

APPROXIMATING INVERSES OF TOEPLITZ MATRICES BY CIRCULANT MATRICES*

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Abstract. With every continuous function a on the complex unit circle one can associate a sequence $\{T_n(a)\}_{n=1}^\infty$ of Toeplitz matrices and a sequence $\{C_n(a)\}_{n=1}^\infty$ of circulant matrices. By employing some advanced results on the finite sections of Toeplitz operators, we prove asymptotic estimates for the central columns of the matrices $T_n^{-1}(a) - C_n^{-1}(a)$ as $n \rightarrow \infty$. Our results generalize and sharpen recent results by T. Strohmer and by F.-W. Sun, Y. Jiang, and J. S. Baras, who also discussed the relevancy of the problem in signal processing.

Key words. Toeplitz matrix, circulant matrix, finite-term strong convergence, signal processing

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1. Introduction. Let a be a continuous complex-valued function on the complex unit circle, $a \in C(\mathbf{T})$, and denote by $\{a_m\}_{m=-\infty}^\infty$ the sequence of the Fourier coefficients,

$$a_m = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-im\theta} d\theta.$$

The $n \times n$ Toeplitz matrix generated by a is the matrix

$$T_n(a) = (a_{j-k})_{j,k=1}^n.$$

An $n \times n$ circulant matrix $C_n(a)$ can be associated with a as follows. Let $\omega_n = \exp(2\pi i/n)$, define the unitary matrix U_n by

$$U_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix},$$

and put

$$C_n(a) = U_n^* \operatorname{diag}\left(a(1), a(\omega_n), \dots, a(\omega_n^{n-1})\right) U_n. \tag{1}$$

It is readily verified that $C_n(a)$ is a circulant matrix (see also Theorem 3.2.3 of [4]).

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Now suppose a has no zeros on the unit circle \mathbf{T} . Then $C_n(a)$ is obviously invertible for all n and the inverse matrix $C_n^{-1}(a) := (C_n(a))^{-1}$ is again a circulant matrix:

$$C_n^{-1}(a) = U_n^* \operatorname{diag} \left(\frac{1}{a(1)}, \frac{1}{a(\omega_n)}, \dots, \frac{1}{a(\omega_n^{n-1})} \right) U_n. \tag{2}$$

Invertibility of Toeplitz matrices is a more delicate issue. However, if in addition the winding number of a about the origin is zero, then $T_n(a)$ is known to be invertible for all sufficiently large n (see, e.g., [3, Theorem 2.11] or [5, Theorem III.2.1]). In several contexts (see [7] for an example from signal processing) it is desirable to replace the inverse $T_n^{-1}(a) := (T_n(a))^{-1}$ of the Toeplitz matrix $T_n(a)$ by the circulant matrix $C_n^{-1}(a)$. This leads to the problem of estimating the difference $T_n^{-1}(a) - C_n^{-1}(a)$ in some sense.

We think of an $n \times n$ matrix as an operator on \mathbf{C}^n in the natural fashion. We equip \mathbf{C}^n with the ℓ^2 norm. The operator norm associated with this vector norm is the spectral norm. We use $\|\cdot\|$ for both the ℓ^2 norm on \mathbf{C}^n and the spectral norm. One cannot expect that $\|T_n^{-1}(a) - C_n^{-1}(a)\| \rightarrow 0$ as $n \rightarrow \infty$. The next question therefore is whether $\|T_n^{-1}(a)x^{(n)} - C_n^{-1}(a)x^{(n)}\|$ goes to 0 as $n \rightarrow \infty$ for certain specific unit vectors $x^{(n)} \in \mathbf{C}^n$. For instance, if $x^{(n)} = (1, 0, \dots, 0)$, we arrive at the question whether the first column of $C_n^{-1}(a)$ is a good approximation to the first column of $T_n^{-1}(a)$ as $n \rightarrow \infty$. There are examples which show that this need not to be the case. Interestingly, F.-W. Sun, Y. Jiang, and J. S. Baras [7] recently observed that if the support of $x^{(n)}$ is concentrated around the midst of $\{1, 2, \dots, n\}$ (for example, if $x^{(n)} = (0, \dots, 0, 1, 0, \dots, 0)$ with about $n/2$ zeros before and after the unit), then $\|T_n^{-1}(a)x^{(n)} - C_n^{-1}(a)x^{(n)}\|$ goes indeed to zero in important cases.

A slight modification of the notion of convergence introduced in [7] is as follows. For every natural number K , let \mathbf{C}_K^n be the set of all nonzero vectors in \mathbf{C}^n of the form

$$x = (0, \dots, 0, x_{[n/2]-K}, \dots, x_{[n/2]+K}, 0, \dots, 0),$$

where $[\cdot]$ denotes the integral part. In other terms, $x \in \mathbf{C}_K^n$ if and only if $x \in \mathbf{C}^n \setminus \{0\}$ and $x_j = 0$ for $j < [n/2] - K$ and $j > [n/2] + K$. Now let $\{A_n\}_{n=1}^\infty$ and $\{B_n\}_{n=1}^\infty$ be two sequences of $n \times n$ matrices A_n and B_n . We say that $\{B_n\}$ approximates $\{A_n\}$ in the sense of finite-term strong convergence if

$$\max_{x \in \mathbf{C}_K^n} \frac{\|A_n x - B_n x\|}{\|x\|} \rightarrow 0 \text{ as } n \rightarrow \infty \tag{3}$$

for each K . (The authors of [7] say that “ A_n converges to B_n in the finite-term strong sense” if (3) holds for each K .)

For $\mu \geq 0$, let W^μ be the set of all $a \in C(\mathbf{T})$ that satisfy

$$\|a\|_{W^\mu} := \sum_{n=-\infty}^\infty (|n| + 1)^\mu |a_n| < \infty.$$

Furthermore, let \mathcal{P} denote the collection of all Laurent polynomials, that is, the collection of all $a \in C(\mathbf{T})$ with only finitely many nonzero Fourier coefficients. Paper [7] concerns positive functions, that is, functions $a : \mathbf{T} \rightarrow (0, \infty)$, and hence positively

definite Hermitian matrices $T_n(a)$ and $C_n(a)$. The main results of [7] say that if a is positive and in W^1 , then

$$\max_{x \in \mathbf{C}_K^n} \frac{\|T_n^{-1}(a)x - C_n^{-1}(a)x\|}{\|x\|} \leq \frac{M_{K,a}}{\sqrt{n}} \tag{4}$$

with some constant $M_{K,a}$ depending only on K and a , and that if a is positive and in \mathcal{P} , then

$$\max_{x \in \mathbf{C}_K^n} \frac{\|T_n^{-1}(a)x - C_n^{-1}(a)x\|}{\|x\|} \leq \frac{M_{K,a}}{n} \tag{5}$$

with some constant $M_{K,a}$ that depends only on K and a . Notice that the positivity of a guarantees that $T_n(a)$ and $C_n(a)$ are invertible for all $n \geq 1$. The approach of [7] is based on more or less straightforward estimation of $\|T_n^{-1}(a)x - C_n^{-1}(a)x\|$.

Earlier Strohmer [6] showed that if a is a positive Laurent polynomial of degree s and $K \leq s$, then one can replace the right-hand side of (5) by $M_{\gamma,a} e^{-\gamma n}$ with some $\gamma > 0$.

We here invoke some advanced results on Toeplitz operators in order to generalize and to sharpen (4) and (5). First, we replace the positivity of a by the requirement that a has no zeros on \mathbf{T} and that the winding number of a about the origin is zero. Secondly, we show that if a is in W^μ with $\mu > 0$, then (4) holds with the right-hand side replaced by $M_{K,\mu,a} n^{-\mu/2}$ and we also prove the right-hand side of (5) can in fact be replaced by $M_{K,\gamma,a} e^{-\gamma n}$, where γ is a positive constant. Notice that we do not need the constraint $K \leq s$ appearing in [6]. Thus, the middle columns of the inverses of banded Toeplitz matrices are approximated by the corresponding columns of appropriate circulant matrices exponentially good. We finally establish that the left-hand sides of (4) and (5) go to zero for arbitrary continuous functions a with no zeros on \mathbf{T} and with winding number zero.

2. Finite sections of Toeplitz operators. In this section we collect some known results that will be employed in the proofs of our main theorems.

We will consider two kinds of norms on \mathbf{C}^n ; one is generated by an exponential weight and the other one by a power weight. Accordingly, we put

$$\|x\|_{E,\alpha}^2 := \sum_{j=1}^n e^{2\alpha j} |x_j|^2, \quad \|x\|_{P,\alpha}^2 := \sum_{j=1}^n j^{2\alpha} |x_j|^2.$$

The corresponding matrix norms will be denoted by $\|\cdot\|_{E,\alpha,\alpha}$ and $\|\cdot\|_{P,\alpha,\alpha}$:

$$\|A\|_{E,\alpha,\alpha} := \max_{x \in \mathbf{C}^n \setminus \{0\}} \frac{\|Ax\|_{E,\alpha}}{\|x\|_{E,\alpha}}, \quad \|A\|_{P,\alpha,\alpha} := \max_{x \in \mathbf{C}^n \setminus \{0\}} \frac{\|Ax\|_{P,\alpha}}{\|x\|_{P,\alpha}}.$$

For $\alpha = 0$, both vector norms become the usual ℓ^2 norm $\|\cdot\|$ on \mathbf{C}^n and both matrix norms are just the spectral norm $\|\cdot\|$.

Let first $b \in \mathcal{P}$. The values $b(z)$ make sense for every $z \in \mathbf{C} \setminus \{0\}$. If b has no zeros on \mathbf{T} , we may choose a number $R > 1$ such that $b(z) \neq 0$ for all complex numbers z satisfying $1/R \leq |z| \leq R$. Put

$$\beta = \log R. \tag{6}$$

THEOREM 1. [1, Theorem 2.4] *Let $b \in \mathcal{P}$ have no zeros on \mathbf{T} and winding number zero about the origin. Define β by (6). Then there is an $n_b \geq 1$ and a constant $N_{b,\beta} < \infty$ such that*

$$\|T_n^{-1}(b)\|_{E,\beta,\beta} \leq N_{b,\beta} \text{ for all } n \geq n_b.$$

For generating functions in W^μ we have the following estimate, which was originally established by Verbitsky and Krupnik [8].

THEOREM 2. [8] or [2, Theorem 7.25] *Let $b \in W^\mu$ ($\mu > 0$) and suppose b has no zeros on \mathbf{T} and winding number zero about the origin. Then there exist an $n_b \geq 1$ and a constant $N_{b,\mu} < \infty$ such that*

$$\|T_n^{-1}(b)\|_{P,\mu,\mu} \leq N_{b,\mu} \text{ for all } n \geq n_b.$$

Finally, under the sole assumption that b be continuous, a classical result by Gohberg and Feldman is as follows.

THEOREM 3. [5, Theorem III.2.1] *If $b \in C(\mathbf{T})$ has no zeros on \mathbf{T} and winding number zero about the origin, then there are $n_b \geq 1$ and $N_b < \infty$ such that*

$$\|T_n^{-1}(b)\| \leq N_b \text{ for all } n \geq n_b.$$

3. Laurent polynomials. In the case where a is a Laurent polynomial, $a \in \mathcal{P}$, the Toeplitz matrix $T_n(a)$ is banded and the circulant matrix $C_n(a)$ results from $T_n(a)$ simply by “periodization”: for example, if

$$a(t) = a_{-2}t^{-2} + a_{-1}t^{-1} + a_0 + a_1t + a_2t^2 + a_3t^3, \quad t = e^{i\theta},$$

then

$$T_8(a) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_0 & a_{-1} & a_{-2} & 0 & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & 0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & 0 \\ 0 & 0 & a_3 & a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ 0 & 0 & 0 & a_3 & a_2 & a_1 & a_0 & a_{-1} \\ 0 & 0 & 0 & 0 & a_3 & a_2 & a_1 & a_0 \end{pmatrix}$$

and

$$C_8(a) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & 0 & 0 & a_3 & a_2 & a_1 \\ a_1 & a_0 & a_{-1} & a_{-2} & 0 & 0 & a_3 & a_2 \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} & 0 & 0 & a_3 \\ a_3 & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & 0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & 0 \\ 0 & 0 & a_3 & a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ a_{-2} & 0 & 0 & a_3 & a_2 & a_1 & a_0 & a_{-1} \\ a_{-1} & a_{-2} & 0 & 0 & a_3 & a_2 & a_1 & a_0 \end{pmatrix}.$$

To see this, let $\text{circ}(b_0, b_1, \dots, b_{n-1})$ denote the circulant matrix whose first column is $(b_0 \ b_1 \ \dots \ b_{n-1})^\top$,

$$\text{circ}(b_0, b_1, \dots, b_{n-1}) = \begin{pmatrix} b_0 & b_{n-1} & \dots & b_1 \\ b_1 & b_0 & \dots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & b_{n-2} & \dots & b_0 \end{pmatrix}.$$

A straightforward computation shows that

$$\text{circ}(b_0, b_1, \dots, b_{n-1}) = U_n^* \text{diag}(b(1), b(\omega_n), \dots, b(\omega_n^{n-1})) U_n, \tag{7}$$

where

$$b(t) = b_0 + b_1 t + \dots + b_{n-1} t^{n-1} \tag{8}$$

(which is a well known result of the theory of circulant matrices; see, e.g., [4, Theorem 3.2.2]). Now suppose we are given a Laurent polynomial

$$a(t) = \sum_{j=-s}^s a_j t^j. \tag{9}$$

What we claim is that

$$C_n(a) = \text{circ}(a_0, a_1, \dots, a_s, 0, \dots, 0, a_{-s}, \dots, a_{-2}, a_{-1})$$

for $n \geq 2s + 1$. But this is immediate from (1) and (7) since in the case at hand the polynomial (8) equals

$$b(t) = a_0 + a_1 t + \dots + a_s t^s + a_{-s} t^{n-s} + \dots + a_{-2} t^{n-2} + a_{-1} t^{n-1}$$

and hence, because $\omega_n^n = 1$,

$$\begin{aligned} b(\omega_n^j) &= a_0 + a_1 \omega_n^j + \dots + a_s \omega_n^{sj} + a_{-s} \omega_n^{-sj} + \dots + a_{-2} \omega_n^{-2j} + a_{-1} \omega_n^{-j} \\ &= a(\omega_n^j) \end{aligned}$$

for all j .

Thus, for $n \geq 2s + 1$ we have $C_n(a) - T_n(a) = G_n(s) + H_n(s)$ where

$$G_n(s) = \begin{pmatrix} 0 & G_s \\ 0 & 0 \end{pmatrix}, \quad H_n(s) = \begin{pmatrix} 0 & 0 \\ H_s & 0 \end{pmatrix}$$

with fixed $s \times s$ matrices G_s and H_s and with zero matrices 0 of appropriate sizes.

THEOREM 4. *Let $a \in \mathcal{P}$ have no zeros on \mathbf{T} and winding number zero about the origin. Choose an $R > 1$ so that $a(z) \neq 0$ for $1/R \leq |z| \leq R$ and put $\beta = \log R$. Then there are an $n_a \geq 1$ and constants $M_{K,\beta,a} < \infty$ such that $T_n(a)$ is invertible for $n \geq n_a$ and*

$$\max_{x \in \mathbf{C}_K^n} \frac{\|T_n^{-1}(a)x - C_n^{-1}(a)x\|}{\|x\|} \leq M_{K,\beta,a} e^{-\beta n/2} \text{ for all } n \geq n_a.$$

Proof. The existence of n_a is guaranteed by Theorem 3. Clearly, it suffices to prove that for each fixed integer k there is a constant $D_{k,\beta,a} < \infty$ such that

$$\|T_n^{-1}(a)e_{[n/2]+k} - C_n^{-1}(a)e_{[n/2]+k}\| \leq D_{k,\beta,a} e^{-\beta n/2} \quad (10)$$

for all $n \geq n_a$, where $e_j \in \mathbf{C}^n$ is the vector whose j th component is 1 and the remaining components of which are zero. With $\Delta_m : \mathbf{C}^n \rightarrow \mathbf{C}^n$ defined by

$$\Delta_m \left(\sum_{j=1}^n x_j e_j \right) = x_m e_m,$$

inequality (10) is equivalent to the estimate

$$\|T_n^{-1}(a)\Delta_{[n/2]+k} - C_n^{-1}(a)\Delta_{[n/2]+k}\| \leq D_{k,\beta,a} e^{-\beta n/2}. \quad (11)$$

We have

$$T_n^{-1}(a) - C_n^{-1}(a) = C_n^{-1}(a) \left(C_n(a) - T_n(a) \right) T_n^{-1}(a).$$

From (2) we infer that $\|C_n^{-1}(a)\| \leq \|a^{-1}\|_\infty$, where $\|\cdot\|_\infty$ is the maximum norm in $C(\mathbf{T})$. We are therefore left with estimating

$$\begin{aligned} & \| (C_n(a) - T_n(a)) T_n^{-1}(a) \Delta_{[n/2]+k} \| \\ &= \| (G_n(s) + H_n(s)) T_n^{-1}(a) \Delta_{[n/2]+k} \|. \end{aligned} \quad (12)$$

Define $P_m : \mathbf{C}^n \rightarrow \mathbf{C}^n$ and $Q_m : \mathbf{C}^n \rightarrow \mathbf{C}^n$ by

$$P_m \left(\sum_{j=1}^n x_j e_j \right) = \sum_{j=1}^m x_j e_j, \quad Q_m \left(\sum_{j=1}^n x_j e_j \right) = \sum_{j=m+1}^n x_j e_j.$$

Obviously,

$$\begin{aligned} & (G_n(s) + H_n(s)) T_n^{-1}(a) \Delta_{[n/2]+k} \\ &= P_s G_n(s) Q_{n-s} T_n^{-1}(a) \Delta_{[n/2]+k} + Q_{n-s} H_n(s) P_s T_n^{-1}(a) \Delta_{[n/2]+k} \end{aligned}$$

and hence (12) does not exceed

$$\begin{aligned} & \| P_s G_n(s) Q_{n-s} T_n^{-1}(a) \Delta_{[n/2]+k} \| + \| Q_{n-s} H_n(s) P_s T_n^{-1}(a) \Delta_{[n/2]+k} \| \\ & \leq \| G_s \| \| Q_{n-s} T_n^{-1}(a) \Delta_{[n/2]+k} \| + \| H_s \| \| P_s T_n^{-1}(a) \Delta_{[n/2]+k} \|. \end{aligned} \quad (13)$$

The operators $\Delta_{[n/2]+k}$, P_s , Q_{n-s} are selfadjoint, while $T_n^*(a) = T_n(\bar{a})$ with

$$\bar{a}(t) := \overline{a(t)} = \sum_{j=-s}^s \bar{a}_j t^{-j}, \quad t = e^{i\theta}.$$

Consequently,

$$\| P_s T_n^{-1}(a) \Delta_{[n/2]+k} \| = \| \Delta_{[n/2]+k} T_n^{-1}(\bar{a}) P_s \|. \quad (14)$$

We define \tilde{a} by

$$\tilde{a}(t) = \sum_{j=-s}^s a_j t^{-j}, \quad t = e^{i\theta}.$$

It is readily verified that $T_n(\tilde{a}) = W_n T(a) W_n$, where $W_n : \mathbf{C}^n \rightarrow \mathbf{C}^n$ acts by the rule

$$W_n \left(\sum_{j=1}^n x_j e_j \right) = \sum_{j=1}^n x_{n-j+1} e_j$$

(reversal of coordinates). Since

$$Q_{n-s} W_n = W_n P_s, \quad W_n \Delta_{[n/2]+k} W_n = \Delta_{n+1-[n/2]-k} =: \Delta_{[n/2]-k^*},$$

it follows that

$$\begin{aligned} Q_{n-s} T_n^{-1}(a) \Delta_{[n/2]+k} &= Q_{n-s} W_n T_n^{-1}(\tilde{a}) W_n \Delta_{[n/2]+k} W_n W_n \\ &= W_n P_s T_n^{-1}(\tilde{a}) \Delta_{[n/2]-k^*} W_n \end{aligned}$$

and hence

$$\begin{aligned} \|Q_{n-s} T_n^{-1}(a) \Delta_{[n/2]+k}\| &= \|W_n P_s T_n^{-1}(\tilde{a}) \Delta_{[n/2]-k^*} W_n\| \\ &= \|P_s T_n^{-1}(\tilde{a}) \Delta_{[n/2]-k^*}\| = \|\Delta_{[n/2]-k^*} T_n^{-1}(\tilde{a}) P_s\|. \end{aligned} \quad (15)$$

We now estimate the right-hand sides of (14) and (15) with the help of Theorem 1. Let $b = \bar{a}$ and $\ell = k$ in the case of (14), and put $b = \tilde{\bar{a}}$ and $\ell = k^*$ in the case of (15). For every $x \in \mathbf{C}^n$ we obtain from Theorem 1 that

$$\begin{aligned} &\|\Delta_{[n/2]\pm\ell} T_n^{-1}(b) P_s x\|^2 \\ &= e^{-2([n/2]\pm\ell)\beta} \|\Delta_{[n/2]\pm\ell} T_n^{-1}(b) P_s x\|_{E,\beta}^2 \\ &\leq e^{-2([n/2]\pm\ell)\beta} N_{b,\beta}^2 \|P_s x\|_{E,\beta}^2 \\ &\leq e^{-2([n/2]\pm\ell)\beta} N_{b,\beta}^2 e^{2s\beta} \|x\|^2, \end{aligned}$$

which gives

$$\|\Delta_{[n/2]\pm\ell} T_n^{-1}(b) P_s\| \leq F_{k,\beta,b} e^{-\beta n/2} \quad (16)$$

with some constant $F_{k,\beta,b} < \infty$. Estimating (14) and (15) by (16) and inserting the result in (13) we arrive at (11). \square

4. Smooth generating functions. The following result is a generalization of estimate (4).

THEOREM 5. *Let $a \in W^\mu$ ($\mu > 0$) and suppose a has no zeros on \mathbf{T} and winding number zero about the origin. Then there are an $n_a \geq 1$ and constants $M_{K,\mu,a} < \infty$ such that*

$$\max_{x \in \mathbf{C}_K^n} \frac{\|T_n^{-1}(a)x - C_n^{-1}(a)x\|}{\|x\|} \leq \frac{M_{K,\mu,a}}{n^{\mu/2}} \text{ for all } n \geq n_a.$$

Proof. By Theorem 3, $T_n(a)$ is invertible for all sufficiently large n . We define $f_n \in \mathcal{P}$ by

$$f_n(t) = \sum_{j=-\lfloor\sqrt{n}\rfloor}^{\lfloor\sqrt{n}\rfloor} a_j t^j.$$

For every $x \in \mathbf{C}^n$, we have

$$\begin{aligned} \|T_n^{-1}(a)x - C_n^{-1}(a)x\| &\leq \|T_n^{-1}(a)x - T_n^{-1}(f_n)x\| \\ &\quad + \|C_n^{-1}(a)x - C_n^{-1}(f_n)x\| \\ &\quad + \|T_n^{-1}(f_n)x - C_n^{-1}(f_n)x\|. \end{aligned} \quad (17)$$

By (2), the second term on the right of (17) is

$$\begin{aligned} \|C_n^{-1}(a)x - C_n^{-1}(f_n)x\| &\leq \|a^{-1} - f_n^{-1}\|_\infty \|x\| \\ &\leq \|a^{-1}\|_\infty \|f_n^{-1}\|_\infty \|a - f_n\|_\infty \|x\|. \end{aligned}$$

Since $\|f_n^{-1}\|_\infty \rightarrow \|a^{-1}\|_\infty$ and

$$\begin{aligned} \|a - f_n\|_\infty &\leq \sum_{|j|>\lfloor\sqrt{n}\rfloor} |a_j| \leq \frac{1}{(\lfloor\sqrt{n}\rfloor + 1)^\mu} \sum_{|j|>\lfloor\sqrt{n}\rfloor} |a_j| (|j| + 1)^\mu \\ &\leq \frac{1}{(\lfloor\sqrt{n}\rfloor + 1)^\mu} \|a\|_{W^\mu}, \end{aligned} \quad (18)$$

we see that

$$\|C_n^{-1}(a)x - C_n^{-1}(f_n)x\| \leq \frac{D_{\mu,a}^{(1)}}{n^{\mu/2}} \|x\| \quad (19)$$

with some constant $D_{\mu,a}^{(1)} < \infty$. For the first term on the right of (17) we have

$$\begin{aligned} \|T_n^{-1}(a)x - T_n^{-1}(f_n)x\| &\leq \|T_n^{-1}(a)\| \|T_n^{-1}(f_n)\| \|T_n(a - f_n)\| \|x\| \\ &\leq \|T_n^{-1}(a)\| \|T_n^{-1}(f_n)\| \|a - f_n\|_\infty \|x\|. \end{aligned}$$

The norm $\|a - f_n\|_\infty$ can be estimated by (18). Furthermore, $\|T_n^{-1}(a)\| \leq N_a$ for all sufficiently large n due to Theorem 3. Write $f_n = a + \delta_n$. Then

$$T_n^{-1}(f_n) = \left(I + T_n^{-1}(a)T_n(\delta_n) \right)^{-1} T_n^{-1}(a), \quad (20)$$

and since $\|T_n^{-1}(a)\| \leq N_a$ and $\|T_n(\delta_n)\| \leq \|\delta_n\|_\infty = o(1)$, it follows that $\|T_n^{-1}(f_n)\|$ also remains bounded as $n \rightarrow \infty$. Thus,

$$\|T_n^{-1}(a)x - T_n^{-1}(f_n)x\| \leq \frac{D_{\mu,a}^{(2)}}{n^{\mu/2}} \|x\| \quad (21)$$

with some constant $D_{\mu,a}^{(2)} < \infty$ for all sufficiently large n .

To tackle the third term on the right of (17), we proceed as in the proof of Theorem 4. We first note that

$$\begin{aligned} &\|(T_n^{-1}(f_n) - C_n^{-1}(f_n))\Delta_{[n/2]+k}\| \\ &\leq \|f_n^{-1}\|_\infty \|G_{\lfloor\sqrt{n}\rfloor}\| \|\Delta_{[n/2]-k^*} T_n^{-1}(\tilde{f}_n) P_{\lfloor\sqrt{n}\rfloor}\| \\ &\quad + \|f_n^{-1}\|_\infty \|H_{\lfloor\sqrt{n}\rfloor}\| \|\Delta_{[n/2]+k} T_n^{-1}(\tilde{f}_n) P_{\lfloor\sqrt{n}\rfloor}\|, \end{aligned} \quad (22)$$

where $G_{[\sqrt{n}]}$ and $H_{[\sqrt{n}]}$ are $[\sqrt{n}] \times [\sqrt{n}]$ matrices whose norms are bounded by $\|f_n\|_{W^0} \leq \|a\|_{W^0} \leq \|a\|_{W^\mu}$. As already noticed, $\|f_n^{-1}\|_\infty \rightarrow \|a^{-1}\|_\infty$ as $n \rightarrow \infty$. To estimate the remaining terms on the right of (22) we use Theorem 2. Let $b_n = \bar{f}_n$, $\ell = k$ or $b_n = \tilde{f}_n$, $\ell = k^*$. Then for every $x \in \mathbf{C}^n$,

$$\begin{aligned} & \|\Delta_{[n/2] \pm \ell} T_n^{-1}(b_n) P_{[\sqrt{n}]} x\|^2 \\ &= \frac{1}{([n/2] \pm \ell)^{2\mu}} \|\Delta_{[n/2] \pm \ell} T_n^{-1}(b_n) P_{[\sqrt{n}]} x\|_{P, \mu}^2 \\ &\leq \frac{1}{([n/2] \pm \ell)^{2\mu}} \|T_n^{-1}(b_n)\|_{P, \mu, \mu}^2 \|P_{[\sqrt{n}]} x\|_{P, \mu}^2 \\ &\leq \frac{1}{([n/2] \pm \ell)^{2\mu}} \|T_n^{-1}(b_n)\|_{P, \mu, \mu}^2 [\sqrt{n}]^{2\mu} \|x\|^2. \end{aligned} \tag{23}$$

We have $b_n = \bar{f}_n = \bar{a} + \delta_n$ or $b_n = \tilde{f}_n = \tilde{a} + \delta_n$ with $\|\delta_n\|_{W^\mu} \rightarrow 0$. Theorem 2 tells us that $\|T_n^{-1}(\bar{a})\|_{P, \mu, \mu}$ and $\|T_n^{-1}(\tilde{a})\|_{P, \mu, \mu}$ remain bounded as $n \rightarrow \infty$. Since

$$\|T_n(\delta_n)\|_{P, \mu, \mu} \leq \|\delta_n\|_{W^\mu} = o(1),$$

we infer from identity (20) (with a replaced by \bar{a} or \tilde{a}) that $\|T_n^{-1}(b_n)\|_{P, \mu, \mu}$ is bounded for all sufficiently large n . Thus, (23) yields

$$\|\Delta_{[n/2] \pm \ell} T_n^{-1}(b_n) P_{[\sqrt{n}]} x\| \leq \frac{D_{k, \mu, a}^{(3)}}{n^{\mu/2}} \|x\|$$

for n sufficiently large. The last estimate together with (22) implies that

$$\|(T_n^{-1}(f_n) - C_n^{-1}(f_n))x\| \leq \frac{D_{K, \mu, a}^{(4)}}{n^{\mu/2}} \|x\| \tag{24}$$

for all $x \in \mathbf{C}_K^n$ and all sufficiently large n . Inserting (19), (21), (24) in (17) we get the assertion. \square

5. Continuous generating functions. The following result concerns arbitrary continuous generating functions. Clearly, as no additional smoothness is required, we cannot expect estimates for the speed of the finite-term strong convergence.

THEOREM 6. *Let $a \in C(\mathbf{T})$ and suppose a has no zeros on \mathbf{T} and winding number zero about the origin. Then*

$$\max_{x \in \mathbf{C}_K^n} \frac{\|T_n^{-1}(a)x - C_n^{-1}(a)x\|}{\|x\|} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. Let $\{f_m\}_{m=1}^\infty$ be a sequence of Laurent polynomials such that f_m is of degree m and $\|a - f_m\|_\infty \rightarrow 0$ as $m \rightarrow \infty$. For $x \in \mathbf{C}_K^n$, we have

$$\begin{aligned} \|T_n^{-1}(a)x - C_n^{-1}(a)x\| &\leq \|T_n^{-1}(a)x - T_n^{-1}(f_m)x\| \\ &\quad + \|C_n^{-1}(a)x - C_n^{-1}(f_m)x\| \\ &\quad + \|T_n^{-1}(f_m)x - C_n^{-1}(f_m)x\|. \end{aligned}$$

Arguing as in the proof of Theorem 5 we first get

$$\begin{aligned} \|T_n^{-1}(a)x - T_n^{-1}(f_m)x\| &\leq \|T_n^{-1}(a)\| \|T_n^{-1}(f_m)\| \|a - f_m\|_\infty \|x\|, \\ \|C_n^{-1}(a)x - C_n^{-1}(f_m)x\| &\leq \|a^{-1}\|_\infty \|f_m^{-1}\|_\infty \|a - f_m\|_\infty \|x\|, \end{aligned}$$

and then the existence of numbers n_0 and m_0 such that

$$\begin{aligned}\|T_n^{-1}(a)x - T_n^{-1}(f_{m_0})x\| &< \frac{\varepsilon}{3} \|x\|, \\ \|C_n^{-1}(a)x - C_n^{-1}(f_{m_0})x\| &< \frac{\varepsilon}{3} \|x\|\end{aligned}$$

for all $n \geq n_0$. We may without loss of generality assume that f_{m_0} has no zeros on \mathbf{T} and that the winding number of f_{m_0} about the origin is zero. Theorem 4 implies that

$$\|T_n^{-1}(f_{m_0})x - C_n^{-1}(f_{m_0})x\| < \frac{\varepsilon}{3} \|x\|$$

for all $x \in \mathbf{C}_K^n$ if only n is large enough. This gives the assertion. \square

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