 0 & -i\omega\mu\hat{\Phi}_2 & -i2\omega\mu\hat{\phi}_{12} \\ 2\sigma_A(x_3')\hat{\phi}_{12} & \sigma_A(x_3')\hat{\Phi}_1 & 0 & 0 \\ -\sigma_A(x_3')\hat{\Phi}_2 & -2\sigma_A(x_3')\hat{\phi}_{12} & 0 & 0 \end{bmatrix} \tag{5.3}$$

The $\hat{\phi}_{ij}$ are the lateral Fourier transforms of the ϕ_{ij} functions characterizing the deformations. Recall that $\hat{\Phi}_1 = -\hat{\phi}_{11} + \hat{\phi}_{22} + \hat{\phi}_{33}$ and $\hat{\Phi}_2 = \hat{\phi}_{11} - \hat{\phi}_{22} + \hat{\phi}_{33}$. Note that $\hat{\phi}_{ij} = -k_ik_j\hat{\phi},\ i,j = 1,2$, where ϕ is given by (3.5). The matrices $\hat{L}_{00}^b$ and $\hat{L}_{01}^b$ are obtained from $\hat{L}_{00}$ and $\hat{L}_{01}$, respectively, by replacing both σ_A and σ_H with the constant basement conductivity σ_b. The previously-mentioned interface continuity and radiation conditions carry over to this transformed problem.

To further simplify the notation, let $\hat{E}_1, \hat{E}_2$ and $\hat{H}_1, \hat{H}_2$ represent the components of $\hat{\mathbf{E}}_{t1}$ and $\hat{\mathbf{H}}_{t1}$, respectively. The 4-dimensional linear system described by equations (5.2), (5.3) can be decomposed into two decoupled 2-dimensional systems by adopting appropriate linear combinations of the field components as new dependent variables. To motivate this change of variables, consider the first of equations (5.2) written in component form.

$$\begin{aligned} \frac{d}{dx_3'}\hat{E}_1 &= \sigma_H^{-1}k_1\left[k_2\hat{H}_1 - k_1\hat{H}_2\right] + i\omega\mu\hat{H}_2 + i2\omega\mu\hat{\phi}_{12}H_{t0_1} + i\omega\mu(-\hat{\phi}_{11} + \hat{\phi}_{22} + \hat{\phi}_{33})H_{t0_2} \\ \frac{d}{dx_3'}\hat{E}_2 &= \sigma_H^{-1}k_2\left[k_2\hat{H}_1 - k_1\hat{H}_2\right] - i\omega\mu\hat{H}_1 - i\omega\mu(\hat{\phi}_{11} - \hat{\phi}_{22} + \hat{\phi}_{33})H_{t0_1} - i2\omega\mu\hat{\phi}_{12}H_{t0_2} \\ \frac{d}{dx_3'}\hat{H}_1 &= (i\omega\mu)^{-1}k_1\left[k_2\hat{E}_1 - k_1\hat{E}_2\right] + \sigma_A\hat{E}_2 + 2\sigma_A\hat{\phi}_{12}E_{t0_1} + \sigma_A(-\hat{\phi}_{11} + \hat{\phi}_{22} + \hat{\phi}_{33})E_{t0_2} \\ \frac{d}{dx_3'}\hat{H}_2 &= (i\omega\mu)^{-1}k_2\left[k_2\hat{E}_1 - k_1\hat{E}_2\right] - \sigma_A\hat{E}_1 - \sigma_A(\hat{\phi}_{11} - \hat{\phi}_{22} + \hat{\phi}_{33})E_{t0_1} - 2\sigma_A\hat{\phi}_{12}E_{t0_2}. \end{aligned} \tag{5.4}$$

Equations (5.4) suggest the following change of variables; as the notation suggests, these new variables are related to TE (transverse electric) and TM (transverse magnetic) modes (see (2.12), (2.13)). Let:

$$\begin{aligned} \hat{E}^{(TE)} &= k_2\hat{E}_1 - k_1\hat{E}_2 \qquad \hat{E}^{(TM)} = k_1\hat{E}_1 + k_2\hat{E}_2 \\ \hat{H}^{(TE)} &= k_1\hat{H}_1 + k_2\hat{H}_2 \qquad \hat{H}^{(TM)} = k_2\hat{H}_1 - k_1\hat{H}_2. \end{aligned} \tag{5.5}$$

Then, letting $k_t^2 = k_1^2 + k_2^2$ and recalling that $\hat{\phi}_{ij} = -k_ik_j\hat{\phi}, j = 1,2$, system (5.4) reduces to the following two decoupled systems.

$$\begin{aligned} \frac{d}{dx_3'}\begin{bmatrix} \hat{E}^{(TE)} \\ \hat{H}^{(TE)} \end{bmatrix} &= \begin{bmatrix} 0 & i\omega\mu \\ (i\omega\mu)^{-1}k_t^2 - \sigma_A & 0 \end{bmatrix}\begin{bmatrix} \hat{E}^{(TE)} \\ \hat{H}^{(TE)} \end{bmatrix} + \begin{bmatrix} i\omega\mu(\hat{\phi}_{33} - k_t^2\hat{\phi})H_{t0}^{(TE)} \\ -\sigma_A(\hat{\phi}_{33} + k_t^2\hat{\phi})E_{t0}^{(TE)} \end{bmatrix} \\ \frac{d}{dx_3'}\begin{bmatrix} \hat{E}^{(TM)} \\ \hat{H}^{(TM)} \end{bmatrix} &= \begin{bmatrix} 0 & -i\omega\mu + \sigma_H^{-1}k_t^2 \\ \sigma_A & 0 \end{bmatrix}\begin{bmatrix} \hat{E}^{(TM)} \\ \hat{H}^{(TM)} \end{bmatrix} + \begin{bmatrix} -i\omega\mu(\hat{\phi}_{33} + k_t^2\hat{\phi})H_{t0}^{(TM)} \\ \sigma_A(\hat{\phi}_{33} - k_t^2\hat{\phi})E_{t0}^{(TM)} \end{bmatrix} \end{aligned} \tag{5.6}$$

where the TE and TM superscripts on the leading order transverse fields represent the same linear combinations as given in (5.5). Equations (5.6) describe behavior in the macrolayer $-L' < x_3' < 0$; equations for the basement region $-\infty < x_3' < -L'$ are obtained from (5.6) by replacing both σ_A and σ_H by basement conductivity σ_b.

The perturbation mean field problem consists of equations (5.6) in the macrolayer, corresponding equations in the basement region, along with interface continuity and radiation conditions at the macrolayer boundaries. We now discuss this problem in greater detail and, in particular, show that the continuity and radiation conditions at the macrolayer boundaries can be recast as equivalent boundary conditions at the macrolayer extremities.

From equations (5.6), the leading order fields are solutions of

$$\frac{d}{dx_3'}\begin{bmatrix} E_{t0}^{(TE)} \\ H_{t0}^{(TE)} \end{bmatrix} = \begin{bmatrix} 0 & i\omega\mu \\ -\sigma_A & 0 \end{bmatrix}\begin{bmatrix} E_{t0}^{(TE)} \\ H_{t0}^{(TE)} \end{bmatrix}$$
$$\frac{d}{dx_3'}\begin{bmatrix} E_{t0}^{(TM)} \\ H_{t0}^{(TM)} \end{bmatrix} = \begin{bmatrix} 0 & -i\omega\mu \\ \sigma_A & 0 \end{bmatrix}\begin{bmatrix} E_{t0}^{(TM)} \\ H_{t0}^{(TM)} \end{bmatrix}, \qquad -L' < x_3' < 0. \tag{5.7}$$

At the air-macrolayer boundary $x_3' = 0$, the leading order fields must satisfy the boundary condition:

$$H_{t0}^{(TE)}(0) - \sqrt{\frac{\epsilon}{\mu}}E_{t0}^{(TE)}(0) = 2H_{inc}^{(TE)},$$
$$H_{t0}^{(TM)}(0) + \sqrt{\frac{\epsilon}{\mu}}E_{t0}^{(TM)}(0) = 2H_{inc}^{(TM)} \tag{5.8}$$

where $H_{inc}^{(TE)}$ and $H_{inc}^{(TM)}$ represent the modal tangential magnetic fields incident upon the boundary. At the macrolayer-basement interface $x_3' = -L'$ the leading order fields satisfy the following boundary condition.

$$H_{t0}^{(TE)}(-L') + \frac{(1+i)}{\sqrt{2}}\sqrt{\frac{\sigma_b}{\omega\mu}}E_{t0}^{(TE)}(-L') = 0,$$
$$H_{t0}^{(TM)}(-L') - \frac{(1+i)}{\sqrt{2}}\sqrt{\frac{\sigma_b}{\omega\mu}}E_{t0}^{(TM)}(-L') = 0. \tag{5.9}$$

The solution of the two-point boundary value problems defined by equations (5.7)-(5.9) yields the leading order fields; these fields, along with the deformation function $\hat{\phi}$, define the effective sources for the perturbation mean field equations.

The problem for the mean perturbation fields can also be rewritten as an equivalent two-point boundary value problem within the macrolayer region, $-L' < x_3' < 0$. The governing differential equations are given by (5.6). The upgoing radiation condition at $x = 0$ leads to:

$$\hat{H}^{(TE)}(0) - \left[\frac{\sqrt{\omega^2\mu\epsilon - k_t^2}}{\omega\mu}\right]\hat{E}^{(TE)}(0) = 0,$$
$$\hat{H}^{(TM)}(0) + \left[\frac{\omega\epsilon}{\sqrt{\omega^2\mu\epsilon - k_t^2}}\right]\hat{E}^{(TM)}(0) = 0 \tag{5.10}$$

where $\sqrt{\omega^2\mu\epsilon - k_t^2} = i\sqrt{k_t^2 - \omega^2\mu\epsilon}$ when $k_t^2 > \omega^2\mu\epsilon$. The boundary condition at $x_3' = -L'$ arises from the continuity condition across this interface along with the fact that the upgoing fields must vanish in the limit as $x_3' \to -\infty$. The governing differential equations in the basement are equations (5.6) with the effective medium conductivities σ_A and σ_H replaced by the constant basement conductivity σ_b. In contrast to the air region, the deformation function $\hat{\phi}$ is not zero in the basement. The boundary conditions are obtained by solving the nonhomogeneous systems in the basement region and requiring that the perturbation mean fields vanish in the limit as $x_3' \to -\infty$. For brevity, let $\kappa = \sqrt{i\omega\mu\sigma_b - k_t^2}$ and $\kappa_0 = \sqrt{i\omega\mu\sigma_b}$, with $\Im\{\kappa\} > 0$ and $\Im\{\kappa_0\} > 0$. Then, we obtain the following boundary conditions.

$$\begin{aligned}\hat{E}^{(TE)}(-L') + \left(\frac{\omega\mu}{\kappa}\right)\hat{H}^{(TE)}(-L') &= -e^{-i(\kappa+\kappa_0)L'}E_{t0}^{(TE)}(-L')\int_{-\infty}^{-L'}\Big[i\kappa_0(\hat{\phi}_{33} - k_t^2\hat{\phi}) \\ &\quad + \left(\frac{\omega\mu\sigma_b}{\kappa}\right)(\hat{\phi}_{33} + k_t^2\hat{\phi})\Big]e^{-i(\kappa+\kappa_0)s}ds \\ \hat{E}^{(TM)}(-L') + \left(\frac{i\kappa}{\sigma_b}\right)\hat{H}^{(TM)}(-L') &= e^{-i(\kappa+\kappa_0)L'}E_{t0}^{(TM)}(-L')\int_{-\infty}^{-L'}\Big[i\kappa(\hat{\phi}_{33} - k_t^2\hat{\phi}) \\ &\quad + \left(\frac{\omega\mu\sigma_b\kappa_0}{\kappa^2}\right)(\hat{\phi}_{33} + k_t^2\hat{\phi})\Big]e^{-i(\kappa+\kappa_0)s}ds. \qquad (5.11)\end{aligned}$$

The solution of this problem is straightforward in principle but certainly nontrivial in practice. In the next section, we shall analyze the simpler problem of the semi-infinite locally-layered medium, obtained by letting $-L' \to -\infty$.

We conclude this section with a brief comment on the quasi-static approximation that is commonly utilized in MT studies (see [1], [14]). This approximation, motivated by the low frequencies employed in MT exploration, amounts to totally neglecting the dielectric permittivity. We have already neglected the displacement current in the conducting half-space $x_3' \le 0$ since it is small compared to the conductivity. We can similarly make the approximation in the upper air region by setting $\epsilon = 0$ in boundary conditions (5.8) and (5.10). In that case, boundary conditions (5.8) reduce to the leading order tangential magnetic fields being known at $x_3' = 0$ while the boundary conditions for the perturbation fields, (5.10), reduce to:

$$\hat{H}^{(TE)}(0) - i\left(\frac{k_t}{\omega\mu}\right)\hat{E}^{(TE)}(0) = 0, \qquad \hat{H}^{(TM)}(0) = 0. \qquad (5.12)$$

6. Locally-layered Conducting Half-space. We now consider the case where the locally-layered conducting slab becomes infinitely thick, that is, we let $-L' \to -\infty$. To obtain explicit expressions for the perturbation mean fields, we restrict our attention to the case where the effective medium conductivities σ_A and σ_H are constants.

From equations (5.6), the leading order fields in $-\infty < x_3' < 0$ are solutions of

$$\begin{aligned}\frac{d}{dx_3'}\begin{bmatrix} E_{t0}^{(TE)} \\ H_{t0}^{(TE)} \end{bmatrix} &= \begin{bmatrix} 0 & i\omega\mu \\ -\sigma_A & 0 \end{bmatrix}\begin{bmatrix} E_{t0}^{(TE)} \\ H_{t0}^{(TE)} \end{bmatrix} \\ \frac{d}{dx_3'}\begin{bmatrix} E_{t0}^{(TM)} \\ H_{t0}^{(TM)} \end{bmatrix} &= \begin{bmatrix} 0 & -i\omega\mu \\ \sigma_A & 0 \end{bmatrix}\begin{bmatrix} E_{t0}^{(TM)} \\ H_{t0}^{(TM)} \end{bmatrix}. \qquad (6.1)\end{aligned}$$

We require these fields to vanish as $x_3' \to -\infty$. If we define $\bar{k} = \sqrt{i\omega\mu\sigma_A} =$

$\left(\frac{1+i}{\sqrt{2}}\right)\sqrt{\omega\mu\sigma_A}$, the solutions of (6.1) can be expressed as:

$$
\begin{aligned}
E_{t0}^{(TE)} &= \frac{i\bar{k}}{\sigma_A} H_{t0}^{(TE)}(0) e^{-i\bar{k}x_3'}, \qquad H_{t0}^{(TE)} = H_{t0}^{(TE)}(0) e^{-i\bar{k}x_3'} \\
E_{t0}^{(TM)} &= -\frac{i\bar{k}}{\sigma_A} H_{t0}^{(TM)}(0) e^{-i\bar{k}x_3'}, \qquad H_{t0}^{(TM)} = H_{t0}^{(TM)}(0) e^{-i\bar{k}x_3'}.
\end{aligned} \tag{6.2}
$$

Perturbation field equations (5.6) therefore become

$$
\begin{aligned}
\frac{d}{dx_3'} \begin{bmatrix} \hat{E}^{(TE)} \\ \hat{H}^{(TE)} \end{bmatrix} &= \begin{bmatrix} 0 & i\omega\mu \\ (i\omega\mu)^{-1}k_t^2 - \sigma_A & 0 \end{bmatrix} \begin{bmatrix} \hat{E}^{(TE)} \\ \hat{H}^{(TE)} \end{bmatrix} \\
&\quad + \begin{bmatrix} i\omega\mu(\hat{\phi}_{33} - k_t^2\hat{\phi}) \\ -i\bar{k}(\hat{\phi}_{33} + k_t^2\hat{\phi}) \end{bmatrix} H_{t0}^{(TE)}(0) e^{-i\bar{k}x_3'} \\
\frac{d}{dx_3'} \begin{bmatrix} \hat{E}^{(TM)} \\ \hat{H}^{(TM)} \end{bmatrix} &= \begin{bmatrix} 0 & -i\omega\mu + \sigma_H^{-1}k_t^2 \\ \sigma_A & 0 \end{bmatrix} \begin{bmatrix} \hat{E}^{(TM)} \\ \hat{H}^{(TM)} \end{bmatrix} \\
&\quad + \begin{bmatrix} -i\omega\mu(\hat{\phi}_{33} + k_t^2\hat{\phi}) \\ -i\bar{k}(\hat{\phi}_{33} - k_t^2\hat{\phi}) \end{bmatrix} H_{t0}^{(TM)}(0) e^{-i\bar{k}x_3'}
\end{aligned} \tag{6.3}
$$

The solutions of (6.3) must vanish as $x_3' \to -\infty$. At $x_3' = 0$ we shall impose quasi-static boundary conditions (5.12).

This problem can be solved directly using differential equation techniques. We shall adopt an alternate, slightly more convenient approach, by introducing an additional Fourier transform. Let

$$
\begin{aligned}
\tilde{E}^{(TE)}(k_1, k_2, k_3) &= \int_{-\infty}^{0} \hat{E}^{(TE)}(k_1, k_2, x_3') e^{-ik_3x_3'} dx_3' \\
\tilde{H}^{(TE)}(k_1, k_2, k_3) &= \int_{-\infty}^{0} \hat{H}^{(TE)}(k_1, k_2, x_3') e^{-ik_3x_3'} dx_3'
\end{aligned} \tag{6.4}
$$

along with an analogous transform defined for the TM fields. We also define

$$
\tilde{\phi}((k_1, k_2, k_3) = \int_{-\infty}^{0} \hat{\phi}(k_1, k_2, x_3') e^{-i[k_3+\bar{k}]x_3'} dx_3' \tag{6.5}
$$

Recall that both $\hat{\phi}$ and $\hat{\phi}_3$ vanish at $x_3' = 0$. Therefore, letting $k^2 = k_t^2 + k_3^2$, the transformed version of equations (6.3) become the following algebraic system.

$$
\begin{aligned}
\begin{bmatrix} -ik_3 & i\omega\mu \\ (i\omega\mu)^{-1}k_t^2 - \sigma_A & -ik_3 \end{bmatrix} \begin{bmatrix} \tilde{E}^{(TE)} \\ \tilde{H}^{(TE)} \end{bmatrix} &= \begin{bmatrix} 1 \\ \frac{ik_t}{\omega\mu} \end{bmatrix} \hat{E}^{(TE)}(k_{1,}, k_2, 0) \\
&\quad + \begin{bmatrix} i\omega\mu([k_3 + \bar{k}]^2 + k_t^2) \\ i\bar{k}(-[k_3 + \bar{k}]^2 + k_t^2) \end{bmatrix} \tilde{\phi} H_{t0}^{(TE)}(0) \\
\begin{bmatrix} -ik_3 & -i\omega\mu + \sigma_H^{-1}k_t^2 \\ \sigma_A & -ik_3 \end{bmatrix} \begin{bmatrix} \tilde{E}^{(TM)} \\ \tilde{H}^{(TM)} \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \hat{E}^{(TM)}(k_{1,}, k_2, 0) \\
&\quad - \begin{bmatrix} i\omega\mu([k_3 + \bar{k}]^2 - k_t^2) \\ i\bar{k}([k_3 + \bar{k}]^2 + k_t^2) \end{bmatrix} \tilde{\phi} H_{t0}^{(TM)}(0).
\end{aligned} \tag{6.6}
$$

Note that equations (6.6) incorporate both the radiation condition and the surface boundary conditions.

We can solve for the quantities of interest, $\hat{E}^{(TE)}(k_1,k_2,0)$ and $\hat{E}^{(TE)}(k_1,k_2,0)$ by first rewriting equations (6.6) as:

$$\begin{aligned}
\left(i\omega\mu\sigma_A - k_t^2 - k_3^2\right)\begin{bmatrix}\tilde{E}^{(TE)}\\ \tilde{H}^{(TE)}\end{bmatrix} &= \begin{bmatrix}-ik_3 & -i\omega\mu\\ -(i\omega\mu)^{-1}k_t^2+\sigma_A & -ik_3\end{bmatrix}\left(\begin{bmatrix}1\\ \frac{ik_t}{\omega\mu}\end{bmatrix}\hat{E}^{(TE)}\right.\\
&\left.\quad + \begin{bmatrix}i\omega\mu([k_3+\bar{k}]^2+k_t^2)\\ i\bar{k}(-[k_3+\bar{k}]^2+k_t^2)\end{bmatrix}\tilde{\phi}H_{t0}^{(TE)}(0)\right)\\
(i\omega\mu\sigma_A - \frac{\sigma_A}{\sigma_H}k_t^2 - k_3^2)\begin{bmatrix}\tilde{E}^{(TM)}\\ \tilde{H}^{(TM)}\end{bmatrix} &= \begin{bmatrix}-ik_3 & i\omega\mu-\sigma_H^{-1}k_t^2\\ -\sigma_A & -ik_3\end{bmatrix}\left(\begin{bmatrix}1\\ 0\end{bmatrix}\hat{E}^{(TM)}\right.\\
&\left.\quad - \begin{bmatrix}i\omega\mu([k_3+\bar{k}]^2-k_t^2)\\ i\bar{k}([k_3+\bar{k}]^2+k_t^2)\end{bmatrix}\tilde{\phi}H_{t0}^{(TM)}(0)\right). \qquad (6.7)
\end{aligned}$$

Consider the first of equations (6.7). The functions $\tilde{E}^{(TE)}, \tilde{H}^{(TE)}$ are Fourier transforms of functions vanishing on the half-line $x_3' > 0$; they are analytic functions in the upper half-plane $\Im\{k_3\} > 0$. Therefore, the right side of this equation vanishes when $k_3 = \gamma_E \equiv \sqrt{i\omega\mu\sigma_A - k_t^2}$, $\Im\{\gamma_E\} > 0$. Setting $k_3 = \gamma_E$ on both sides of the equation enables one to solve for $\hat{E}^{(TE)}$. By setting $k_3 = \gamma_M \equiv \sqrt{i\omega\mu\sigma_A - \frac{\sigma_A}{\sigma_H}k_t^2}$, $\Im\{\gamma_M\} > 0$ in the second of equations (6.7), one can similarly solve for $\hat{E}^{(TM)}$. We obtain:

$$\begin{aligned}
&\hat{E}^{(TE)}(k_1,k_2,0) = 0\\
&\hat{E}^{(TM)}(k_1,k_2,0) = -\frac{\bar{k}}{\sigma_A}\left(1-\frac{\sigma_A}{\sigma_H}\right)\left(\gamma_M+\bar{k}\right)k_t^2\tilde{\phi}(k_1,k_2,\gamma_M)H_{t0}^{(TM)}. \qquad (6.8)
\end{aligned}$$

The perturbation mean field at the surface of the locally-layered half-space is therefore TM in nature. Since $\hat{H}^{(TM)}(k_1,k_2,0) = 0$, only the mean TM electric field is perturbed. Note, moreover, that this perturbation vanishes if either $\sigma_A = \sigma_H$ or $\tilde{\phi} = 0$. This is to be expected; the perturbation is a consequence of the effective medium anisotropy produced by the undulating random layering.

Noting equations (5.5), we can reconstruct the transverse field components at $x_3' = 0$.

$$\begin{aligned}
\begin{bmatrix}\hat{E}_1\\ \hat{E}_2\end{bmatrix} &= \begin{bmatrix}k_1\\ k_2\end{bmatrix}k_t^{-2}\tilde{E}^{(TM)} \qquad (6.9)\\
&= -\begin{bmatrix}k_1\\ k_2\end{bmatrix}\frac{\bar{k}}{\sigma_A}\left(1-\frac{\sigma_A}{\sigma_H}\right)\left(\gamma_M+\bar{k}\right)\tilde{\phi}(k_1,k_2,\gamma_M)\left(k_2H_1-k_1H_2\right).
\end{aligned}$$

In (6.9) $\hat{E}_j = \hat{E}_j(k_1,k_2,0)$ and the H_j are the transverse magnetic field components at the surface. We discuss the impact of these computations upon the surface impedance in the next section.

7. Half-space Surface Impedance. The surface impedance is a 2×2 matrix relating the transverse electric and magnetic fields.

$$\begin{bmatrix}E_1(x_1,x_2)\\ E_2(x_1,x_2)\end{bmatrix} = \begin{bmatrix}Z_{11}(x_1,x_2) & Z_{12}(x_1,x_2)\\ Z_{21}(x_1,x_2) & Z_{22}(x_1,x_2)\end{bmatrix}\begin{bmatrix}H_1(x_1,x_2)\\ H_2(x_1,x_2)\end{bmatrix}. \qquad (7.1)$$

We have dropped the primes on the transverse coordinates since they equal their unprimed counterparts on the surface.

In our case, the magnetic field on the surface is constant; recall that the perturbations $\hat{H}^{(TE)}$ and $\hat{H}^{(TM)}$ both vanished on the surface. Noting (6.9) let ζ denote the following inverse Fourier transform.

$$\zeta(x_1, x_2) = \mathcal{F}^{-1}\left(\bar{k}(\gamma_M + \bar{k})\tilde{\phi}(k_1, k_2, \gamma_M)\right) \tag{7.2}$$
$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{k}(\gamma_M + \bar{k})\tilde{\phi}(k_1, k_2, \gamma_M)e^{i(k_1x_1+k_2x_2)}dk_1dk_2.$$

Recall that ξ, the solution of stochastic Riccati equation (4.13), embodies the leading order deterministic and zero mean random perturbation contributions to the surface impedance. (From (4.11), $E'_{t1} = \xi H'_{t2}$ and $E'_{t2} = -\xi H'_{t1}$). To these terms must be added the $O(\delta^{\frac{1}{2}})$ depolarizing perturbation terms arising from the local layering. Noting equation (6.9), these terms will involve partial derivatives of ζ. We obtain

$$\begin{bmatrix} Z_{11}(x_1, x_2) & Z_{12}(x_1, x_2) \\ Z_{21}(x_1, x_2) & Z_{22}(x_1, x_2) \end{bmatrix} \tag{7.3}$$
$$= \begin{bmatrix} \delta^{\frac{1}{2}}(\rho_A - \rho_H)\partial^2_{x_1x_2}\zeta & \xi - \delta^{\frac{1}{2}}(\rho_A - \rho_H)\partial^2_{x_1x_1}\zeta \\ -\xi + \delta^{\frac{1}{2}}(\rho_A - \rho_H)\partial^2_{x_2x_2}\zeta & -\delta^{\frac{1}{2}}(\rho_A - \rho_H)\partial^2_{x_1x_2}\zeta \end{bmatrix}$$

where $\rho_A = \sigma_A^{-1}, \;\; \rho_H = \sigma_H^{-1}$ are the effective medium resistivities. In the next section we evaluate the impedance perturbations for a simple specific local layering model.

8. A Model Problem. We consider an undulation model characterized by the function

$$\Psi_3(x_1, x_2, x_3) = e^{-\frac{x_1^2}{2\beta_1^2}}e^{-\frac{x_2^2}{2\beta_2^2}}x_3e^{\frac{x_3}{\beta_3}}, \quad -\infty < x_1, x_2 < \infty, -\infty < x_3 \le 0. \tag{8.1}$$

This choice of Ψ_3 leads to layering that has an elliptical depression. The function has a minimum value of $-\beta_3 e^{-1}$ at $x_1 = 0, \;\; x_3 = -\beta_3$. The corresponding coordinate level surfaces are given by

$$x_3' = x_3 + \delta^{\frac{1}{2}}\Psi_3 = x_3 + \delta^{\frac{1}{2}}e^{-\frac{x_1^2}{2\beta_1^2}}e^{-\frac{x_2^2}{2\beta_2^2}}x_3e^{\frac{x_3}{\beta_3}}. \tag{8.2}$$

Figure 1 shows the edges of some representative level surfaces as a function of x_1, with $x_2 = 0$, for the parameter values chosen below.

Noting equation (3.5), it follows from (6.5) and (8.1) that

$$\left(\gamma_M + \bar{k}\right)\tilde{\phi}(k_1, k_2, \gamma_M) \tag{8.3}$$
$$= -i2\pi\beta_1\beta_2 e^{\frac{-k_1^2\beta_1^2}{2}}e^{\frac{-k_2^2\beta_2^2}{2}}\left[\beta_3 + \bar{d}^{-1} - i(\gamma_M + \bar{d}^{-1})\right]^{-2}$$

where $\bar{d} = \sqrt{\frac{2}{\omega\mu\sigma_A}}$ is an effective medium skin depth.

For our calculations we assume $\beta_1 = 100m$, $\beta_2 = 300m$ and $\beta_3 = 500m$. The undulation amplitude therefore reaches a maximum value at a depth of $500m$. We use the effective medium conductivities $\sigma_A, \;\; \sigma_H$ arising from the random layering model used in [12]. This model assumes conductivity microlayers 3 meters thick; in each microlayer the conductivity is an independent, identically-distributed random variable uniformly distributed between 0.01 and 0.10 Siemens/m. It follows that

$$\delta = 0.003, \qquad \sigma_A = 0.055\, S/m, \qquad \sigma_H = 0.0390865\, S/m.$$

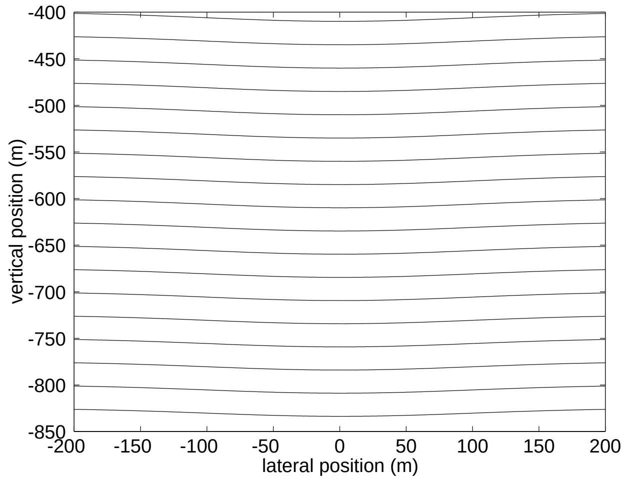

FIG. 1. *Representative level surface edges as a function of x_1, with $x_2 = 0$*

The frequency is assumed to be 1 Hz; the value of permeability adopted is $\mu = 4\pi(10)^{-7}\ henry/m$.

To exhibit depolarization effects, it was necessary to evaluate the impedance perturbations at surface spatial locations displaced from the undulation symmetry axes. Figures 2, 3 and 4 exhibit the normalized resistive and reactive impedance perturbations to matrix elements Z_{11}, Z_{12} and Z_{21} as a function of transverse coordinate x_1 with $x_2 = 100m$. As these figures indicate, the perturbations are small (1 percent or less) for the parameter values chosen.

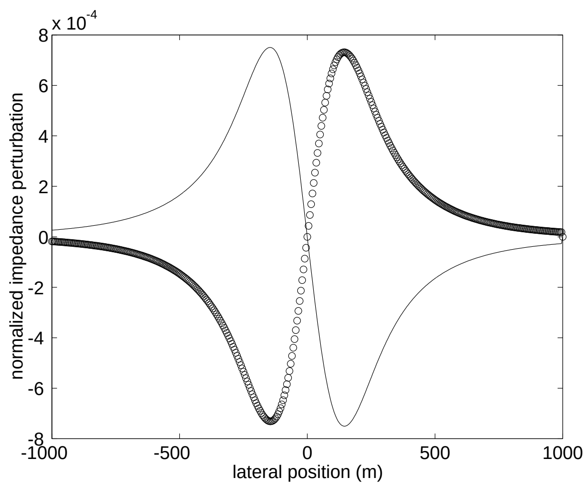

FIG. 2. *Normalized Z_{11} perturbation, $\delta^{\frac{1}{2}}(\rho_A - \rho_H)\partial^2_{x_1 x_2}\zeta/\sqrt{\omega\mu\sigma_A^{-1}}$, as a function of x_1 with $x_2 = 100$ m. The real and imaginary parts are given by the solid curve and circles, respectively.*

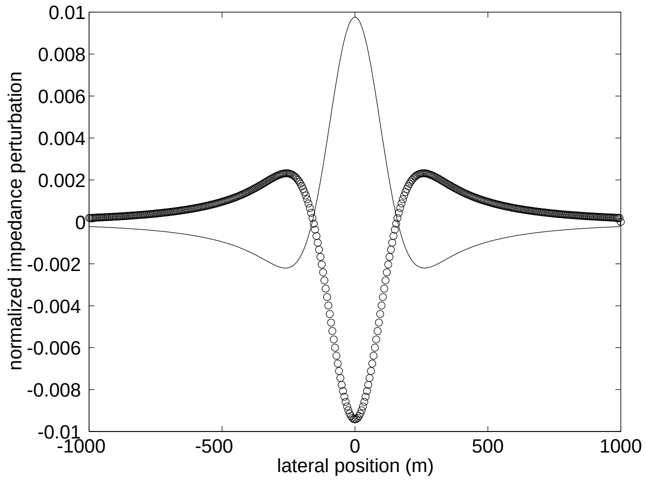

FIG. 3. *Normalized Z_{12} perturbation, $-\delta^{\frac{1}{2}}(\rho_A - \rho_H)\partial^2_{x_1 x_1}\zeta/\sqrt{\omega\mu\sigma_A^{-1}}$, as a function of x_1 with $x_2 = 100$ m. The real and imaginary parts are given by the solid curve and circles, respectively.*

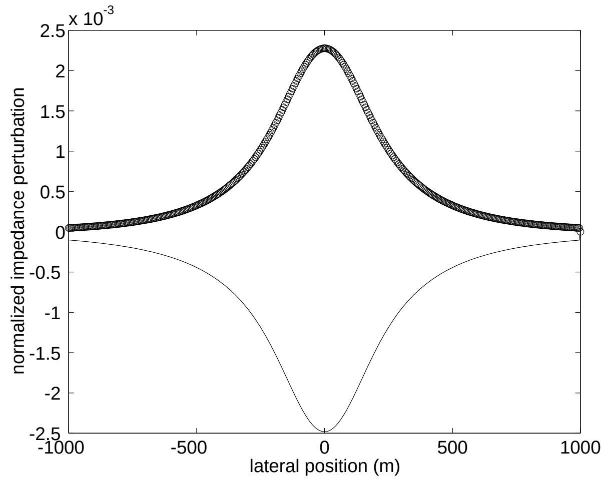

FIG. 4. *Normalized Z_{21} perturbation, $\delta^{\frac{1}{2}}(\rho_A - \rho_H)\partial^2_{x_2 x_2}\zeta/\sqrt{\omega\mu\sigma_A^{-1}}$, as a function of x_1 with $x_2 = 100$ m. The real and imaginary parts are given by the solid curve and circles, respectively.*

9. Conclusions. We have considered the predicted results of an MT survey over a subsurface which has lateral variations caused by nonplanar wandering beds. Furthermore, we have incorporated rapid random fluctuations in the local conductivity vs. depth profile, consistent with what is observed in typical well logs. Under appropriate hypotheses, we have shown that the surface impedance is then a sum of three terms: (1) a mean field impedance, giving the response to a strictly plane-layered model, without the small scale random fluctuations; (2) a gaussian random process which characterizes scattering by the random inhomogeneities; and (3) a term that results from the 3-dimensional geometry of the wandering beds. The first two of these terms are familiar from previous plane-layered theory while the third term is new.

The new term characterizing the effects of the wandering beds may be computed in general by solving a two-point linear boundary value problem for a system of ordinary differential equations. These equations show how waves are scattered into all angles by undulations in the layering. Using these equations, the effects of wandering beds on the surface impedance matrix could be computed for any hypothesized earth model and the results compared with data.

As a special case we have considered in detail an earth model which is a homogeneous random half-space. In this case, the effect of the nonplanar undulations can be computed with explicit formulas, which complement formulas previously obtained for the strictly plane-layered case. We have illustrated these new formulas with numerical calculations.

The 2×2 surface impedance matrix is the primary measurement in MT surveys. Our example shows how the structure of this matrix changes in response to the lateral variations. In particular, the sum of the off-diagonal terms does not generally vanish, as it does in the strictly plane-layered case. Moreover, although the the trace of this matrix vanishes, the two diagonal elements do not vanish individually, as they do in the plane-layered case; the undulations typically produce some depolarization. Thus, the effect of the wandering beds is immediately apparent by inspection of the impedance matrix structure and more detailed calculations can be done to confirm any hypothesized earth model.

REFERENCES

[1] Berdichevsky, M. N. and Dmitriev, *Magnetotellurics in the Context of Ill-Posed Problems*, Society of Exploration Geophysicists, 2002.

[2] Khasminskii, R. Z., *On stochastic processes defined by differential equations with a small parameter*, Theory of Probability and its Applications, 11 (1966), pp. 211–228.

[3] Mehanee, S. and Zhdanov, M., *Two-dimensional magnetotelluric inversion of blocky geoelectrical structures, J. Geophysical Research*, 107 (2002), EPM 2-1 - 2-12.

[4] Parker, R. L., *Geophysical Inverse Theory*, Princeton University Press, 1994.

[5] Tikhonov, A. N., *The stability of inverse problems*, Dokl. Akad. Nauk. SSSR, 39 (1943), pp. 195–198.

[6] Tikhonov, A. N., *On determination of electrical characteristics of deep layers of the Earth's crust*, Dokl. Akad. Nauk. SSSR, 73 (1950), pp. 295–297.

[7] Tikhonov, A. N., *On solution of ill-posed problems and method of regularization*, Dokl. Akad. Nauk. SSSR, 151 (1963), pp. 501–504.

[8] Tikhonov, A. N., *On regularization of ill-posed problems*, Dokl. Akad. Nauk. SSSR, 153 (1963), pp. 49–52.

[9] Vozoff, K., *The magnetotelluric method*, 1991, Electromagnetic Methods in Applied Geophysics - Applications, Chapter 8, edited by Nabighian, M. N., Society of Exploration Geophysicists.

[10] Ward, S. H. and Hohmann, G. W., *Electromagnetic Theory for Geophysical Applications - Theory*, 1987, Electromagnetic Methods in Applied Geophysics - Applications, Chapter 4, edited by Nabighian, M. N., Society of Exploration Geophysicists.

[11] Weidelt, P., *The inverse problem of geomagnetic induction*, Zeitschrift fur Geophysik, 38 (1972), pp. 257–289.

[12] White, B. S., Kohler, W. E. and Srnka, L. J., *Random scattering in magnetotellurics*, Geophysics, 66 (2001), pp. 188–202.

[13] Zhdanov, M. S., *Geophysical Inverse Theory and Regularization Problems*, 2002, Elsevier.

[14] *Zhdanov, M. S. and Keller, G. V., The Geoelectrical Methods in Geophysical Exploration*, 1994, Elsevier.

[15] Zhdanov, M. and Tolstaya, E., *Minimum support nonlinear parametrization in the solution of a 3D magnetotelluric inverse problem*, Inverse Problems, 20 (2004), pp. 937–952.

Appendix A. Coordinate System Construction. We show constructively that equations (2.2) and (2.3) determine $\overline{X}_1, \overline{X}_2$ uniquely. Define $\mathbf{R}(s,\mathbf{x}) = (R_1, R_2, R_3)^T$ as the solution of the final value problem

$$\begin{aligned} \frac{\partial}{\partial s}\mathbf{R}(s,\mathbf{x}) &= -\frac{\nabla\overline{X}_3(\mathbf{R}(s,\mathbf{x})}{|\nabla\overline{X}_3(\mathbf{R}(s,\mathbf{x}))|^2}, \qquad s<0 \\ \mathbf{R}(s,\mathbf{0}) &= \mathbf{x} \end{aligned} \tag{A.1}$$

First assume that the functions $\overline{X}_1, \overline{X}_2$ exist and satisfy equations (2.2) and (2.3). Then from equations (2.3) and (A.1) it follows that

$$\frac{\partial}{\partial s}\overline{X}_j(\mathbf{R}(s,\mathbf{x})) = -\delta_{j,3}, \qquad j=1,2,3. \tag{A.2}$$

Integration of equation (A.2) yields

$$\overline{X}_j(\mathbf{R}(s,\mathbf{x})) = \overline{X}_j(\mathbf{x}), \qquad j=1,2 \tag{A.3}$$

$$\overline{X}_3(\mathbf{R}(s,\mathbf{x})) = \overline{X}_3(\mathbf{x}) - s. \tag{A.4}$$

Letting $s = \overline{X}_3(\mathbf{x})$ in equation (A.4) yields that

$$\overline{X}_3(\mathbf{R}(\overline{X}_3(\mathbf{x}),\mathbf{x})) = 0. \tag{A.5}$$

Comparison of equation (A.5) with equation (2.1) yields that

$$R_3(\overline{X}_3(\mathbf{x}),\mathbf{x}) = 0 \tag{A.6}$$

assuming that $\overline{X}_3 = 0$ only on the surface $x_3 = 0$. Putting equation (A.6) into equation (A.3) now gives that

$$\overline{X}_j(R_1(\overline{X}_3(\mathbf{x}),\mathbf{x}), R_2(\overline{X}_3(\mathbf{x}),\mathbf{x}), 0) = \overline{X}_j(\mathbf{x}), \qquad j=1,2. \tag{A.7}$$

Finally, comparison of (A.7) with (2.2) gives formulas for $\overline{X}_1, \overline{X}_2$

$$\overline{X}_j(\mathbf{x}) = R_j(\overline{X}_3(\mathbf{x}),\mathbf{x}), \qquad j=1,2. \tag{A.8}$$

Next, we show that if $\overline{X}_j(\mathbf{x})$ are given by equation (A.8) then they satisfy equation (2.3). Differentiating equation (A.8) gives, for $j=1,2$

$$\nabla\overline{X}_j(\mathbf{x}) = \nabla R_j(\overline{X}_3(\mathbf{x}),\mathbf{x}) + \frac{\partial R_j(\overline{X}_3(\mathbf{x}),\mathbf{x})}{\partial s}\nabla\overline{X}_3(\mathbf{x}) \tag{A.9}$$

where

$$\nabla R_j = \nabla R_j(s,\mathbf{x})|_{s=\overline{X}_3(\mathbf{x})}.$$

Therefore

$$\begin{aligned} \nabla\overline{X}_3(\mathbf{x})\cdot\nabla\overline{X}_j(\mathbf{x}) &= \nabla\overline{X}_3(\mathbf{x})\cdot\nabla R_j(\overline{X}_3(\mathbf{x}),\mathbf{x}) \\ -\frac{|\nabla\overline{X}_3(\mathbf{x})|^2}{|\nabla\overline{X}_3(R_j(\overline{X}_3(\mathbf{x}),\mathbf{x}))|^2}&\frac{\partial\overline{X}_3}{\partial x_j}(R_j(\overline{X}_3(\mathbf{x}),\mathbf{x})), \qquad j=1,2. \end{aligned} \tag{A.10}$$

To see that the right hand side of equation (A.10) vanishes, we use the semigroup property of differential equation (A.1).

$$\mathbf{R}(s_1+s_2,\mathbf{x}) = \mathbf{R}(s_2,\mathbf{R}(s_1,\mathbf{x})). \tag{A.11}$$

Differentiation of equation (A.11) with respect to s_1 gives

$$-\frac{\nabla\overline{X}_3(\mathbf{R}(s_1+s_2,\mathbf{x}))}{|\nabla\overline{X}_3(\mathbf{R}(s_1+s_2,\mathbf{x}))|^2} = -\frac{\nabla\overline{X}_3(\mathbf{R}(s_1,\mathbf{x}))}{|\nabla\overline{X}_3(\mathbf{R}(s_1,\mathbf{x}))|^2}\cdot\nabla\mathbf{R}(s_2,\mathbf{R}(s_1,\mathbf{x})). \quad \text{(A.12)}$$

Now putting $s_1=0, s_2=\overline{X}_3(\mathbf{x})$ yields

$$\frac{\nabla\overline{X}_3(\mathbf{R}(\overline{X}_3(\mathbf{x}),\mathbf{x}))}{|\nabla\overline{X}_3(\mathbf{R}(\overline{X}_3(\mathbf{x}),\mathbf{x}))|^2} = \frac{\nabla\overline{X}_3(\mathbf{x})}{|\nabla\overline{X}_3(\mathbf{x})|^2}\cdot\nabla\mathbf{R}(\overline{X}_3(\mathbf{x}),\mathbf{x}) \quad \text{(A.13)}$$

which substituted into equation (A.10) shows that $\nabla\overline{X}_j\cdot\nabla\overline{X}_3=0, \quad j=1,2.$