METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 11, No. 2, pp. 179–210, June 2004

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BIFURCATION OF NONLINEAR EQUATIONS: II. DYNAMIC BIFURCATION*

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Abstract. We study in this article dynamic bifurcation of nonlinear evolution equations due to higher order nonlinear terms, focusing on detailed bifurcation behavior of nonlinear evolution equations in the cases where the algebraic multiplicity of the eigenvalues of the linearized problem is one or two.

Key words. dynamic bifurcation, attractor bifurcation, periodic solutions, stability of bifurcated solutions

AMS subject classifications. 37L, 37G, 47J, 58E

1. Introduction.

This article, which is Part II of a series of two articles, studies dynamic bifurcation of nonlinear evolution equations, and Part I deals with steady state bifurcations of nonlinear equations. The main objective of these articles is to study (both steady state and dynamic) bifurcations when the eigenvalue of the linearized problem may have even multiplicity. The key idea is to analyze precisely the effect of the higher-order nondegenerate nonlinear terms.

The main focus of this article is on the cases where the eigenvalue of the linearized problem has either (algebraic) multiplicity one or two. In the case where the eigenvalue is simple, our main theorems include (a) Theorems 3.1, 3.2 and 3.6, in the case where the eigenvalue is simple, and (b) Theorems 4.2, 4.3 and 5.1 in the case where the eigenvalue has multiplicity two. These theorems provide a complete characterization of the bifurcation and the stability of the bifurcated solutions in this two cases, and bifurcated attractors are classified.

The main results obtained can be easily applied to bifurcation problems in partial differential equations from science and engineering. To demonstrate the applications, we present an example of a system of two second order parabolic equations. Bifurcation is obtained at the first eigenvalue, which has either multiplicity 1 or 2.

This article is organized as follows. In Section 2, we introduce some preliminary results. Section 3 studies bifurcation when the eigenvalue is simple. Sections 4 and 5 are concerned with bifurcation when the eigenvalue has multiplicity two, and in particular, bifurcation to periodic solutions from a real eigenvalue with multiplicity two is given. Section 6 gives an application to a system of nonlinear parabolic equations.

2. Preliminaries.

For convenience, we recall in this section some basic results and concepts which will be used in the throughout of this article.

* Received May 10, 2004; accepted for publication September 28, 2004. The authors are grateful for the referees' insightful comments. The work was supported in part by the Office of Naval Research, by the National Science Foundation, and by the National Science Foundation of China.

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2.1. Set-up. Let H and H_1 be two Hilbert spaces, and $H_1 \to H$ be a dense and compact inclusion. We consider the following nonlinear evolution equation

$$\frac{du}{dt} = L_\lambda u + G(u, \lambda), \tag{2.1}$$

$$u(0) = u_0, \tag{2.2}$$

where $L_\lambda : h_1 \to H$ is a family of linear completely continuous fields depending continuously on $\lambda \in \mathbb{R}$, such that

$$\begin{cases} L_\lambda = -A + B_\lambda \text{ is a sectorial operator,} \\ A : H_1 \to H \text{ a linear homeomorphism,} \\ B_\lambda : H_1 \to H \text{ a linear compact operator.} \end{cases} \tag{2.3}$$

It is known that L_λ generates an analytic semigroup $\{e^{-tL_\lambda}\}_{t\geq 0}$, and we can define fractional power operators L_λ^α for $\alpha \in \mathbb{R}$ with domain $H_\alpha = D(L_\lambda^\alpha)$ such that $H_{\alpha_1} \subset H_{\alpha_2}$ is compact if $\alpha_1 > \alpha_2, H_0 = H$, and $H_1 = H_{\alpha=1}$.

Furthermore, we assume that for some $\theta < 1$ the nonlinear operator $G(\cdot, \lambda) = H_\theta \to H_0$ is C^r bounded operator ($r \geq 1$), and

$$G(u, \lambda) = o(\|u\|_\theta), \forall\ \lambda \in \mathbb{R}. \tag{2.4}$$

Let $\{S_\lambda(t)\}_{t\geq 0}$ be an operator semigroup generated by (2.1), and the solution of (2.1) and (2.2) can be expressed as

$$u(t, u_0) = S_\lambda(t)u_0,\ \forall\ t \geq 0.$$

Definition 2.1.

(1) *We say that the equation (2.1) bifurcates from* $(u, \lambda) = (0, \lambda_0)$ *an invariant set* Ω_λ*, if there exists a sequence of invariant sets* $\{\Omega_{\lambda_n}\}$ *of (2.1),* $0 \notin \Omega_{\lambda_n}$ *such that*

$$\lim_{n\to\infty} \lambda_n = \lambda_0,$$
$$\lim_{n\to\infty} \max_{x\in\Omega_{\lambda_n}} |x| = 0.$$

(2) *If the invariant sets* Ω_λ *are attractors of (2.1), then the bifurcation is called attractor bifurcation.*

2.2. A spectral theorem. A complex number $\beta = \alpha + i\rho \in \mathbb{C}$ is called an eigenvalue of a linear operator $L : H_1 \to H$ if there exist $x, y \in H_1$ with $x \neq 0$ such that

$$Lz = \beta z \qquad (z = x + iy). \tag{2.5}$$

The space

$$E_\beta = \{x, y \in H_1 \quad | \quad (L - \beta)^n z = 0, z = x + iy,\ \text{for some } n \in \mathbb{N}\}$$

is called the eigenspace of L corresponding to β, and $x, y \in E_\beta$ are called eigenvectors of L.

A linear mapping $L^* : H_1 \to H$ is called the conjugate operator of $L : H_1 \to H$, if

$$\langle Lx, y\rangle_H = \langle x, L^* y\rangle_H, \forall x, y \in H_1.$$

A linear operator $L : H_1 \to H$ is symmetric if $L = L^*$.

The following spectral theorem for a completely continuous field can be found in [5], which can be regarded as a unified version of the classical Jordan theorem and the Fredholm alternative theorem.

THEOREM 2.2. *Let $L = -A + B = H_1 \to H$ be a linear completely continuous field. Then the following assertions hold true.*

(1) *If $\{\beta_k \mid k \geq 1\} \subset \mathbb{C}$ are the eigenvalues of L, then we can take the eigenvectors $\{\varphi_k\} \subset H_1$ of L and eigenvectors $\{\varphi_k^*\} \subset H_1$ of the conjugate operator L^* such that*

$$\langle \varphi_i, \varphi_j^*\rangle_H = \delta_{ij}, \qquad \delta_{ij} \text{ the Kronecker symbol.}$$

(2) *H can be decomposed into the following direct sum*

$$\begin{aligned} &H = E_1 \oplus E_2,\\ &E_1 = \text{span}\,\{\varphi_k \mid k \geq 1\},\\ &E_2 = \{v \in H \mid \langle v, \varphi_k^*\rangle_H = 0, \forall\ k \geq 1\}. \end{aligned}$$

(3) *For any $u \in H$ we have the generalized Fourier expansion.*

$$u = \sum_k u_k \varphi_k + v, \qquad u_k = \langle u, \varphi_k^*\rangle_H, \qquad v \in E_2.$$

In particular, if L is symmetric, then

$$u = \sum_{k=1}^{\infty} u_k \varphi_k, \qquad u_k = \langle u, \varphi_k\rangle_H.$$

2.3. Higher order nondegeneracy. Let the nonlinear operator $G(\cdot, \lambda) : H_1 \to H$ in (2.1) has the Taylor expansion near $u = 0$ as follows

$$G(u, \lambda) = G_1(u, \lambda) + o(\|u\|_1^k), \qquad k \geq 2 \text{ an integer}, \tag{2.6}$$

where $G_1 : H_1 \times \cdots \times H_1 \to H$ is a k multilinear mapping, and we set

$$G_1(u, \lambda) = G_1(u, \cdots, u, \lambda). \tag{2.7}$$

Let $\beta_j(\lambda) \in \mathbb{C}$ be the eigenvalues (counting the multiplicity) of L_λ. Assume that $\beta_i(\lambda) (1 \leq i \leq m)$ are real, and

$$\beta_i(\lambda) \begin{cases} < 0, & \lambda < \lambda_0,\\ = 0, & \lambda = \lambda_0, \qquad 1 \leq i \leq m,\\ > 0, & \lambda > \lambda_0, \end{cases} \tag{2.8}$$

$$\begin{cases} \text{Re}\beta_j(\lambda_0) > 0, & m < j \leq m + n,\\ \text{Re}\beta_j(\lambda_0) < 0, & \forall\ m + n < j. \end{cases} \tag{2.9}$$

Let $\{e_1, \cdots, e_r\}$ and $\{e_1^*, \cdots, e_r^*\} \subset H_1$ be the eigenvectors of L_λ and L_λ^* respectively at $\lambda = \lambda_0$ satisfying

$$L_{\lambda_0} e_j = 0, \qquad L_{\lambda_0}^* e_j^* = 0, \qquad 1 \leq j \leq r, \tag{2.10}$$

where $r \leq m$ is the geometric multiplicity of $\beta_1(\lambda_0)$. Let

$$a^i_{j_1 \cdots j_k}(\lambda) = \langle G_1(e_{j_1}, \cdots, e_{j_k}, \lambda), e_i^* \rangle_H.$$

Definition 2.3. *Under the conditions (2.6)-(2.9), the operator $L_\lambda + G(\cdot, \lambda)$ is called k-th order nondegenerate at $(u, \lambda) = (0, \lambda_0)$, if $x = (x_1, \cdots, x_r) = 0$ is an isolated singular point of the following r-dimensional algebraic equations*

$$\sum_{j_1, \cdots, j_k = 1}^{r} a^i_{j_1 \cdots j_k}(\lambda_0) x_{j_1} \cdots x_{j_k} = 0, \qquad 1 \leq i \leq r. \tag{2.11}$$

In this case, $u = 0$ is also called a k-th order nondegenerate singular point of $L_\lambda + G(\cdot, \lambda)$ at $\lambda = \lambda_0$.

2.4. Attractor bifurcation theorem. We consider the finite system given by

$$\frac{dx}{dt} = A_\lambda x + G(x, \lambda), \quad \lambda \in \mathbb{R}, \quad x \in \mathbb{R}^n \quad (n \geq 2), \tag{2.12}$$

where $G = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is C^r $(r \geq 1)$ on $x \in \mathbb{R}^n$, and continuous on $\lambda \in \mathbb{R}$,

$$G(x, \lambda) = o(|x|), \ \forall \lambda \in \mathbb{R}, \tag{2.13}$$

and

$$A_\lambda = \begin{pmatrix} a_{11}(\lambda) \dots a_{1n}(\lambda) \\ \vdots \qquad \vdots \\ a_{n1}(\lambda) \dots a_{nn}(\lambda) \end{pmatrix}, \tag{2.14}$$

where $a_{ij}(\lambda)$ are continuous functions of λ.

Let all eigenvalues (counting the multiplicity) of (2.12) be denoted by $\beta_1(\lambda), \cdots, \beta_n(\lambda)$.

Assume that

$$\mathrm{Re}\beta_i(\lambda) \begin{cases} < 0, & \lambda < \lambda_0, \\ = 0, & \lambda = \lambda_0, \\ > 0, & \lambda > \lambda_0, \end{cases} \qquad 1 \leq i \leq m, \tag{2.15}$$

$$\mathrm{Re}\beta_j(\lambda_0) < 0, \qquad \forall m + 1 \leq j \leq n. \tag{2.16}$$

Let the eigenspace of A_λ at λ_0 be

$$E_0 = \cup_{i=1}^m \{x \in \mathbb{R}^n \mid (A_{\lambda_0} - \beta_i(\lambda_0))^k x = 0, k = 1, 2, \cdots\}.$$

The following attractor bifurcation theorem for (2.12) was proved in Ma and Wang [4, 6].

Theorem 2.4. *Under the conditions (2.15) and (2.16), if $x = 0$ is locally asymptotically stable for (2.12) at $\lambda = \lambda_0$, then the following assertions hold true.*

(1) *The system (2.12) bifurcates from* $(0,\lambda_0)$ *on* $\lambda > \lambda_0$ *an attractor* Σ_λ *with* $m-1 \le \dim \Sigma_\lambda \le m$, *which is connected as* $m \ge 2$.
(2) Σ_λ *is a limit of a sequence of* m*-dimensional annulus* M_k *with* $M_{k+1} \subset M_k$, *i.e.* $\Sigma_\lambda = \cap_{k=1}^{\infty} M_k$.
(3) *If* Σ_λ *is a finite simplicial complex, then* Σ_λ *has the homotopy type of* S^{m-1}.
(4) *For any* $x_\lambda \in \Sigma_\lambda, x_\lambda$ *can be expressed as*

$$x_\lambda = z_\lambda + o(|z_\lambda|), z_\lambda \in E_0.$$

2.5. Morse index and nondegeneracy of singular points. In order to investigate the dynamic bifurcation of (2.1), it is necessary to consider the regularity of bifurcated branches for the following stationary equation of (2.1)

$$L_\lambda u + G(u,\lambda) = 0. \tag{2.17}$$

Definition 2.5. *Under conditions (2.8) and (2.9), a bifurcated branch* $\Gamma(\lambda) \subset H_1$ *of (2.17), from* $(0,\lambda_0)$, *is called regular if for any* $|\lambda - \lambda_0| \neq 0$ *sufficiently small, each singular point* $u_\lambda \in \Gamma(\lambda)$ *of (2.17) is nondegenerate, i.e. the derivative operator,*

$$L_\lambda + DG(u_\lambda,\lambda) : H_1 \to H, \tag{2.18}$$

of (2.17) at u_λ *is a linear homeomorphism.*

Hereafter, we always assume (2.3) and (2.4). Thus, the number of eigenvalues with positive real part (counting multiplicities) is finite. Hence, we can define the Morse index for any nondegenerate singular point $u_\lambda \in \Gamma(\lambda)$ of (2.17) as follows:

$$k = \text{ number of all eigenvalues having positive real part.}$$

It is known that if a nondegenerate singular point u_λ of (2.17) has Morse index zero, then u_λ is an attractor. If a nondegenerate singular point u_λ of (2.17) has Morse index k ($k \ge 1$), then u_λ is called a saddle point of (2.1). We know that a saddle point u_λ of (2.1) with Morse index k has a k-dimensional unstable manifold and a stable manifold with codimension k in H.

By (2.8) and (2.9), near $\lambda = \lambda_0$, the spaces H_1 and H can be decomposed into the form

$$\begin{aligned} &H_\alpha = E_1^\lambda \oplus F_\alpha^\lambda, \qquad (\alpha = 1,0),\\ &E_1^\lambda = \text{span }\{e_1(\lambda),\cdots,e_m(\lambda)\},\\ &F_\alpha^\lambda = \text{ the complement of } E_1^\lambda \text{ in } H_\alpha, \end{aligned}$$

where $e_i(\lambda)$ ($1 \le i \le m$) are the eigenvectors of L_λ corresponding to $\beta_i(\lambda)$. Near λ_0, (2.17) can be decomposed into

$$\mathcal{L}_1^\lambda v + P_1 G(v+w,\lambda) = 0, \tag{2.19}$$

$$\mathcal{L}_2^\lambda w + P_2 G(v+w,\lambda) = 0, \tag{2.20}$$

where $v \in E_1^\lambda, w \in F_1^\lambda$, and

$$\begin{aligned} &\mathcal{L}_1^\lambda = L_\lambda|_{E_1^\lambda} : E_1^\lambda \to E_1^\lambda,\\ &\mathcal{L}_2^\lambda = L_\lambda|_{F_1^\lambda} : F_1^\lambda \to F_0^\lambda, \end{aligned}$$

and $P_1 : H \to E_1^\lambda, P_2 : H \to F_0^\lambda$ the canonical projections.

By (2.9), the operator $\mathcal{L}_2^\lambda$ is invertible, therefore by the implicit function theorem, there exists a C^r implicit function near $(v, \lambda) = (0, \lambda_0)$ defined by

$$w = f(v, \lambda), \qquad v \in E_1^\lambda, \tag{2.21}$$

which is a solution of (2.20).

By the Lyapunov-Schmidt method, if $v_0(\lambda)$ is a bifurcation solution from $(0, \lambda_0)$ of the equation

$$\mathcal{L}_1^\lambda v + P_1 G(v + f(v, \lambda), \lambda) = 0, \tag{2.22}$$

then $(v_0(\lambda), f(v_0(\lambda), \lambda)) = u(\lambda)$ is a bifurcation solution of (2.19) and (2.20).

The following theorem is useful to verify the regularity of a bifurcated branch of (2.17).

THEOREM 2.6. *Let $u_0(\lambda) = (v_0(\lambda), f(v_0(\lambda), \lambda))$ be a bifurcation solution of (2.17) from $(0, \lambda_0)$. Then $u_0(\lambda)$ is a nondegenerate singular point of (2.17) if and only if $v_0(\lambda)$ is a nondegenerate singular point of (2.22).*

Proof. The derivative operator of (2.22) at v_0 is given by

$$\mathcal{L}_1^\lambda + D_v P_1 G + D_w P_1 G \circ Df|_{v=v_0} : E_1^\lambda \to E_1^\lambda. \tag{2.23}$$

On the other hand, the derivative operator of (2.17) at $u_0 = (v_0, f(v_0, \lambda))$ is invertible if and only if the following equations has no nonzero solution $u = (v, w) \in H_1$:

$$(\mathcal{L}_1^\lambda + D_v P_1 G) \cdot v + D_w P_1 G \cdot w = 0, \tag{2.24}$$

$$(\mathcal{L}_2^\lambda + D_w P_2 G) \cdot w + D_v P_2 G \cdot v = 0, \tag{2.25}$$

where the derivative is taken at $u_0 = (v_0, f(v_0, \lambda))$

Because $\|u_0\|$ is small near λ_0, by (2.4), $\|D_w P_2 G\|$ is also small. Therefore the operator

$$B = \mathcal{L}_2^\lambda + D_w P_2 G(u_0, \lambda) : H_1 \to H$$

is invertible. Thus it follows from (2.25) that

$$w = -B^{-1} \circ D_v P_2 G \cdot v. \tag{2.26}$$

Putting (2.26) in (2.24), we get

$$(\mathcal{L}_1^\lambda + D_v P_1 G - D_w P_1 G \circ B^{-1} \circ D_v P_2 G) \cdot v = 0, \qquad v \in E_1^\lambda. \tag{2.27}$$

We deduce from (2.20) that

$$Df \cdot v = (-B^{-1} \circ D_v P_2 G) \cdot v.$$

Hence, (2.23) is invertible if and only if (2.27) has no nonzero solution in E_1^λ. The proof is complete. ∎

2.6. An index formula. In order to investigate dynamic bifurcations of (2.1) from eigenvalues with multiplicity two, it is necessary to discuss the index of the following vector field at $x = 0$.

$$u = \begin{pmatrix} a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 \\ b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2 \end{pmatrix}. \tag{2.28}$$

We assume that the vector field (2.28) is 2nd order nondegenerate at $x = 0$, which implies that $a_{11}^2 + b_{11}^2 \neq 0$. Without loss of generality, we assume that $a_{11} \neq 0$.

Let

$$\triangle = a_{12}^2 - 4a_{11}a_{22},$$

and if $\triangle \geq 0$, let

$$\begin{aligned} \alpha_1 &= \frac{-a_{12} + \sqrt{\triangle}}{2a_{11}}, \\ \alpha_2 &= \frac{-a_{12} - \sqrt{\triangle}}{2a_{11}}, \\ \beta_i &= b_{11}\alpha_i^2 + b_{12}\alpha_i + b_{22}, \quad i = 1, 2. \end{aligned}$$

The following index theorem will be useful in studying dynamic bifurcation of (2.28) hereafter.

THEOREM 2.7. *Let the vector field (2.28) be 2nd order nondegenerate at $x = 0$, and $a_{11} \neq 0$. Then*

$$\operatorname{ind}(u, 0) = \begin{cases} 0, & \text{if } \triangle < 0 \text{ or } \beta_1\beta_2 > 0, \\ 2, & \text{if } a_{11}\beta_1 > 0 \text{ and } a_{11}\beta_2 < 0, \\ -2, & \text{if } a_{11}\beta_1 < 0 \text{ and } a_{11}\beta_2 > 0. \end{cases} \tag{2.29}$$

Proof. We proceed in several steps as follows.

STEP 1. When $\triangle = a_{12}^2 - 4a_{11}a_{22} < 0$, the following quadratic form is either positively or negatively definite:

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 > 0 \ (\text{or} < 0), \ \forall x \in \mathbb{R}^2, x \neq 0,$$

depending on the sign of a_{11}. Hence the following system of equations

$$\begin{cases} a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 = -\varepsilon^2 (\text{or} = \varepsilon^2), \\ b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2 = 0, \end{cases}$$

has no solution for any $\varepsilon \neq 0$, which implies that

$$\operatorname{ind}(u, 0) = 0, \text{ as } \ \triangle < 0. \tag{2.30}$$

STEP 2. In the case where $\triangle \geq 0$, the vector field u given in (2.28) can be rewritten as

$$u = \begin{pmatrix} a_{11}(x_1 - \alpha_1 x_2)(x_1 - \alpha_2 x_2) \\ b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2 \end{pmatrix}. \tag{2.31}$$

Since u is 2nd order nondegenerate at $x = 0$, $\beta_1 \cdot \beta_2 \neq 0$. By (2.31), $u = (0, \pm\varepsilon^2)^t$, with $\varepsilon \neq 0$, is equivalent to

$$x_1 = \alpha_i x_2, \qquad \beta_i x_2^2 = \pm\varepsilon^2 \quad (i = 1, 2). \tag{2.32}$$

If $\beta_1 \cdot \beta_2 > 0$, then one of the systems in (2.32), for either $+\varepsilon^2$ or $-\varepsilon^2$, has no solution, which means that the index of u at $x = 0$ is zero.

Step 3. When $\beta_1 \cdot \beta_2 < 0$, it is easy to see that $\alpha_1 \neq \alpha_2$ and $\triangle > 0$. The vector field $u = (u_1, u_2)^t$ given in (2.31) takes the following form:

$$\begin{cases} u_1 = a_{11}(x_1 - \alpha_1 x_2)(x_1 - \alpha_2 x_2), \\ u_2 = \dfrac{1}{(\alpha_1 - \alpha_2)^2}[\beta_1(x_1 - \alpha_2 x_2)^2 + \beta_2(x_1 - \alpha_1 x_2)^2 \\ \qquad\qquad + \gamma(x_1 - \alpha_1 x_2)(x_1 - \alpha_2 x_2)], \end{cases} \tag{2.33}$$

where $\gamma = -(2b_{11}\alpha_1\alpha_2 + b_{12}\alpha_1 + b_{12}\alpha_2 + 2b_{22})$. Let

$$\begin{aligned} &\beta_1 > 0, \quad \beta_2 < 0 \qquad \text{if } a_{11} > 0, \\ &\beta_1 < 0, \quad \beta_2 > 0 \qquad \text{if } a_{11} < 0. \end{aligned}$$

Then the solutions $y = (y_1, y_2)$ of (2.32) are given by

$$\begin{cases} y_1^{\pm} = \begin{cases} \alpha_1 y_2^{\pm}, & \text{if } a_{11} > 0, \\ \alpha_2 y_2^{\pm}, & \text{if } a_{11} < 0, \end{cases} \\ y_2^{\pm} = \begin{cases} \pm\beta_1^{-1/2}\varepsilon, & \text{if } a_{11} > 0, \\ \pm\beta_2^{-1/2}\varepsilon, & \text{if } a_{11} < 0. \end{cases} \end{cases} \tag{2.34}$$

Let

$$z_1 = x_1 - \alpha_1 x_2, \quad z_2 = x_1 - \alpha_2 x_2. \tag{2.35}$$

Then the Jacobian matrix of u is given by

$$J(u) = \begin{pmatrix} \dfrac{\partial u_1}{\partial z_1} & \dfrac{\partial u_1}{\partial z_2} \\ \dfrac{\partial u_2}{\partial z_1} & \dfrac{\partial u_2}{\partial z_2} \end{pmatrix} \begin{pmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \end{pmatrix}.$$

It is easy to see that

$$\det \begin{pmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \end{pmatrix} = \det \begin{pmatrix} 1 & -\alpha_1 \\ 1 & -\alpha_2 \end{pmatrix} = \alpha_1 - \alpha_2 > 0.$$

Hence we infer from (2.33) and (2.35) that

$$\det Ju(x) = (\alpha_1 - \alpha_2) \det \begin{pmatrix} a_{11} z_2 & a_{11} z_1 \\ \dfrac{2\beta_2 z_1 + \gamma z_2}{(\alpha_1 - \alpha_2)^2} & \dfrac{2\beta_1 z_2 + \gamma z_1}{(\alpha_1 - \alpha_2)^2} \end{pmatrix}. \tag{2.36}$$

On the other hand, by (2.34) we deduce that

$$\begin{cases} z_1^{\pm} = y_1^{\pm} - \alpha_1 y_2^{\pm} = \begin{cases} 0, & \text{if } a_{11} > 0, \\ \pm(\alpha_2 - \alpha_1)\beta_2^{-1/2}\varepsilon, & \text{if } a_{11} < 0, \end{cases} \\ z_2^{\pm} = y_1^{\pm} - \alpha_2 y_2^{\pm} = \begin{cases} \pm(\alpha_1 - \alpha_2)\beta_1^{-1/2}\varepsilon, & \text{if } a_{11} > 0, \\ 0, & \text{if } a_{11} < 0. \end{cases} \end{cases} \tag{2.37}$$

Therefore, by (2.36) and (2.37) we arrive

$$\det Ju(y^{\pm}) = \begin{cases} 2a_{11}\beta_1(\alpha_1 - \alpha_2)^{-1}(z_2^{\pm})^2, & \text{if } a_{11} > 0, \\ -2a_{11}\beta_2(\alpha_1 - \alpha_2)^{-1}(z_1^{\pm})^2, & \text{if } a_{11} < 0. \end{cases} \tag{2.38}$$

By the Brouwer degree theory, we know that

$$\operatorname{ind}(u, 0) = \deg(u, B_r, x_0), x_0 = (0, \varepsilon^2) \in B_r \tag{2.39}$$

where $B_r = \{x \in \mathbb{R}^2 | \ |x| < r\}$, and $r > 0$ sufficiently small.

It follows from (2.38) and (2.39) that

$$\begin{aligned} \operatorname{ind}(u, 0) =& \text{sign det } \mathrm{J}u(y^+) + \text{sign det } \mathrm{J}u(y^-) \\ =& 2, \text{ for } a_{11}\beta_1 > 0 \text{ and } a_{11}\beta_2 < 0. \end{aligned}$$

We can obtain in the same fashion that

$$\operatorname{ind}(u, 0) = -2, \text{ for } a_{11}\beta_1 < 0 \text{ and } a_{11}\beta_2 > 0.$$

Thus, the formula (2.29) is proved. The proof of the theorem is complete. ∎

Remark 2.8. If $a_{11} = 0$ and $b_{11} \neq 0$, we let

$$\widetilde{\triangle} = b_{12}^2 - 4b_{11}b_{12}.$$

If $\widetilde{\triangle} \geq 0$, we define

$$\begin{aligned} \widetilde{\alpha}_1 &= \frac{-b_{12} - \sqrt{\widetilde{\triangle}}}{2b_{11}}, \\ \widetilde{\alpha}_2 &= \frac{-b_{12} + \sqrt{\widetilde{\triangle}}}{2b_{11}}, \\ \widetilde{\beta}_i &= a_{11}\widetilde{\alpha}_i^2 + a_{12}\widetilde{\alpha}_i + a_{22}, i = 1, 2, \end{aligned}$$

then, the formula (2.29) is written as

$$\operatorname{ind}(u, 0) = \begin{cases} 0, & \widetilde{\triangle} < 0 \text{ or } \widetilde{\beta}_1\widetilde{\beta}_2 > 0, \\ 2, & b_{11}\widetilde{\beta}_1 < 0 \text{ and } \ b_{11}\widetilde{\beta}_2 > 0, \\ -2, & b_{11}\widetilde{\beta}_1 > 0 \text{ and } \ b_{11}\widetilde{\beta}_2 < 0. \end{cases}$$

Remark 2.9. The index formula (2.29) shows that a two dimensional vector field, which is 2nd order nondegenerate at $x = 0$, takes only values $\{0, \pm 2\}$ as its indices at $x = 0$. In fact, let u be an m-dimensional vector field, which is k-order nondegenerate at $x = 0$, defined by

$$u = \begin{pmatrix} \sum\limits_{j_1+\cdots+j_m=k} a^1_{j_1\cdots j_m} x_1^{j_1}\cdots x_m^{j_m} \\ \vdots \\ \sum\limits_{j_1+\cdots+j_m=k} a^m_{j_1\cdots j_m} x_1^{j_1}\cdots x_m^{j_m} \end{pmatrix}, \tag{2.40}$$

then its index at $x = 0$ is given by

$$\mathrm{ind}(u, 0) = \begin{cases} 0, & \text{if } m = \text{ odd}, k = \text{ even}, \\ \text{even}, & \text{if } m = \text{ even}, k = \text{ even}, \\ \text{odd}, & \text{if } k = \text{ odd}, \forall m \geq 1. \end{cases} \tag{2.41}$$

Moreover, the index of (2.40) at $x = 0$ takes values in the following range.

$$\mathrm{ind}(u, 0) = \begin{cases} 0, \pm 2, \cdots, \pm k^{m-1}, & \text{as } k = \text{even}, m = \text{even}, \\ \pm 1, \cdots, \pm k^{m-1}, & \text{as } k = \text{odd}. \end{cases} \tag{2.42}$$

The formula (2.41) was known [5], and (2.42) will be proved elsewhere.

3. Bifurcation From Simple Eigenvalues.

3.1. Main theorems. Now we study the dynamic bifurcation of (2.1) from a simple eigenvalue. We assume $n = 0$ in (2.9) for attractor bifurcation, i.e.

$$\mathrm{Re}\beta_j(\lambda_0) < 0, \ \forall j \geq m + 1. \tag{3.1}$$

Let $m = 1$ in (2.8), and

$$\mathcal{L}_{\lambda_0} e_1 = 0, \quad \mathcal{L}^*_{\lambda_0} e^*_1 = 0, \quad < e_1, e^*_1 >_H = 1.$$

Let

$$\alpha =< G_1(e_1, \lambda_0), e^*_1 >_H, \tag{3.2}$$

where G_1 is the k-multilinear operator defined by (2.7). Then we have the following bifurcation theorems.

THEOREM 3.1. *Assume (2.6)-(2.8), (3.1), $m = 1$, and $k =$ odd. Then the following assertions hold true.*

(1) *If $\alpha > 0$, then (2.1) bifurcates from $(0, \lambda_0)$ exactly two saddle points $v_1(\lambda), v_2(\lambda) \in H_1$ with Morse index one on $\lambda < \lambda_0$, and (2.1) has no bifurcation on $\lambda_0 < \lambda$.*
(2) *If $\alpha < 0$, then (2.1) bifurcates from $(0, \lambda_0)$ exactly two singular points $v_1(\lambda)$ and $v_2(\lambda)$, which are attractors on $\lambda_0 < \lambda$, and (2.1) has no bifurcation on $\lambda < \lambda_0$.*
(3) *If $\alpha < 0$ and $\lambda_0 < \lambda$, there is an open set $U \subset H$ with $0 \in U$ which can be decomposed into two open sets U^λ_1 and U^λ_2*

$$\overline{U} = \overline{U}^\lambda_1 + \overline{U}^\lambda_2, \qquad U^\lambda_1 \cap U^\lambda_2 = \emptyset,$$

such that $\Gamma = \partial U_1^\lambda \cap \partial U_2^\lambda$ *is the stable manifold of* $u = 0$ *with codimension one in* H, $v_i(\lambda) \in U_i^\lambda (i = 1, 2)$, *and*

$$\lim_{t\to\infty} \|u(t, u_0) - v_i\|_H = 0, \quad \textit{if} \ \ u_0 \in U_i^\lambda \quad (i = 1, 2),$$

where $u(t, u_0)$ *is the solution of (2.1) and (2.2).*

(4) *The bifurcated singular points* $v_1(\lambda)$ *and* $v_2(\lambda)$ *in above cases can be expressed as the following form*

$$v_{1,2}(\lambda) = \pm|\beta_1(\lambda)/\alpha\ |^{1/(k-1)} e_1 + o(|\beta_1/\alpha\ |^{1/(k-1)}).$$

THEOREM 3.2. *Assume (2.6)- (2.8) and (3.1) with* $m = 1$, $k =$*even, and* $\alpha \neq 0$. *Then the following assertions hold true.*

(1) *(2.1) bifurcates from* $(0, \lambda_0)$ *a unique saddle point* $v(\lambda)$ *with Morse index one on* $\lambda < \lambda_0$, *and a unique attractor* $v(\lambda) \in H_1$ *on* $\lambda_0 < \lambda$.

(2) *If* $\lambda_0 < \lambda$, *there is an open set* $U \subset H$ *with* $0 \in U$, *and* U *is divided into two open sets* U_1^λ *and* U_2^λ *by the stable manifold* Γ *having codimension one of* $u = 0$*:*

$$\overline{U} = \overline{U}_1^\lambda + \overline{U}_2^\lambda, \qquad U_1^\lambda \cap U_2^\lambda = \emptyset, \qquad \Gamma = \partial U_1^\lambda \cap \partial U_2^\lambda,$$

such that $v(\lambda) \in U_1^\lambda$, *and*

$$\lim_{t\to\infty} \|u(t, u_0) - v(\lambda)\|_H = 0 \ \ \textit{if} \ \ u_0 \in U_1^\lambda.$$

(3) *The bifurcated singular points* $v(\lambda)$ *of (2.1) can be expressed as*

$$v(\lambda) = -(\beta_1(\lambda)/\alpha)^{1/k-1} e_1 + o(|\beta_1/\alpha|^{1/k-1}).$$

REMARK 3.3. Theorem 3.1 corresponds to the classical pitchfork bifurcation.

REMARK 3.4. In general, if we replace (3.2) by

$$\langle G(xe_1 + f(xe_1, \lambda_0), \lambda_0), e_1^*\rangle_H = \alpha x^k + o(|x|^k), \qquad x \in \mathbb{R},$$

where $f(v, \lambda)$ is given by (2.21), then for $\alpha \neq 0$ and $k > 1$, Theorems 3.1 and 3.2 are valid.

REMARK 3.5. The topological structure of dynamic bifurcation of (2.1) is schematically shown in the center manifold in Figures 3.1-3.3.

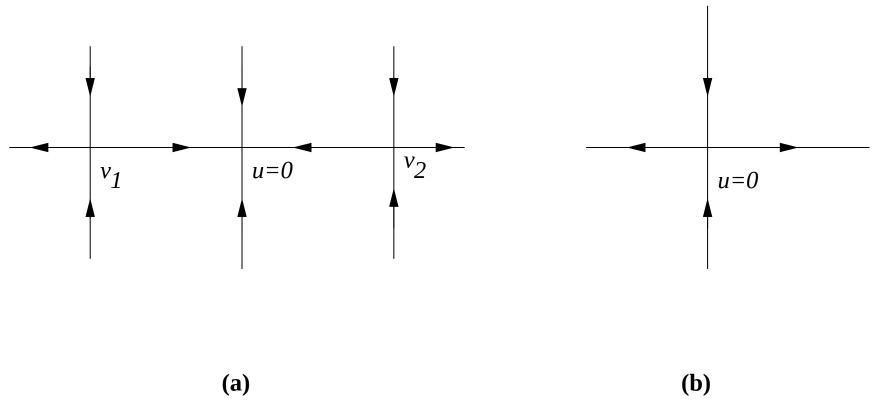


FIG. 3.1. *Topological structure of dynamic bifurcation of (2.1) when* $k =$ *odd and* $\alpha > 0$*: (a)* $\lambda < \lambda_0$*; (b)* $\lambda \geq \lambda_0$*. Here the horizontal line represents the center manifold.*

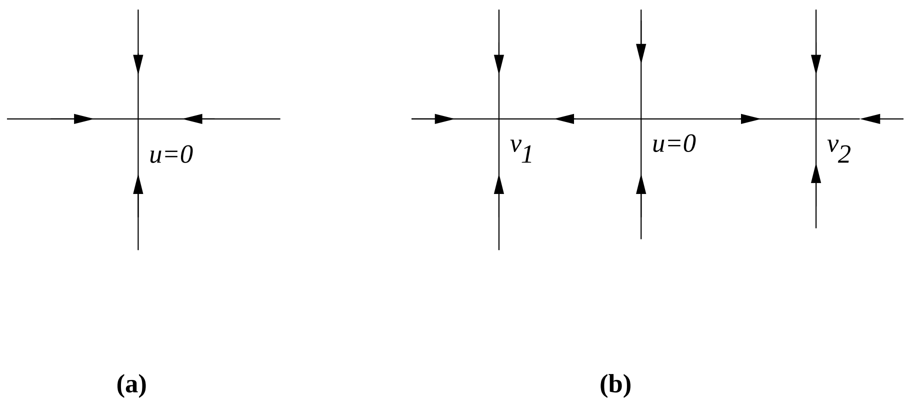


FIG. 3.2. *Topological structure of dynamic bifurcation of (2.1) when* $k =$ *odd and* $\alpha < 0$.

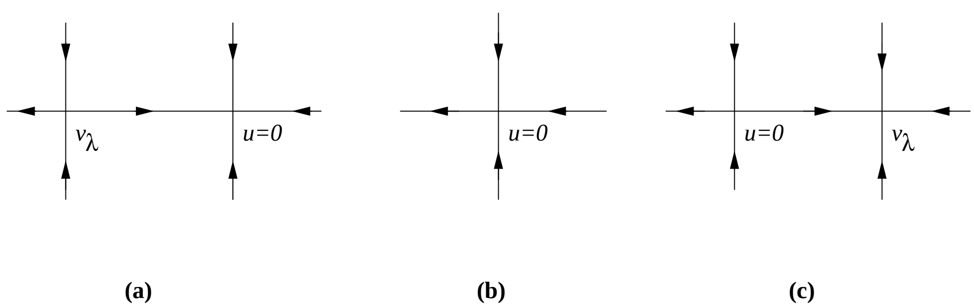


FIG. 3.3. *Topological structure of dynamic bifurcation of (2.1) when* $k =$ *even and* $\alpha \neq 0$.

If we replace the condition (3.1) in Theorems 3.1 and 3.2 by (2.9), then the bifurcated singular points of (2.1) are saddle points, which are characterized in the following theorem. The proof is trivial, and we omit the details.

THEOREM 3.6. *Assume the conditions (2.6)-(2.9) with* $m = 1$ *and* $\alpha \neq 0$ *in (3.2).*

Then the bifurcated singular points of (2.1) from $(0, \lambda_0)$ *have Morse index* $n+1$ *on* $\lambda < \lambda_0$*, and have Morse index* n *on* $\lambda_0 < \lambda$*. Moreover,* $u = 0$ *has Morse index* n *on* $\lambda < \lambda_0$ *and has Morse index* $n+1$ *on* $\lambda_0 < \lambda$*.*

3.2. Proof of Theorems 3.1, 3.2 and 3.6. By Theorem 2.2, (2.8) and $m = 1$, near $\lambda = \lambda_0$ we let $e_1(\lambda)$ and $e_1^*(\lambda)$ be the eigenvectors of L_λ and L_λ^* respectively such that

$$\begin{aligned} &L_\lambda e_1(\lambda) = \beta_1(\lambda), \\ &L_\lambda^* e_1^*(\lambda) = \beta_1(\lambda) e_1^*(\lambda), \\ &\langle e_1(\lambda), e_1^*(\lambda)\rangle_H = 1, \end{aligned}$$

and set

$$\begin{aligned} &H_1 = E^\lambda \oplus F_1^\lambda, \\ &H = E^\lambda \oplus F_0^\lambda, \\ &E^\lambda = \{x e_1(\lambda) \mid x \in \mathbb{R}\}, \\ &F_1^\lambda = \{u \in H_1 \mid \langle u, e_1^*(\lambda)\rangle_H = 0\}, \\ &F_0^\lambda = \{u \in H \mid \langle u, e_1^*(\lambda)\rangle_H = 0\}. \end{aligned}$$

Furthermore, we let $\mathcal{L}_1^\lambda : E^\lambda \to E^\lambda$ and $\mathcal{L}_2^\lambda : F_1^\lambda \to F_0^\lambda$ by $\mathcal{L}_1^\lambda(x e_1(\lambda)) = x\beta_1(\lambda) e_1(\lambda)$, and $\mathcal{L}_2^\lambda$ has eigenvalues $\beta_j(\lambda)$ $(j \geq 2)$ such that

$$L_\lambda = \mathcal{L}_1^\lambda \oplus \mathcal{L}_2^\lambda.$$

Hence, by the center manifold theorem [2], the dynamic bifurcation of (2.1) is equivalently reduced to

$$\frac{dx}{dt} = \beta_1(\lambda)x + \langle G(x e_1(\lambda) + h(x,\lambda), \lambda), e_1^*(\lambda)\rangle_H. \tag{3.3}$$

where h is the center manifold function satisfying

$$h(x, \lambda) = o(|x|), \forall \lambda \in \mathbb{R}. \tag{3.4}$$

By (2.6) and (2.7), (3.3) can be rewritten, near $\lambda = \lambda_0$, as

$$\frac{dx}{dt} = \beta_1(\lambda)x + \alpha_\lambda x^k + o(|x|^k), \tag{3.5}$$

where

$$\alpha_\lambda = \langle G_1(e_1(\lambda), \lambda), e_1^*(\lambda)\rangle_H \to \alpha, \quad \text{if} \quad \lambda \to \lambda_0. \tag{3.6}$$

On the other hand, the stationary bifurcation equation (2.22) can be written as

$$\beta_1(\lambda)x + \langle G(x e_1 + f(x e_1, \lambda), \lambda), e_1^*(\lambda)\rangle_H = 0. \tag{3.7}$$

By (2.6) and (2.7), the implicit function f satisfies

$$f(x e_1, \lambda) = o(|x|). \tag{3.8}$$

Therefore, we infer from (2.6), (2.7) and (3.8) that (3.7) takes the following form

$$\beta_1(\lambda)x + \alpha_\lambda x^k + o(|x|^k) = 0. \tag{3.9}$$

By Theorem 2.6, Assertions (1), (2) and (4) in Theorem 3.1, and Assertions (1) and (3) in Theorem 3.2 follow from (3.9). In addition, Assertion (3) in Theorem 3.1 and Assertion (2) in Theorem 3.2 can be deduced from (3.5) and (3.6).

The proofs of Theorems 3.1 and 3.2 are complete, and Theorem 3.6 can be proved in the same fashion. □

4. Bifurcation from Eigenvalues with Multiplicity Two.

4.1. Main theorems. Under the conditions (2.8)-(2.10), the integers m and r are the algebraic and geometric multiplicities of the eigenvalue $\beta_1(\lambda_0) = \beta_2(\lambda_0)$ of L_λ at $\lambda = \lambda_0$. Here, we assume that $m = r = k = 2$, and the operator $\mathcal{L}_\lambda + G(\cdot, \lambda)$ is second-order nondegenerate at $(u, \lambda) = (0, \lambda_0)$.

Let

$$\begin{aligned}
a_{11}(\lambda) &= \langle G_1(e_1(\lambda), e_1(\lambda), \lambda), e_1^*(\lambda)\rangle_H,\\
a_{22}(\lambda) &= \langle G_1(e_2(\lambda), e_2(\lambda), \lambda), e_1^*(\lambda)\rangle_H,\\
a_{12}(\lambda) &= \langle G_1(e_1(\lambda), e_2(\lambda), \lambda) + G_1(e_2(\lambda), e_1(\lambda), \lambda), e_1^*\rangle_H,\\
b_{11}(\lambda) &= \langle G_1(e_1(\lambda), e_1(\lambda), \lambda), e_2^*(\lambda)\rangle_H,\\
b_{22}(\lambda) &= \langle G_1(e_2(\lambda), e_2(\lambda), \lambda), e_2^*(\lambda)\rangle_H,\\
b_{12}(\lambda) &= \langle G_1(e_1(\lambda), e_2(\lambda), \lambda) + G_1(e_2(\lambda), e_1(\lambda), \lambda), e_2^*(\lambda)\rangle_H,
\end{aligned}$$

where G_1 is given by (2.7), and $e_i(\lambda), e_j^*(\lambda)$ $(i, j = 1, 2)$ are the eigenvectors of L_λ and L_λ^* near λ_0:

$$L_\lambda e_i(\lambda) = \beta_i(\lambda)e_i(\lambda),\ L_\lambda^* e_j^*(\lambda) = \beta_j(\lambda)e_j^*(\lambda),\ i, j = 1, 2.$$

Thus, we obtain a vector field

$$u_0(\lambda) = \begin{pmatrix} a_{11}(\lambda)x_1^2 + a_{12}(\lambda)x_1x_2 + a_{22}(\lambda)x_2^2 \\ b_{11}(\lambda)x_1^2 + b_{12}(\lambda)x_1x_2 + b_{22}(\lambda)x_2^2 \end{pmatrix} \tag{4.1}$$

By assumption, u_0 is second order nondegenerate at $x = 0$ near λ_0.

To proceed, we need to recall a theorem on steady state bifurcation given in Part I of this series [5].

THEOREM 4.1. *Let (2.6)-(2.9) with $r = k = 2$ hold true, and that $L_\lambda + G(\cdot, \lambda)$ be second-order nondegenerate at $(u, \lambda) = (0, \lambda_0)$, and the two vectors $(a_{11}(\lambda), a_{12}(\lambda), a_{22}(\lambda))$ and $(b_{11}(\lambda), b_{12}(\lambda), b_{22}(\lambda))$, are linearly independent near λ_0. Then we have the following assertions.*

1. *There are at most 3 bifurcated branches of (2.17) from $(0, \lambda_0)$ on each side of $\lambda = \lambda_0$.*
2. *If all bifurcated branches on one side are regular, then the number of branches on this side is either 1 or 3.*
3. *If the number of branches on one side is 3, then all branches on this side must be regular.*
4. *If the number of branches on one side is 2, then one of them is regular.*

By Theorem 2.7, the index of u_0 given by (4.1) at $x = 0$ is either 0, or 2 or -2. Now we state the main dynamic bifurcations of (2.1) in each situation. We start with the case where $\text{ind}(u_0(\lambda_0), 0) = -2$.

THEOREM 4.2. *Let the conditions (2.6)-(2.9) with $m = r = k = 2$ hold true, $L_\lambda + G(\cdot, \lambda)$ be second order nondegenerate at $(u, \lambda) = (0, \lambda_0)$, and $ind(u_0(\lambda_0), 0) = -2$ for u_0 defined in (4.1). Then (2.1) bifurcates exactly 3 saddle points with Morse index $n + 1$ from $(0, \lambda_0)$ on each side of $\lambda = \lambda_0$, where n is given in (2.9).*

For other two cases, we need to introduce a notation. A $S(\theta) \subset \mathbb{R}^2$ is called a sectorial region with angle $\theta \in [0, 2\pi]$, if $S(\theta)$ is enclosed by two curves γ_1, γ_2 starting with $x = 0$ and an arc Γ, and the angle between the two tangent lines L_1 and L_2 of γ_1 and γ_2 at $x = 0$ is θ; see Figure 4.1. Let $S_r(\theta)$ be the sectorial domain with angle θ and radius $r > 0$ given by

$$S_r(\theta) = \{x \in \mathbb{R}^2 |\ |x| < r, \quad \text{and } x \in S(\theta)\}.$$

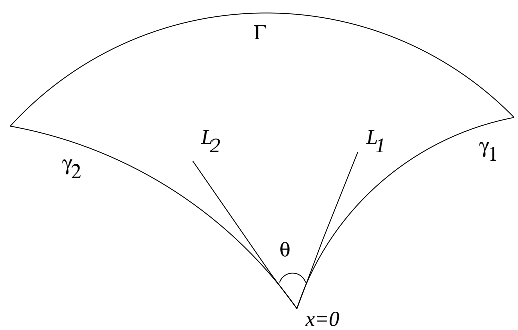


FIG. 4.1.

THEOREM 4.3. *Assume (2.6)-(2.8), (3.1), $m = r = k = 2$, and $\beta_1(\lambda) = \beta_2(\lambda)$ near λ_0. Let $L_\lambda + G(\cdot, \lambda)$ be 2nd order nondegenerate at $(0, \lambda_0)$, and $u_0(\lambda)$ be given by (4.1). Then the following assertions hold true.*

(1) *If $ind(u_0(\lambda_0), 0) = 2$, then (2.1) bifurcates an attractor $\mathcal{A}_\lambda$ with $\dim \mathcal{A}_\lambda \leq 1$ from $(0, \lambda_0)$ on $\lambda_0 < \lambda$, and $\mathcal{A}_\lambda$ attracts a sectorial region $S_r(\theta)$ in H with angle $\theta \in (\pi <, 2\pi]$, and radius $r > 0$.*
(2) *The attractor $\mathcal{A}_\lambda$ contains minimal attractors, which are singular points, as shown in Figure 4.2 (a) - (c).*
(3) *If $ind(u_0(\lambda_0), 0) = 0$ and (2.1) bifurcates from $(0, \lambda_0)$ three singular points on $\lambda_0 < \lambda$, then one of them is an attractor, which attracts a sectorial region $S_r(\theta)$ with $0 < \theta < \pi$, as shown in Figure 4.3 (a) and (b).*

REMARK 4.4. If $\beta_1(\lambda) \neq \beta_2(\lambda)$ near $\lambda = \lambda_0$ and $\lambda \neq \lambda_0$, then Theorem 4.3 may not be valid. Consider for instance

$$(4.2) \qquad \begin{cases} \dfrac{dy_1}{dt} = 4\lambda y_1 + \lambda y_2 + y_1^2 - 4y_1 y_2 - y_2^2, \\ \dfrac{dy_2}{dt} = -\lambda y_1 + y_1 y_2. \end{cases}$$

It is easy to see that $y = (0, \lambda)$ is a unique bifurcated singular point of (4.2) from $\lambda_0 = 0$, which is not an attractor on $\lambda_0 < \lambda$ near $(y, \lambda) = (0, \lambda_0)$, which has the topological structure as shown in Figure 4.4. ∎

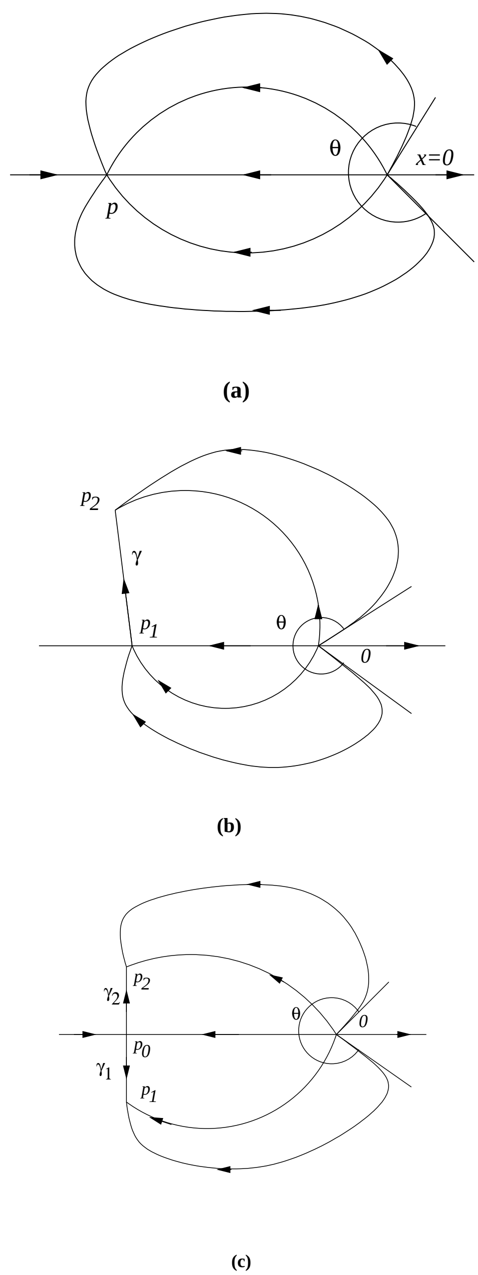
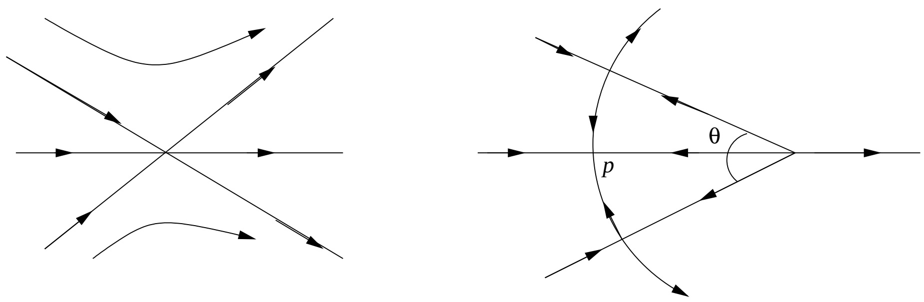


FIG. 4.2. *(a) If (2.1) bifurcates one singularity* p*, the attractor* $\mathcal{A}_\lambda = \{p\}$*; (b) If (2.1) bifurcates two singularities* p_1 *and* p_2*, then* $\mathcal{A}_\lambda = \gamma \cup \{p_1, p_2\}$*, where* γ *is the orbit connecting* p_1 *and* p_2*; and (c) If (2.1) bifurcates three singularities* p_0*,* p_1 *and* p_3*, then* $\mathcal{A}_\lambda = \gamma_1 \cup \gamma_2 \cup \{p_1, p_2, p_3\}$*, where* γ_i *are the orbits connecting* p_0 *and* p_i*.*



(a) **(b)**

FIG. 4.3. *(a)* $\lambda = \lambda_0$, *(b)* $\lambda_0 < \lambda$ *with* $\{p\}$ *being an attractor.*

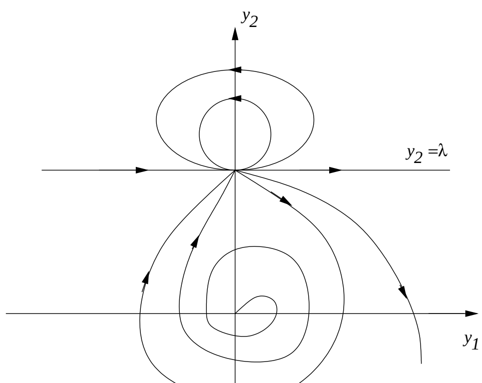


FIG. 4.4.

4.2. Proof of Theorems 4.2 and 4.3.

4.2.1. Center manifold reduction. By the center manifold theorem, the dynamic bifurcation of (2.1) is equivalently reduced to that of the following equations

$$\begin{cases} \dfrac{dx_1}{dt} = \beta_1(\lambda)x_1 + \langle G(x_1e_1 + x_2e_2 + h(x,\lambda), e_1^*(\lambda)\rangle_H, \\ \dfrac{dx_2}{dt} = \beta_2(\lambda)x_2 + \langle G(x_1e_1 + x_2e_2 + h(x,\lambda), e_2^*(\lambda)\rangle_H, \end{cases} \tag{4.3}$$

where $h(x, \lambda)$ is the center manifold function satisfying (3.4) for $x \in \mathbb{R}^2$. Thus, near $(x, \lambda) = (0, \lambda_0)$, (4.3) can be written as

$$\frac{dx}{dt} = B_\lambda x + F(x,\lambda) + o(|x|^2), \tag{4.4}$$

where

$$B_\lambda x = \begin{pmatrix} \beta_1(\lambda) & 0 \\ 0 & \beta_2(\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \beta_1(\lambda)x_1 \\ \beta_2(\lambda)x_2 \end{pmatrix},$$

$$F(x,\lambda) = \begin{pmatrix} a_{11}(\lambda)x_1^2 + a_{12}(\lambda)x_1x_2 + a_{22}(\lambda)x_2^2 \\ b_{11}(\lambda)x_1^2 + b_{12}(\lambda)x_1x_2 + b_{22}(\lambda)x_2^2 \end{pmatrix},$$

and a_{ij}, b_{ij} are as in (4.1).

Since F is 2nd order nondegenerate at $(x, \lambda) = (0, \lambda_0)$, the vector field on the right hand side of (4.4) is a perturbation of $B_\lambda + F$ near $x = 0$. Hence, it suffices to prove Theorems 4.2 and 4.3 for the following system

$$\frac{dx}{dt} = B_\lambda x + F(x, \lambda). \tag{4.5}$$

4.2.2. Proof of Theorem 4.2. The proof can be achieved by Theorem 2.6 and the following lemma.

LEMMA 4.5. *If ind$(F(\cdot, \lambda_0), 0) = -2$, then (4.5) bifurcates from $(x, \lambda) = (0, \lambda_0)$ exactly three saddle points with Morse index one on each side of $\lambda = \lambda_0$.*

Proof. By Theorem 2.7, as $\text{ind}(F, 0) = -2$, the two vectors (a_{11}, a_{12}, a_{22}) and (b_{11}, b_{12}, b_{22}) are linearly independent. Therefore it follows from Theorem 4.1 that (4.5) has at most three bifurcated singular points from $(0, \lambda_0)$. We shall prove that (4.5) has just three bifurcated singular points on each side of $\lambda = \lambda_0$.

It is known that

$$\begin{aligned}
&\text{ind } (B_\lambda + F, 0) = \text{sign } [\beta_1(\lambda) \cdot \beta_2(\lambda)] = 1, \text{ if } \ \lambda \neq \lambda_0, \\
&\sum_{i=1}^{k} \text{ind } (B_\lambda + F, p_i) + \text{ind } (B_\lambda + F, 0) = \text{ind } (F, 0) = -2,
\end{aligned}$$

where p_i $(1 \leq i \leq k)$ are the bifurcated singular points of (4.5) from $(0, \lambda_0)$. Hence, we have

$$\sum_{i=1}^{k} \text{ind } (B_\lambda + F, p_i) = -3, \qquad \text{if } \lambda \neq \lambda_0. \tag{4.6}$$

If the number $k < 3$ in (4.6), then one of bifurcated singular points, say p_1, of (4.5) satisfies that

$$| \text{ ind } (B_\lambda + F, p_1)| \geq 2. \tag{4.7}$$

By the Brouwer degree theory, if

$$J(B_\lambda + F)(p_1) \neq 0,$$

then the index is reduced to the index of a one-dimensional operator at the isolated singular point, which can only be 0 and ± 1. Hence

$$|\text{ind}(B_\lambda + F, p_1)| \leq 1.$$

Therefore, it follows from (4.7) that the Jacobian matrix of $B_\lambda + F$ at p_1 is zero:

$$J(B_\lambda + F)(p_1) = B_\lambda + \left(\frac{\partial F_i(p_1)}{\partial x_j} \right) = 0. \tag{4.8}$$

Let $p_1 = (z_1, z_2)$, then we infer from (4.8) that

$$\begin{cases}
\beta_1 + 2a_{11}z_1 + a_{12}z_2 = 0, \\
a_{12}z_1 + 2a_{22}z_2 = 0, \\
\beta_2 + 2b_{22}z_2 + b_{12}z_1 = 0, \\
b_{12}z_2 + 2b_{11}z_1 = 0,
\end{cases} \tag{4.9}$$

which, together with $B_\lambda p_1 + F(p_1, \lambda) = 0$, imply that $p_1 = 0$, a contradiction to $p_1 \neq 0$. Thus, we have shown that $k = 3$. From (4.6) and Theorem 4.1 we have

$$\text{ind}(B_\lambda + F, p_i) = -1, \ i = 1, 2, 3,$$

which implies that p_i $(1 \leq i \leq 3)$ are saddle points with Morse index one. This proof is complete. □

4.2.3. An index formula and stability of extended orbits. In order to prove Theorem 4.3, we need the following two lemmas. The first one is known as the Poincare formula; see [1].

LEMMA 4.6. *Let v be a two dimensional $C^r (r \geq 0)$ vector field with $v(0) = 0$. Then*

$$ind(v, 0) = 1 + \frac{1}{2}(e - h), \tag{4.10}$$

where e is the number of elliptic regions, and h number of hyperbolic regions. Here the elliptic, hyperbolic and parabolic regions E, H and P in a neighborhood $U \subset \mathbb{R}^2$ of $x = 0$ are defined as follows; see Figure 4.5.

$$\begin{aligned} &E = \{x \in U | \ \textit{the orbits} \ \ S(t)x \ \ \textit{and} \ \ S(-t)x \to 0 \ \textit{as} \ t \to \infty\}, \\ &H = \{x \in U | \ S(t)x, S(-t)x \notin U \ \textit{for some} \ \ t > t_0 > 0\}, \\ &P = \{x \in U | \ \textit{either} \ S(t)x \to 0 (t \to \infty), S(-t)x \notin U (t > t_0), \\ &\qquad \textit{or} \ S(-t)x \to 0, S(t)x \notin U, \textit{or} \ S(t)x, S(-t)x \in U, \forall t \geq 0\}. \end{aligned}$$

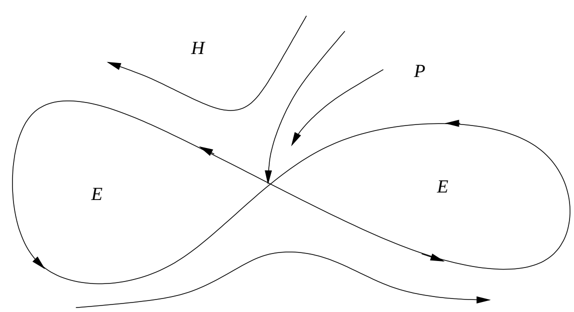


FIG. 4.5.

Next we need a technical lemma on stability of extended orbits for vector fields. Let $v \in C^r(U, \mathbb{R}^n)$ be a vector field where $U \subset \mathbb{R}^n$ is an open set. A curve $\gamma \subset U$ is called an extended orbit of v, if γ is a union of curves

$$\gamma = \bigcup_{i=1} \gamma_i$$

such that either γ_i is an orbit of v, or γ_i consists of singular points of v, and if γ_i and γ_{i+1} are orbits of v, then the ω–limit set of γ_i is the α-limit set of γ_{i+1},

$$\omega(x) = \alpha(y), \qquad \forall x \in \gamma_i, \ y \in \gamma_{i+1}.$$

Namely, endpoints of γ_i are singular points of v, and the starting endpoint of γ_{i+1} is the finishing endpoint of γ_i; see Figure 4.6.

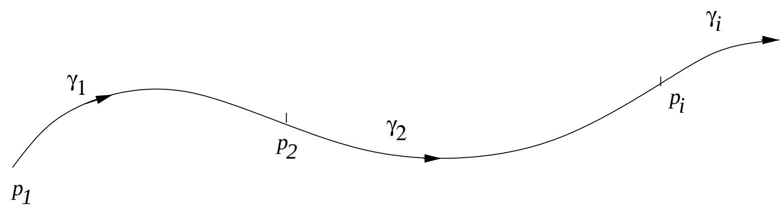


FIG. 4.6.

Then we have the following stability lemma of extended orbits. The result of this lemma has been proved and used in Step 2 of the proof of Lemma 4.5 in [3]. Here we only state the result as a lemma.

LEMMA 4.7. (Stability of Extended Orbits [3]). *Let $v_k \in C^r(U, \mathbb{R}^n)$ be a consequence of vector fields with $\lim_{k\to\infty} v_k = v_0 \in C^r(U, \mathbb{R}^n)$. Suppose that $\gamma_k \subset U$ is an extended orbit of v_k with finite length uniformly with respect to k, and the starting points p_1^k of γ_k converge to p_1, then the extended orbits γ_k of v_k converge, up to taking a subsequence, to an extended orbit γ of v_0 with starting point p_1.*

4.2.4. Proof of Theorem 4.3. The proof of Theorem 4.3 can be derived in a few lemmas as follows. Here, we always assume that $\beta(\lambda) = \beta_1(\lambda) = \beta_2(\lambda)$ for λ near λ_0.

First, by the homotopy invariance of indices, for λ near λ_0,

$$\operatorname{ind}(F(\cdot, \lambda), 0) = \operatorname{ind}(F(\cdot, \lambda_0), 0). \tag{4.11}$$

LEMMA 4.8. *Let* $\operatorname{ind} F(\cdot, \lambda), 0) = 0$ *or* 2. *Then for λ near λ_0, the vector fields $F(x, \lambda)$ have k straight orbit lines with $1 \le k \le 3$:*

$$\alpha_i x_1 + \beta_i x_2 = 0, \quad \alpha_i^2 + \beta_i^2 \neq 0, \quad i = 1, \cdots, k, \tag{4.12}$$

where $\sigma_i = \alpha_i/\beta_i$ or $\sigma_i = -\beta_i/\alpha_i$ are the solutions of the following algebraic equation:

$$\begin{cases} a_{22}\sigma^3 + (a_{12} - b_{22})\sigma^2 + (a_{11} - b_{12})\sigma - b_{11} = 0, \\ or \\ b_{11}\sigma^3 + (b_{12} - a_{11})\sigma^2 + (b_{22} - a_{12})\sigma - a_{22} = 0. \end{cases} \tag{4.13}$$

Proof. When $F(x, \lambda)$ are second-order nondegenerate at $x = 0$ near λ_0, $a_{11}^2 + b_{11}^2 \neq 0$. We assume that $a_{11} \neq 0$. By the homogeneity of $F(x, \lambda)$, a straight line $x_2 = \sigma x_1$ is an orbit line of $F(x, \lambda)$ if and only if

$$\begin{aligned} \sigma &= \frac{F_2(x, \lambda)}{F_1(x, \lambda)} \\ &= \frac{b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2}{a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2} \\ &= \frac{b_{11} + b_{12}\sigma + b_{22}\sigma^2}{a_{11} + a_{12}\sigma + a_{22}\sigma^2}. \end{aligned}$$

Hence, the straight lines (4.12) satisfying (4.13) are orbit lines of $F(x, \lambda)$. Obviously, one of the two equations in (4.13) has a solution. Thus we obtain that the number of solutions of (4.13) is k $(1 \leq k \leq 3)$. The proof is complete. □

LEMMA 4.9. *If* $\mathrm{ind}(F(\cdot, \lambda_0), 0) = 2$, *then we have*

1. $F(x, \lambda)$ *has no hyperbolic regions at* $x = 0$,
2. $F(x, \lambda)$ *has exactly two elliptic regions* E_1 *and* E_2,
3. $F(x, \lambda)$ *has no parabolic regions if* $k = 1$, *which is the number of solutions of (4.13), and has exactly two parabolic regions* P_1 *and* P_2, *if* $k \geq 2$,
4. *the elliptic and parabolic regions* E *and* P *are sectorial regions* $E = S(\theta_1), P = S(\theta_2)$ *with* $0 < \theta_1, \theta_2 < \pi, \theta_1 + \theta_2 = \pi$, *and the edges of* $S_r(\theta_1)$ *and* $S_r(\theta_2)$ *are the straight orbit lines of* $F(x, \lambda)$; *see Figure 4.7 (a)-(c).*

Proof. Based on Lemma 4.8, we take an orthogonal coordinate transformation $y = Ax$ with a straight orbit line of $F(x, \lambda)$ as the y_1-axis. Under this transformation, the vector field $F(x, \lambda)$ is changed into the following form

$$\widetilde{F}(y, \lambda) = \begin{pmatrix} \widetilde{a}_{11}y_1^2 + \widetilde{a}_{12}y_1y_2 + \widetilde{a}_{22}y_2^2 \\ y_2(\widetilde{b}_1y_1 + \widetilde{b}_2y_2) \end{pmatrix}. \tag{4.14}$$

Since $\mathrm{ind}(\widetilde{F}(\cdot, \lambda), 0) = 2$, $\widetilde{b}_1 \neq 0$. Otherwise, there is no solution for $\widetilde{F} = (0, -\mathrm{sign}(\widetilde{b}_1)\varepsilon)^t$ for any $\varepsilon > 0$ small. Hence the index is zero, a contradication.

Take another coordinate transformation as follows

$$\begin{aligned} x_1' &= \widetilde{b}_1y_1 + \widetilde{b}_2y_2, \\ x_2' &= y_2. \end{aligned}$$

Then, by Theorem 2.7 the vector field in (4.14) is transformed into

$$F'(x', \lambda) = \begin{pmatrix} F_1' \\ F_2' \end{pmatrix} = \begin{pmatrix} a(x_1' - \alpha_1x_2')(x_1' + \alpha_2x_2') \\ bx_1'x_2' \end{pmatrix} \tag{4.15}$$

where $a \cdot b > 0, \alpha_1, \alpha_2 > 0$.

It is known that $F(x, \lambda)$ and $F'(x', \lambda)$ have the same topological structure. It is easy to see that (4.15) has the topological structure as shown in Figure 4.7(a) - (c) for $a, b > 0$ in (4.15).

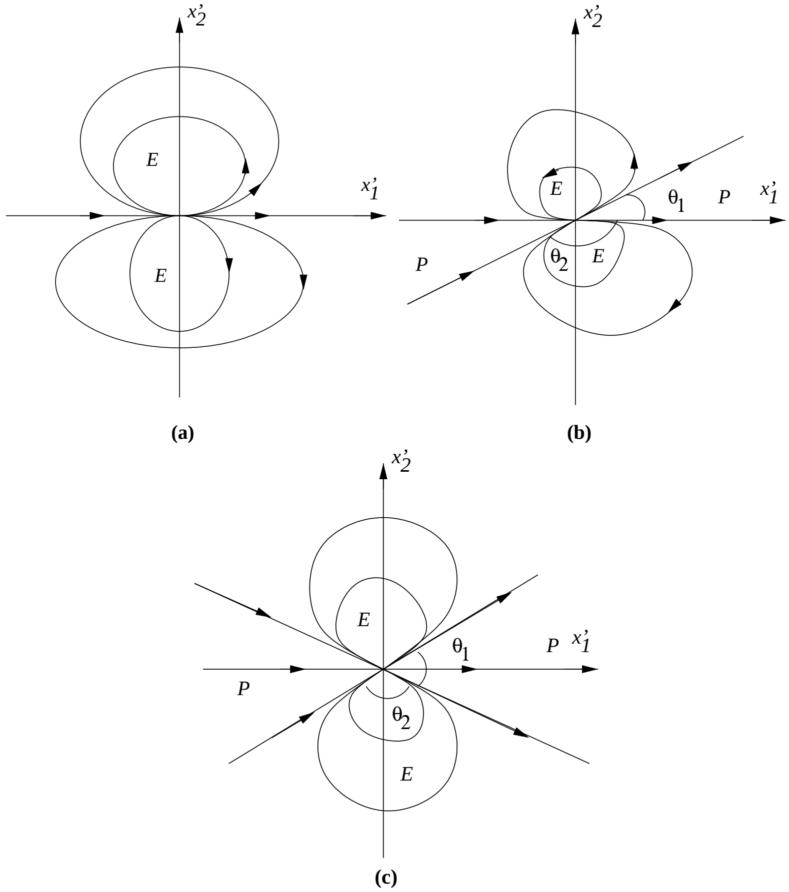


FIG. 4.7. *Toplogical structure of (4.15): (a) The number of straight orbit lines $k = 1$, (b) $k = 2$, and (c) $k = 3$.*

To derive the topological structure in Figure 4.7(a)-(c) of (4.15). Let D_1, D_2, D_3 and D_4 be the 4 open quadrants in $\mathbb{R}^2$, and the two straight lines $x_1 - \alpha_1 x_2 = 0, x_1 + \alpha_2 x_2 = 0$ also divide the plane $\mathbb{R}^2$ into four regions

$$
\begin{aligned}
Q_1 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - \alpha_1 x_2 > 0, x_1 + \alpha_2 x_2 > 0\},\\
Q_2 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - \alpha_1 x_2 < 0, x_1 + \alpha_2 x_2 > 0\},\\
Q_3 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - \alpha_1 x_2 > 0, x_1 + \alpha_2 x_2 < 0\},\\
Q_4 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - \alpha_1 x_2 < 0, x_1 + \alpha_2 x_2 < 0\}.
\end{aligned}
$$

It is easy to see that

$$
\left\{
\begin{array}{l}
F_1' > 0 \ \text{ in } \ Q_1 \ \text{ and } \ Q_4,\\
F_1' < 0 \ \text{ in } \ Q_2 \ \text{ and } \ Q_3,\\
F_2' > 0 \ \text{ in } \ D_1 \ \text{ and } \ D_3,\\
F_2' < 0 \ \text{ in } \ D_2 \ \text{ and } D_4.
\end{array}
\right.
\tag{4.16}
$$

The properties (4.16) ensure that (4.15) has only two elliptic regions E_1 and E_2, with $E_1 \subset \mathbb{R}^2_+ = \{(x_1, x_2) | x_2 > 0\}$ and $E_2 \subset \mathbb{R}^2_- = \{(x_1, x_2) | x_2 < 0\}$; see Figure 4.8.

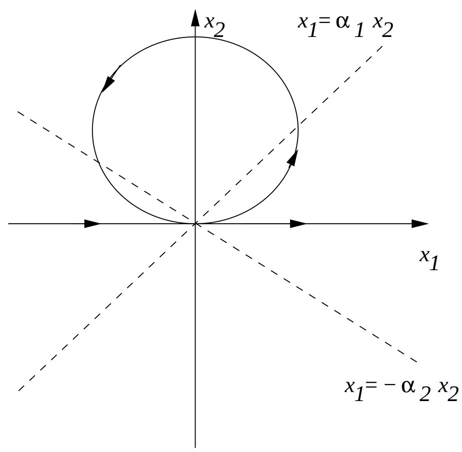


FIG. 4.8.

Thus, by Lemma 4.6, F' has no hyperbolic regions, and Assertions (1) and (2) are proved.

By (4.13), it follows from (4.15) that the straight orbit lines $L_i (i = 0, 1, 2)$ of $F'(x', \lambda)$ are given by

$$L_0 : x_2 = 0, \qquad L_1 : x_2 = \sigma_1 x_1, \qquad L_2 : x_2 = \sigma_2 x_1,$$

and if σ_1, σ_2 are real, then

$$\sigma_1 = \frac{1}{\alpha_1} - \varepsilon_1, \qquad \sigma_2 = -\frac{1}{\alpha_2} + \varepsilon_2,$$

for some real numbers $0 < \varepsilon_1 < 1/\alpha_1$ and $0 < \varepsilon_2 < 1/\alpha_2$. Hence we have

$$L_i \subset Q_1 \cup Q_3. \tag{4.17}$$

Therefore, Assertions (3) and (4) follow from (4.17) and the symmetry of $F(x, \lambda)$, i.e. $F(-x, \lambda) = F(x, \lambda)$. The proof is complete. ∎

LEMMA 4.10. *If ind$(F, 0) = 2$, then (4.5) bifurcates from $(x, \lambda) = (0, \lambda_0)$ an attractor $\mathcal{A}_\lambda$ on $\lambda_0 < \lambda$, which attracts a sectorial region $S_r(\theta)$ with $\pi < \theta \le 2\pi$. Actually, $S_r(\theta) \subset (E_1 \cup E_2 \cup P_1) \cap B_r$, where E_1, E_2 are the elliptic regions of F. P_1 is the parabolic region where all orbits of F reach $x = 0$, $B_r = \{x \in \mathbb{R}^2 | |x| < r\}$, $\theta = 2\pi - \theta_0$, and θ_0 the angle of the parabolic region.*

Proof. We know that under an orthogonal coordinate system transformation, the linear operator

$$B_\lambda = \begin{pmatrix} \beta(\lambda) & 0 \\ 0 & \beta(\lambda) \end{pmatrix}$$

is invariant. Therefore, without loss of generality, we take the vector F as given by (4.14). By Theorem 2.7, $F(x, \lambda)$ can be written as

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} a(x_1 - \alpha_1 x_2)(x_1 + \alpha_2 x_2) \\ b x_2 (x_1 - \sigma x_2) \end{pmatrix}, \tag{4.18}$$

where $a \cdot b > 0, \alpha_1^2 + \alpha_2^2 \neq 0$.

We proceed with the case where $a, b > 0$ and $\sigma \geq 0$. The other case can be proved in the same fashion. By Theorem 2.7 we know that $\alpha_1 > \sigma > \alpha_2$, which implies that the lines $x_1 - \alpha_i x_2 = 0 (i = 1, 2)$, $x_2 = 0$, and $x_1 - \sigma x_2 = 0$ are alternatively positioned in $\mathbb{R}^2$.

Based on the definition of elliptic and parabolic regions, by Lemma 4.9 we obtain that

$$\lim_{t\to\infty} S_\lambda(t)x = 0, \ \forall x \in E_1 \cup E_2 \cup P_1, \tag{4.19}$$

where $S_\lambda(t)$ is the operator semigroup generated by $F(x, \lambda)$.

On the other hand, we obtain from (4.18) that for any $x \in E_1 \cup E_2 \cup P_1$, there is a $t_0(x) \geq 0$ such that

$$S_\lambda(t)x \in D = \{x \in \mathbb{R}^2 \mid x_1 - \sigma x_2 < 0\}, \ \forall\ t \geq t_0(x); \tag{4.20}$$

see Figure 4.9.

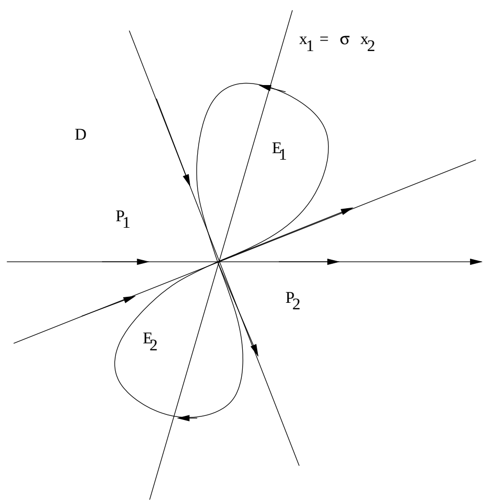


FIG. 4.9.

It is clear that $P_1 \subset D \subset E_1 \cup E_2 \cup D$. Let

$$D(r) = \{x \in D \mid |x| < r\},$$
$$D(r_1, r_2) = \{x \in D \mid 0 < r_1 < |x| < r_2\}.$$

Let $T_\lambda(t)$ be the operator semigroup generated by $B_\lambda + F(\cdot, \lambda)$. It is known that for $\lambda > \lambda_0$ all orbits of $B_\lambda x$ are straight lines emitting outward from $x = 0$. Therefore, by (4.20) we deduce that

$$T_\lambda(t)x \in D, \quad \forall\ t > 0, \quad x \in \partial D, x \neq 0. \tag{4.21}$$

Now, we shall prove that for any $\lambda - \lambda_0 > 0$ sufficiently small there are $r_1, r_2, r_3 > 0$ with $r_1 < r_2 < r_3$ such that

$$T_\lambda(t)x \in D(r_1, r_2), \ \forall\ x \in D(r_3), t > t_x, \tag{4.22}$$

for some $t_x \geq 0$.

We know that for $\lambda > \lambda_0$, the singular point $x = 0$ of $B_\lambda + F$ has an unstable manifold M^u with $\dim M^u = 2$. We take $r_1 > 0$ such that the ball $B_{r_1} \subset M^u$. Then, by (4.21) we obtain that

$$T_\lambda(t)x \in D(r_1, r_2),\ \forall\ x \in D(r_1), t > t_x; \tag{4.23}$$

see Figure 4.9.

If (4.22) is not valid, then by (4.21) and (4.23) there exist $\lambda_n \to \lambda_0 + 0, t_n \to \infty$ and $\{x_n\} \subset D(r_3)$ such that

$$|\ T_{\lambda_n}(t_n)x_n\ | \geq r_2,\ \forall\ n \geq 1. \tag{4.24}$$

Let $x_n \to x_0 \in D(r_3)$. Then by Lemma 4.7 and (4.24) there is an orbit line γ of $F(\cdot, \lambda_0)$ with starting point $x_0 \in D(r_3)$ which does not reach to $x = 0$. This is a contradiction to (4.19).

It follows from (4.21) and (4.22) that $D(r_1, r_2)$ is an absorbing set in a neighborhood U of $D(r_1, r_2)$. Hence, by the existence theorem of attractors, for $\lambda > \lambda_0$, the set

$$\mathcal{A}_\lambda = \omega(D(r_1, r_2), \lambda),$$

with $0 \notin \mathcal{A}_\lambda$, is an attractor of (4.5), which attracts $D(r_3)$. Here the ω-limit set $\omega(D, \lambda)$ of a set $D \subset \mathbb{R}^2$ for $B_\lambda + F(\cdot, \lambda)$ is defined by

$$\omega(D, \lambda) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} T_\lambda(t)D}.$$

Applying Lemma 4.7 again we infer from (4.19) that

$$\lim_{\lambda \to \lambda_0} \max_{x \in \mathcal{A}_\lambda} |x| = 0.$$

Thus $\mathcal{A}_\lambda$ is a bifurcated attractor of (4.5) from $(0, \lambda_0)$.

We can deduce from (4.20) that $\mathcal{A}_\lambda$ attracts a sectorial region $S_r(\theta) \subset E_1 \cup E_2 \cup P_1$, with $\theta = 2\pi - \theta_0$, where θ_0 is the angle of the parabolic region P_2. The proof is complete. □

Lemma 4.11. *The attract* $\mathcal{A}_\lambda$ *has dimension* $\dim \mathcal{A}_\lambda \leq 1$, *and* $\mathcal{A}_\lambda$ *contains minimal attractors consisting of singular points.*

Proof. It is clear that $\mathcal{A}_\lambda$ contains all singular points of (4.5). We shall prove that $\mathcal{A}_\lambda$ does not contain closed orbit line.

By Lemma 4.8, all singular points of (4.5) must be in the straight orbit lines L of $F(x, \lambda)$, and L are invariant sets of (4.5) which consist of orbits and singular points.

Use the method as in the proof of Lemma 4.9, for any straight orbit line L we can take an orthogonal coordinate system transformation with L as its x_1-axis. Thus, the vector field $F(x, \lambda)$ take the form of (4.18), and the singular point $x_0 = (x_1^0, x_2^0)$ of $B_\lambda + F$ on L is given by

$$x_0 = (x_1^0, x_2^0) = (-\beta(\lambda)/a, 0),$$

and the Jacobian matrix of $B_\lambda + F$ at x_0 is given by

$$J(B_\lambda + F)(x_0) = \begin{pmatrix} -\beta(\lambda) & * \\ 0 & (1 - \frac{b}{a})\beta(\lambda) \end{pmatrix}. \tag{4.25}$$

Hence, on each straight orbit line there is only one singular point x_0 of $B_\lambda + F$, and there are two orbits γ_1 and γ_2 in L reaching to x_0. Moreover one of γ_1 and γ_2 connects from $x = 0$ to x_0. It follows that the attractor $\mathcal{A}_\lambda$ containing all singular points has no closed orbit lines. By the Poincare-Bendixon theorem we obtain that $\dim \mathcal{A}_\lambda \leq 1$.

When $B_\lambda + F$ has three singular points z_i $(1 \leq i \leq 3)$, by Theorem 4.1 they are regular, and

$$\text{ind}\,(B_\lambda + F, z_1) = -1, \text{ ind}\,(B_\lambda + F, z_2) = \text{ind}\,(B_\lambda + F, z_3) = 1. \tag{4.26}$$

It follows from (4.25) and (4.26) that z_1 is a saddle point, z_2 and z_3 are attractors. In this case the attractor $\mathcal{A}_\lambda$ has the structure as shown in Figure 4.2 (c).

When $B_\lambda + F$ has two singular points z_1 and z_2,

$$\text{ind}(B_\lambda + F, z_1) = 0, \qquad \text{ind}(B_\lambda + F, z_2) = 1,$$

which implies, by (4.25), that z_2 is an attractor, and z_1 has exactly two hyperbolic regions. Thus, $\mathcal{A}_\lambda$ has the topological structure as shown in Figure 4.2 (b).

When $B_\lambda + F$ has only one singular point z, then

$$\text{ind}(B_\lambda + F, z) = 1,$$

which implies by (4.25) that z is an attractor, and $\mathcal{A}_\lambda = \{z\}$ has the topological structure as shown in Figure 4.2 (a). The proof is complete. □

Note that Lemmas 4.9-4.11 are still valid. Hence if there is a higher order nonlinear perturbation for the vector field $F(x, \lambda)$, Assertions (1) and (2) of Theorem 4.3 follows from Lemma 4.10 and Lemma 4.11. Assertion (3) of Theorem 4.3 is an immediately consequence of the following lemma.

Lemma 4.12. *If ind$(F, 0) = 0$, and (4.5) bifurcates three singular points from $(0, \lambda_0)$ on $\lambda_0 < \lambda$, then one of them is an attractor which attracts a sectorial region $D_r(\theta)$ with $0 < \theta < \pi$.*

Proof. By Theorem 4.1 the three bifurcated singular points p_i $(1 \leq i \leq 3)$ are nondegenerate, and

$$\text{ind}\,(B_\lambda + F, p_1) = \text{ind}\,(B_\lambda + F, p_2) = -1, \text{ ind}\,(B_\lambda + F, p_3) = 1.$$

Then as in the proof of Lemma 4.11, we can deduce that p_3 is an attractor.

Since p_1 and p_2 are in the other two straight orbit lines which enclose the parabolic region P, the singular point $p_3 \in P$ and attracts a domain $P \cap B_r = D_r(\theta)$ for some $r > 0$. The proof of the lemma is complete. □

5. Bifurcation to Periodic Solutions. In this subsection, we consider the case where $m = 2, r = 1$ and $k = odd \geq 3$ in (2.7) - (2.10). Since $m = 2$ and $r = 1$, the two eigenvectors v_1, v_2 of L_λ at $\lambda = \lambda_0$ enjoy the following properties; see Theorem 2.2:

$$\begin{aligned} & L_{\lambda_0} v_1 = 0, \quad L_{\lambda_0} v_2 = v_1, \\ & L^*_{\lambda_0} v^*_2 = 0, \quad L_{\lambda_0} v^*_1 = v^*_2, \\ & \langle v_i, v^*_j \rangle_H \begin{cases} > 0, & i = j, \\ = 0, & i \neq j. \end{cases} \end{aligned}$$

Let $\alpha \in \mathbb{R}$ be the number defined by

$$\alpha = \langle G_1(v_1, \lambda_0), v^*_2 \rangle_H, \tag{5.1}$$

where G_1 is as in (2.7).

Then, we have the following bifurcation theorem of periodic orbits from the real eigenvalues with $m = 2$ and $r = 1$.

THEOREM 5.1. *Assume the conditions (2.6)-(2.9) with $m = 2, r = 1$ and $k =$ odd ≥ 3. Let α be given by (5.1). If $\alpha < 0$, then (2.1) bifurcates from $(u, \lambda_0) = (0, \lambda_0)$ a periodic orbit.*

Proof. STEP 1. By the center manifold theorem, it suffices to consider the bifurcation of the following equations

$$\begin{cases} \frac{dx_1}{dt} = \beta_1(\lambda)x_1 + ax_2 + \langle G(x + h(x,\lambda),\lambda), v_1^*(\lambda)\rangle_H, \\ \frac{dx_2}{dt} = \qquad \beta_2(\lambda)x_2 + \langle G(x + h(x,\lambda),\lambda), v_2^*(\lambda)\rangle_H, \end{cases} \tag{5.2}$$

where $x = x_1 v_1(\lambda) + x_2 v_2(\lambda), h(x,\lambda)$ is the center manifold function,

$$\begin{aligned} & a =< v_1(\lambda), v_1^*(\lambda) >_H > 0, \\ & L_\lambda v_1(\lambda) = \beta_1(\lambda) v_1(\lambda), \\ & L_\lambda v_2(\lambda) = \beta_2(\lambda) v_2(\lambda) + v_1(\lambda), \\ & L_\lambda^* v_1^*(\lambda) = \beta_1(\lambda) v_1^*(\lambda) + v_2^*(\lambda), \\ & L_\lambda^* v_2^*(\lambda) = \beta_2(\lambda) v_2^*(\lambda), \\ & \langle v_i(\lambda),\ v_j^*(\lambda)\rangle_H \begin{cases} > 0, \text{ if } i = j, \\ = 0, \text{ if } i \neq j. \end{cases} \end{aligned}$$

By (2.6) and (5.1), equation (5.2) at $\lambda = \lambda_0$ reads as

$$\frac{dx}{dt} = F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}, \tag{5.3}$$

where

$$\begin{aligned} & F_1(x) = ax_2 + 0(|x_1|^k, |x_2|^k), \\ & F_2(x) = \alpha x_1^k + 0(|x_1|^{k+1}, |x_1|^1 |x_2|^{k-1}, |x_1|^2 |x_2|^{k-2} \cdots, |x_2|^k). \end{aligned}$$

Since $a > 0, \alpha < 0$ and $k = odd \geq 3$, we have

$$\operatorname{ind}(F, 0) = 1. \tag{5.4}$$

STEP 2. We now prove that the number of elliptic regions of F at $x = 0$ is zero, i.e. $e = 0$. Assume otherwise, then there exist an orbit γ of (5.3) connected to $x = 0$, i.e.

$$\lim_{t\to\infty} S(t)x = 0, \forall x \in \gamma,$$

where $S(t)$ is the operator semigroup generated by (5.3). Let γ be expressed near $x = 0$ as

$$x_2 = f(x_1), (x_1, x_2) \in \gamma.$$

From (5.3) it follows that for any $(x_1, x_2) \in \gamma$

$$\frac{dx_2}{dx_1} = \frac{\alpha x_1^k + 0(|x_1|^{k+1}, |x_1|^1|f(x_1)|^{k-1}, |x_1|^2|f(x_1)|^{k-2} \cdots , |f(x_1)|^k)}{af(x_1) + 0(|x_1|^k, |f(x_1)|^k)}.$$

Thus we obtain

$$af(x_1)f'(x_1) + 0(|x_1|^k, |f|^k)f' = \alpha x_1^k + 0(|x_1|^{k+1}, |x_1|^{k-i}|f|^i), \tag{5.5}$$

which implies that

$$f(x) = \beta x^m + o(|x|^m), \quad 2 \le m = \frac{k+1}{2} < k, \beta \neq 0. \tag{5.6}$$

Therefore, from (5.5) and (5.6) we get

$$am\beta^2 = \alpha(\alpha < 0, m \geq 2 \text{ and } \beta \neq 0).$$

It is a contradiction. Hence $e = 0$.

STEP 3. By (5.4) and the Poincaré formula (4.10), $h = 0$. Therefore $x = 0$ must be a degenerate singular point, and is either (a) a stable focus, or (b) an unstable focus or (c) a singular point having infinite periodic orbits in its neighborhood.

The case (c) implies a bifurcation of periodic orbits for (5.2).

For the case (a), $x = 0$ is an asymptotically stable singular point of (5.3). Then by Theorem 2.4, the equation (5.2) bifurcates from $(x, \lambda) = (0, \lambda_0)$ an S^1-attractor Σ_λ on $\lambda > \lambda_0$. For the case (b), $x = 0$ is an asymptotically stable singular point of the vector field $-F(x)$, therefore the vector field

$$\begin{pmatrix} \beta_1(\lambda) & 0 \\ 0 & \beta_2(\lambda) \end{pmatrix} x - F(x)$$

bifurcates from $(0, \lambda_0)$ an S^1-attractor Σ_λ on $\lambda > \lambda_0$, which implies that (5.2) bifurcates from $(0, \lambda_0)$ on $\lambda < \lambda_0$ an S^1- repelor Σ_λ, which is an invariant set.

STEP 4. Now, we need to prove that the S^1-invariant set Σ_λ contains no singular points. Consider the following equations

$$\beta_1(\lambda)x_1 + ax_2 + 0(|x_1|^k, |x_2|^k) = 0, \tag{5.7}$$

$$\beta_2(\lambda)x_2 + \alpha x_1^k + 0(|x_1|^{k+1}, |x_2|^{k-i}|x_1|^i) = 0. \tag{5.8}$$

Hence

$$x_2 = -\beta_1(\lambda)a^{-1}x_1 + 0(|x_1|^k, |\beta_1|^k), \tag{5.9}$$

$$\alpha x_1^{k-1} - a^{-1}\beta_1(\lambda)\beta_2(\lambda) + o(|x_1|^{k-1}, |\beta_1|^2) = 0. \tag{5.10}$$

By $\alpha < 0, a > 0$ and $\beta_1(\lambda)\beta_2(\lambda) > 0$, (5.10) has no solution near $(x, \lambda) = (0, \lambda_0)$. Hence, there is no singular points in Σ_λ, which means Σ_λ must contain a periodic orbit. The proof is complete. $\square$

6. An Application. As an example we consider the following equations

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \triangle u_1 + \alpha_1 \lambda u_1 + a u_2 + g_1(u_1, u_2), \\ \dfrac{\partial u_2}{\partial t} = \triangle u_2 + \alpha_2 \lambda u_2 + g_2(u_1, u_2), \end{cases} \tag{6.1}$$

where λ_1, λ_2 are parameters, $a > 0$ a constant, $\Omega \subset \mathbb{R}^n$ $(n \le 3)$ is bounded smooth domain, and

$$\begin{aligned} g_1 &= \sum_{i+j=2} a^1_{ij} u_1^i u_2^j + \sum_{i+j=3} a^2_{ij} u_1^i u_2^j + o(|u|^3), \\ g_2 &= \sum_{i+j=2} b^1_{ij} u_1^i u_2^j + \sum_{i+j=3} b^2_{ij} u_2^i u_2^j + o(|u|^3). \end{aligned}$$

Equations (6.1) are supplemented with the Dirichlet boundary conditions

$$u_1|_{\partial\Omega} = 0, \quad u_2|_{\partial\Omega} = 0. \tag{6.2}$$

Let $\lambda_1 > 0$ and $h_1(x)$ be the first eigenvalue and eigenvector of the Laplacian operator with the Dirichlet boundary condition

$$\begin{cases} -\triangle h_1 = \lambda h_1, \\ h_1|_{\partial\Omega} = 0. \end{cases}$$

Let

$$\begin{aligned} H &= L^2(\Omega, \mathbb{R}^2), \\ H_1 &= H^2_0(\Omega, \mathbb{R}^2). \end{aligned}$$

Then we define corresponding operators $L_\lambda = -A + B_\lambda : H_1 \to H$ and $G : H_1 \to H$ by

$$\begin{aligned} -Au &= \begin{pmatrix} \triangle u_1 \\ \triangle u_2 \end{pmatrix}, \\ B_\lambda &= \begin{pmatrix} \alpha_1 \lambda & a \\ 0 & \alpha_2 \lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ Gu &= \begin{pmatrix} g_1(u_1, u_2) \\ g_2(u_1, u_2) \end{pmatrix}. \end{aligned}$$

Case $0 < \alpha_1 < \alpha_2$. In this case, $\beta_1(\lambda) = \alpha_1 \lambda - \lambda_1$, $\lambda_0 = \alpha_1^{-1} \lambda_1$, and the first eigenvectors e_1 and e_1^* of L_λ and L_λ^* corresponding to $\beta_1(\lambda)$ are given by

$$e_1 = \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \qquad e_1^* = \begin{pmatrix} h_1 \\ -\dfrac{a h_1}{(\alpha_2 - \alpha_1)\lambda} \end{pmatrix}.$$

Let G_1 be the second-order homogeneous term of G given by

$$G_1 = \begin{pmatrix} \sum\limits_{i+j=2} a^1_{ij} u_1^i u_2^j \\ \sum\limits_{i+j=2} b^1_{ij} u_1^i u_2^j \end{pmatrix}. \tag{6.3}$$

Let

$$\alpha = \langle G_1(e_1), e_1^* \rangle_H = \left[a_{20}^1 - \frac{\alpha_1 a_1}{(\alpha_2 - \alpha_1)\lambda_1} b_{20}^1 \right] \int_\Omega h_1^3 dx.$$

If

$$\alpha \neq 0,$$

or equivalently

$$a_{20}^1 - \frac{\alpha_1 a_1}{(\alpha_2 - \alpha_1)\lambda_1} b_{20}^1 \neq 0, \tag{6.4}$$

then by Theorem 3.2, as $\lambda_0 = \alpha_1^{-1}\lambda_1 < \lambda$, the problem (6.1) and (6.2) bifurcate from $(0, \alpha^{-1}\lambda_1)$ an attractor

$$v_\lambda = -(\beta_1(\lambda) \cdot \alpha^{-1}) \left(\beta_1(\lambda) \cdot \alpha^{-1} \right) \begin{pmatrix} h_1 \\ 0 \end{pmatrix} + o\left(|\beta_1 \alpha^{-1}| \right).$$

Case $\alpha_1 = \alpha_2 > 0$ and $a = 0$. We have $\beta_1(\lambda) = \beta_2(\lambda) = \alpha_1\lambda - \lambda_1$, and

$$e_1 = e_1^* = \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \qquad e_2 = e_2^* = \begin{pmatrix} 0 \\ h_1 \end{pmatrix}.$$

Let u_0 be a vector field u_0 as in (4.1) defined by

$$u_0 = \begin{pmatrix} C(a_{20}^1 x_1^2 + a_{11}^1 x_1 x_2 + + a_{02}^1 x_2^2) \\ C(b_{20}^1 x_1^2 + b_{11}^1 x_1 x_2 + + b_{02}^1 x_2^2) \end{pmatrix},$$

where $c = \int_\Omega h_1^3 dx$. If u_0 is second-order nondegenerate, then Theorems 4.2 and 4.3 are applicable to the problem (6.1) and (6.2).

Case $\alpha_1 = \alpha_2 > 0$, $a_{ij}^1 = 0, b_{ij}^1 = 0$, and $a > 0$. In this case, we set

$$G_1 = \begin{pmatrix} \sum\limits_{i+j=3} a_{ij}^2 u_1^i u_2^i \\ \sum\limits_{i+j=3} b_{ij}^2 u_1^i u_2^i \end{pmatrix}.$$

Then $\beta_1 = \beta_2(\lambda) = \alpha_1\lambda - \lambda_1, \lambda_0 = \alpha_1^{-1}\lambda_1$. Let

$$v_1 = \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ a^{-1}h_1 \end{pmatrix},$$
$$v_2^* = \begin{pmatrix} 0 \\ h_1 \end{pmatrix}, \qquad v_1^* = \begin{pmatrix} a^{-1}h_1 \\ 0 \end{pmatrix}.$$

Then

$$L_{\lambda_0} v_2 = v_1,$$
$$L_{\lambda_0}^* v_1^* = v_2^*.$$

Let

$$\begin{aligned} \alpha &= \langle G_1(v_1), v_2^* \rangle_H \\ &= \int_\Omega b_{30}^2 h_1^3 \cdot a^{-1} h_1 dx = a^{-1} b_{30}^2 \int_\Omega h_1^4 dx. \end{aligned}$$

If $b_{30}^2 < 0$, then by Theorem 5.1, the problem (6.1) and (6.2) bifurcates from $(0, \lambda_0)$ a periodic orbit.

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